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Pointwise Fourier inversion with Cesàro means

Francisco Javier González Vieli

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Abstract. Conditions for pointwise Fourier inversion using spherical Cesàro means of a given degree are established in euclidean and hyperbolic spaces.

1. Introduction

To solve the Fourier inversion problem, that is, to reconstruct an integrable function f on $\mathbf{R}^n$ from its Fourier transform $\mathscr{F}f$ one has in general to use summation methods. For example it is known that the kth Cesàro means $\int_{\|t\|\leq N}(1-\|t\|/N)^k \mathscr{F}f(t)e^{2\pi i(x|t)}dt$ converge, when N tends to infinity, to $f(x)$ at every Lebesgue point x of f if $k > (n-1)/2$.

This is in general no more the case if $k \leq (n-1)/2$. For example, if f is the indicator function of the unit ball in $\mathbf{R}^3$ and $k = 0$, there is convergence everywhere except at $x = 0$, which is a Lebesgue point. In this work we determine for a large class of functions, including the above indicator, the least value of k implying convergence at a given point.

We do this not only on $\mathbf{R}^n$ but also on the real hyperbolic space $\mathbf{H}^n$. Our results: the more differentiable the spherical mean of the function, the smaller the degree k insuring convergence, are natural and show a complete parallelism between both spaces. We emphasize that still little is known about summability for Fourier transforms on $\mathbf{H}^n$ (see [5] and its bibliography). Forming the basis of our reasonings are those of [7], specified and corrected (see the remark at the end of §6).

2. Cesàro summability: definition and elementary properties

Definition 1. Let $b \in L^1_{loc}(\mathbf{R}_+)$, $k \geq 0$ and $B \in \mathbf{C}$. We say that b is *(C,k)-summable to B* if $\lim_{x\to+\infty}\int_0^x(1-(t/x))^k b(t)\,dt = B$ and we write

$$\int_0^{+\infty} b(t)\,dt = B \quad (C,k).$$

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REMARK. 1. If b is integrable on $\mathbf{R}_+$, it is $(C,0)$-summable to $\int_0^{+\infty} b(t)\,dt$.

LEMMA 1. *Let $b \in L^1_{loc}(\mathbf{R}_+)$, $k \geq 0$ and $B \in \mathbf{C}$. If b is (C,k)-summable to B, it is (C,k')-summable to B for all $k' > k$.*

PROOF: [4] p. 111.

PROPOSITION 1. *Let $k \geq 0$, $\lambda > -1$ and $a > 0$. Then*
i) $\int_0^{+\infty} t^\lambda e^{-ait}dt = e^{-(\lambda+1)\pi i/2}\Gamma(\lambda+1)a^{-\lambda-1}(C,k)$, *if $k > \lambda$;*
ii) $\int_0^x (1-t/x)^k t^\lambda e^{-ait}dt \sim \Gamma(\lambda+1)a^{-\lambda-1}(e^{-(\lambda+1)\pi i/2} + e^{(\lambda+1)\pi i/2}e^{-aix})$ *as $x \to +\infty$, if $k = \lambda$.*

PROOF: According to [4] p. 353,

$$\int_0^x (1-t/x)^k t^l e^{-ait}dt = e^{-(\lambda+1)\pi i/2}\int_0^{+\infty}(1+is/x)^k s^\lambda e^{-as}\,ds$$
$$+ e^{(k+1)\pi i/2}e^{-aix}x^{\lambda-k}\int_0^{+\infty} s^k(1-is/x)^\lambda e^{-as}\,ds.$$

When $x \to +\infty$, the first term on the right tends to

$$e^{-(\lambda+1)\pi i/2}\int_0^{+\infty} s^\lambda e^{-as}ds = e^{-(\lambda+1)\pi i/2}\Gamma(\lambda+1)a^{-\lambda-1}$$

and the second term behaves like

$$e^{(k+1)\pi i/2}e^{-aix}x^{\lambda-k}\int_0^{+\infty} s^k e^{-as}ds = e^{(k+1)\pi i/2}e^{-aix}x^{\lambda-k}\Gamma(k+1)a^{-k-1}.$$

The result follows.

REMARKS. 2. In particular $t^\lambda e^{-ait}$ is (C,k)-summable if and only if $k > \lambda$.
3. As special cases of i) we have for all $m \in \mathbf{Z}_+$:

$$\int_0^{+\infty} x^{2m}\cos x\,dx = 0\ (C,2m+1) \quad \text{and} \quad \int_0^{+\infty} x^{2m+1}\sin x\,dx = 0\ (C,2m+2).$$

4. Also $t^{-1}e^{-ait}\chi_{[1,+\infty[}(t)$ is $(C,0)$-summable. This is easily obtained with an integration by parts.

3. Summability for Bessel functions

PROPOSITION 2. *Let $\nu > -1$, J_ν the Bessel function of first kind and order ν and $k \geq 0$; then $\int_0^{+\infty} t^{\nu+1}J_\nu(t)\,dt = 0\ (C,k)$ if and only if $k > \nu + \frac{1}{2}$.*

PROOF: We show first that $t^{\nu+1}J_\nu(t)$ is (C,k)-summable if and only if $k > \nu + \frac{1}{2}$. According to [11] p. 199, when $z \to +\infty$

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\left[\cos(z - \nu\pi/2 - \pi/4)\sum_{m=0}^{+\infty}\frac{(-1)^m\Gamma(\nu+2m+\frac{1}{2})}{(2m)!\Gamma(\nu-2m+\frac{1}{2})(2z)^{2m}}\right.$$

$$\left. - \sin(z - \nu\pi/2 - \pi/4)\sum_{m=0}^{+\infty}\frac{(-1)^m\Gamma(\nu+2m+\frac{3}{2})}{(2m+1)!\Gamma(\nu-2m-\frac{1}{2})(2z)^{2m+1}}\right].$$

We note $\theta = -\nu\pi/2 - \pi/4$ and let m_1 be an integer greater than $\frac{\nu}{2}+\frac{5}{4}$. There exist $K > 0$ and $\varphi : [1, +\infty[\to \mathbf{R}$ analytic such that, for all $z \geq 1$, $|\varphi(z)| \leq K \cdot z^{-\nu-5/2}$ and

$$J_\nu(z) = z^{-1/2}\left[\cos(z+\theta)\sum_{m=0}^{m_1}\frac{c_m}{z^{2m}} - \sin(z+\theta)\sum_{m=0}^{m_1}\frac{d_m}{z^{2m+1}} + \varphi(z)\right]$$

(with $c_0, \ldots, c_{m_1}, d_0, \ldots, d_{m_1}$ real constants). Hence

$$\begin{aligned}
&\int_1^N \left(1-\frac{t}{N}\right)^k t^{\nu+1}J_\nu(t)\,dt \\
&= c_0\int_1^N\left(1-\frac{t}{N}\right)^k t^{\nu+(1/2)}\cos(t+\theta)\,dt \\
&+\sum_{m=1}^{m_1} c_m\int_1^N\left(1-\frac{t}{N}\right)^k t^{\nu+(1/2)-2m}\cos(t+\theta)\,dt \\
&+\sum_{m=0}^{m_1} d_m\int_1^N\left(1-\frac{t}{N}\right)^k t^{\nu+(1/2)-2m-1}\sin(t+\theta)\,dt + \int_1^N\left(1-\frac{t}{N}\right)^k t^{\nu+(1/2)}\varphi(t)\,dt.
\end{aligned}$$

The last integral of the right hand converges when $N \to +\infty$, whatever k we take, by the decay condition on φ: $t^{\nu+(1/2)}\varphi(t)$ is integrable. If we take $\nu + \frac{1}{2} \geq k > \nu - \frac{1}{2}$, the integrals in the two sums converge when $N \to +\infty$ but not the first integral of the right hand, by proposition 1; so $t^{\nu+1}J_\nu(t)$ is not (C,k)-summable. On the contrary, if $k > \nu + \frac{1}{2}$, all integrals of the right hand converge and $t^{\nu+1}J_\nu(t)$ is therefore (C,k)-summable.

That $t^{\nu+1}J_\nu(t)$ is (C,k)-summable to 0 for sufficiently great k is shown in [3].

REMARK. Suppose $k = \nu + \frac{1}{2}$; then reasoning as above we see that $\int_0^N (1-t/N)^k t^{\nu+1}J_\nu(t)\,dt$ behaves, when $N \to +\infty$, as $\int_0^N (1-t/N)^k t^{\nu+1/2}\cdot \cos(t+\theta)\,dt$, that is, oscillates as $\sin N$ (point ii) of proposition 1).

4. Summability for Legendre functions

We note P_ν the Legendre function of first kind, order 0 and degree ν. Using formula 7.4.7 p. 173 in [6] we have, for $t > 0$ and $x \in \mathbf{R}$,

$$x \text{ th } \pi x \, P_{-(1/2)+ix}(\operatorname{ch} t) = (\sqrt{2}/\pi) x \int_0^{+\infty} \frac{\sin(x \operatorname{Argch}(u + \operatorname{ch} t))}{\sqrt{u}\sqrt{(u+\operatorname{ch} t)^2 - 1}} \, du.$$

PROPOSITION 3. *Let $t > 0$ and $l \in \mathbf{Z}_+$; we have*

$$\int_0^{+\infty} \left(\frac{1}{\operatorname{sh} t} \frac{\partial}{\partial t}\right)^l [x \text{ th } \pi x \, P_{-(1/2)+ix}(\operatorname{ch} t)] \, dx = 0 \ (C, l+2).$$

PROOF: One easily shows that

$$\left(\frac{1}{\operatorname{sh} t} \frac{\partial}{\partial t}\right)^l \left[\frac{\sin(x \operatorname{Argch}(u + \operatorname{ch} t))}{\sqrt{(u + \operatorname{ch} t)^2 - 1}}\right]$$
$$= \sum_{j=0}^{l} x^j \sin(x \operatorname{Argch}(u + \operatorname{ch} t) + j\pi/2) \cdot F_j^l(u + \operatorname{ch} t)$$

where $F_j^l(y)$ has the form $cy^\lambda(y^2-1)^{(-\lambda-l-1)/2}$ with $\lambda \in \mathbf{Z}_+$ and c a real constant. So it will suffice to show that the functions of x

$$\int_0^{+\infty} \frac{du}{\sqrt{u}} \, x^{2m+2} \cos(x \operatorname{Argch}(u + \operatorname{ch} t)) \cdot F_{2m+1}^{2m+1}(u + \operatorname{ch} t)$$

and

$$\int_0^{+\infty} \frac{du}{\sqrt{u}} \, x^{2m+1} \sin(x \operatorname{Argch}(u + \operatorname{ch} t)) \cdot F_{2m}^{2m}(u + \operatorname{ch} t)$$

are $(C, 2m+3)$ and $(C, 2m+2)$-summable respectively to 0. We will do this in detail for the first function only. Note that $F_{2m+1}^{2m+1}(u + \operatorname{ch} t)$ behaves as cu^{-2m-2} at infinity; hence $F_{2m+1}^{2m+1}(u + \operatorname{ch} t) \cdot u^{-1/2}$ is integrable on $\mathbf{R}_+$ and by Fubini

$$\int_0^N \left(1 - \frac{x}{N}\right)^k \left[\int_0^{+\infty} \frac{du}{\sqrt{u}} \, x^{2m+2} \cos(x \operatorname{Argch}(u + \operatorname{ch} t)) \cdot F_{2m+1}^{2m+1}(u + \operatorname{ch} t)\right] dx$$
$$= \int_0^{+\infty} \frac{du}{\sqrt{u}} \, F_{2m+1}^{2m+1}(u + \operatorname{ch} t) \int_0^N \left(1 - \frac{x}{N}\right)^k x^{2m+2} \cos(x \operatorname{Argch}(u + \operatorname{ch} t)) \, dx.$$

Let $\theta = \operatorname{Argch}(u + \operatorname{ch} t)$; we have $\theta \geq t$ for $u \geq 0$ and

$$\int_0^N \left(1 - \frac{x}{N}\right)^k x^{2m+2} \cos(\theta x) \, dx = \frac{1}{\theta^{2m+3}} \int_0^{\theta N} \left(1 - \frac{y}{\theta N}\right)^k y^{2m+2} \cos(y) \, dy.$$

Assume $k = 2m + 3$. The function $s \mapsto \int_0^s (1 - (y/s))^{2m+3} y^{2m+2} \cos(y)\, dy$ is continuous on $\mathbf{R}_+$ and vanishes when $s \to +\infty$ (proposition 1), hence is bounded in absolute value by a constant $M > 0$. So the functions of $u : \int_0^N (1 - x/N)^{2m+3} x^{2m+2} \cos(\theta x)\, dx$ are bounded in absolute value by M/t^{2m+3} for all $N \geq 0$ and converge everywhere to 0 as $N \to +\infty$. The conclusion follows from Lebesgue dominated convergence theorem.

5. Inversion formula in euclidean space

We fix $n \geq 2$. For $f \in L^1(\mathbf{R}^n)$, we denote by $\mathcal{M}(f, x, r)$ the mean value of f on the sphere with centre x and radius $r : \mathcal{M}(f, x, r) = \omega_n^{-1} \int_{S^{n-1}} \cdot f(x + ru)\, d\sigma(u)$, and ω_n the area of the unit sphere $S^{n-1} : \omega_n = 2\pi^{n/2}/\Gamma(n/2)$.

We say that a function $h :]0, +\infty[\to \mathbf{C}$ is *piecewise* C^q for a $q \in \mathbf{Z}_+$ if there exist $0 = a_0 < a_1 < \cdots < a_{K+1} = +\infty$ such that h is C^q on $\bigcup_{j=1}^{K+1}]a_{j-1}, a_j[$ and the limits of $h^{(i)}$ at a_j $(j = 1, \ldots, K)$ from the right and the left and at $a_0 = 0$ from the right exist for all $0 \leq i \leq q$.

We write, for $z \in \mathbf{R}$, $\lceil z \rceil = \min\{m \in \mathbf{Z} \mid m \geq z\}$ and $\lfloor z \rfloor = \max\{m \in \mathbf{Z} \mid m \leq z\}$.

Theorem 1. *Let $f \in L^1(\mathbf{R}^n)$ and x in $\mathbf{R}^n$ such that $h : r \mapsto \mathcal{M}(f, x, r)$ is piecewise $C^{\lceil n/2 \rceil}$ and $h^{(j)}(r) = O(r^{-(n+1+\varepsilon)/2})$ as $r \to +\infty$, for all $0 \leq j < \lceil n/2 \rceil$ ($\varepsilon > 0$ arbitrary). Define $l = \max\{0 \leq j \leq (n-3)/2 | h^{(j)}$ is continuous$\}$ if h is continuous and $l = -1$ if it is not, and take $k \geq 0$. Then*

$$\lim_{N \to +\infty} \int_{\|y\| \leq N} \left(1 - \frac{\|y\|}{N}\right)^k \mathcal{F} f(y) \mathrm{e}^{2\pi i (x|y)} dy = \mathcal{M}(f, x, 0+)$$

if and only if $k > \dfrac{n-3}{2} - l - 1$.

Proof: We have

$$\begin{aligned}
&\int_{\|y\| \leq N} \left(1 - \frac{\|y\|}{N}\right)^k \mathcal{F} f(y) \mathrm{e}^{2\pi i (x|y)} dy \\
&= \int_{\|y\| \leq N} \left(1 - \frac{\|y\|}{N}\right)^k \mathrm{e}^{2\pi i (x|y)} dy \int_{\mathbf{R}^n} f(v) \mathrm{e}^{-2\pi i (v|y)} dv \\
&= \int_{\mathbf{R}^n} f(v) \left[\int_{\|y\| \leq N} \left(1 - \frac{\|y\|}{N}\right)^k \mathrm{e}^{2\pi i (x - v|y)} dy\right] dv \\
&= \int_0^{+\infty} \int_{S^{n-1}} f(x + ru) \left[\frac{2\pi}{r^{(n-2)/2}} \int_0^N \left(1 - \frac{\rho}{N}\right)^k J_{(n-2)/2}(2\pi r\rho) \rho^{n/2} d\rho\right] r^{n-1}\, d\sigma(u)\, dr \\
&= \int_0^N \left(1 - \frac{\rho}{N}\right)^k 2\pi\omega_n \rho^{n/2} \int_0^{+\infty} \mathcal{M}(f, x, r) J_{(n-2)/2}(2\pi r\rho) r^{n/2}\, dr\, d\rho
\end{aligned}$$

(For the Fourier transform of a radial function (third equality) see [1] p. 89.) Hence, integrating by parts on each interval $[a_{j-1}, a_j]$,

$$2\pi\omega_n \rho^{n/2} \int_0^{+\infty} h(r) J_{(n-2)/2}(2\pi r\rho) r^{n/2} dr$$

$$= 2\pi\omega_n \rho^{n/2} \sum_{j=1}^{K+1} \left[h(r) r^{n-2} \left(-\frac{r^{(4-n)/2}}{2\pi\rho} J_{(n-4)/2}(2\pi r\rho) \right) \Big|_{a_{j-1}}^{a_j} \right.$$

$$\left. + \int_{a_{j-1}}^{a_j} \{h'(r) r^{n-2} + (n-2) h(r) r^{n-3}\} \frac{r^{(4-n)/2}}{2\pi\rho} J_{(n-4)/2}(2\pi r\rho)\, dr \right]$$

$$= \omega_n \sum_{j=1}^{K} a_j^{n/2} \rho^{(n-2)/2} J_{(n-4)/2}(2\pi a_j \rho) \{h(a_j+) - h(a_j-)\}$$

$$+ 2\pi\omega_{n-2} \rho^{(n-2)/2} \int_0^{+\infty} \left\{ \frac{h'(r) r}{n-2} + h(r) \right\} J_{(n-4)/2}(2\pi r\rho) r^{(n-2)/2} dr$$

where for the first equality we have used the identity $\frac{d}{dz}(z^{-\mu} J_\mu(z)) = -z^{-\mu} J_{\mu+1}(z)$ and for the second one the facts that $a_0 = 0$, $(n-2)\omega_n / 2\pi = \omega_{n-2}$ and $J_\mu(z) = O(z^{-1/2})$ as $z \to +\infty$.

We define for every integer $0 \le j < \lceil n/2 \rceil$ a piecewise $C^{\lceil n/2 \rceil - j}$ function $h^{[j]}$ on $\mathbf{R}_+^*$ by

$$h^{[j]}(r) = \begin{cases} h(r) & \text{if } j = 0, \\ \dfrac{r}{n-2j} (h^{[j-1]})'(r) + h^{[j-1]}(r) & \text{if } 1 \le j < \lceil n/2 \rceil. \end{cases}$$

One has $h^{[j]}(0+) = h^{[j-1]}(0+)$ and $h^{[j]}(r) = O(r^{j-(n+1+\varepsilon)/2})$ as $r \to +\infty$; $h^{[j]}$ can be written $h^{[j]}(r) = c r^j h^{(j)}(r) + \sum_{i=0}^{j-1} p_i(r) h^{(i)}(r)$ where c is a non zero constant and the p_i's polynomials of degree $\le i$.

By iteration we obtain for $n = 2m+1$ or $n = 2m+2$:

$$\int_{\|t\| \le N} \left(1 - \frac{\|t\|}{N}\right)^k \mathscr{F} f(t) e^{2\pi i (x|t)} dt \tag{1}$$

$$= \sum_{i=l+1}^{m-1} \left[\omega_{n-2i} \sum_{j=1}^{K} a_j^{(n-2i)/2} \Delta h^{[i]}(a_j) \int_0^N \left(1 - \frac{\rho}{N}\right)^k \rho^{(n-2i-2)/2} J_{(n-2i-4)/2}(2\pi a_j \rho)\, d\rho \right]$$

$$+ 2\pi\omega_{n-2m} \int_0^N \left(1 - \frac{\rho}{N}\right)^k \rho^{(n-2m)/2} \left(\int_0^{+\infty} h^{[m]}(r) J_{(n-2m-2)/2}(2\pi r\rho) r^{(n-2m)/2} dr \right) d\rho.$$

$\left(\text{where the sum } \sum_{i=l+1}^{m-1} \text{ is empty if } l+1 > m-1 \text{ i.e. if } l > \frac{n-5}{2}\right)$ since $\Delta h^{[i]}(a_j) := h^{[i]}(a_j+) - h^{[i]}(a_j-) = 0$ for all $1 \le j \le K$ and $0 \le i \le l$ by continuity of these $h^{[i]}$. If $l < \frac{n-3}{2}$, we also know that there exists at least one j in $\{1, \ldots, K\}$ such that $h^{[l+1]}(a_j+) \neq h^{[l+1]}(a_j-)$.

The last term in (1) converges to $h^{[m]}(0+) = h(0+)$ when $N \to +\infty$ for $k = 0$ (hence for all $k \ge 0$): for $n = 2m+1$, we have, since $J_{-1/2}(z) = \sqrt{2/\pi z}\cos z$,

$$\int_0^N 2\pi\omega_1\sqrt{\rho}\left(\int_0^{+\infty} h^{[m]}(r)\frac{\cos(2\pi r\rho)}{\pi\sqrt{\rho r}}\sqrt{r}\,dr\right)d\rho = (2/\pi)\int_0^{+\infty} h^{[m]}(t/2\pi)\frac{\sin Nt}{t}\,dt,$$

which converges to $h^{[m]}(0+)$ when $N \to +\infty$, as $h^{[m]}$ is of bounded variation in $[0, a_1]$ (see [1] 4.7 p. 28); for $n = 2m+2$ it is proved in [7] p. 657.
If $l > \frac{n-5}{2}$, the theorem is then proved. Suppose now $l \le \frac{n-5}{2}$.
Let $k > \frac{n-3}{2} - l - 1$; then, for all $l+1 \le i \le m-1$, proposition 3 gives

$$\lim_{N\to+\infty}\int_0^N \left(1-\frac{\rho}{N}\right)^k \rho^{(n-2i-2)/2} J_{(n-2i-4)/2}(2\pi a_j\rho)\,d\rho = 0.$$

Let $k = \frac{n-3}{2} - l - 1$. Then the same limit holds for $i > l+1$. Moreover

$$\int_0^N \left(1-\frac{\rho}{N}\right)^k \rho^{(n-2(l+1)-2)/2} J_{(n-2(l+1)-4)/2}(2\pi a_j\rho)\,d\rho$$

oscillates like $\sin(2\pi a_j N)$ (Remark at the end of paragraph 3). Hence, by linear independence,

$$\lim_{N\to+\infty}\sum_{j=1}^{K} a_j^{(n-2(l+1))/2}\Delta h^{[l+1]}(a_j)\int_0^N \left(1-\frac{\rho}{N}\right)^k \rho^{(n-2(l+1)-2)/2} J_{(n-2(l+1)-4)/2}(2\pi a_j\rho)\,d\rho$$

does not exist, and the theorem is established.

Remarks. 1. In any case, for the class of functions considered there is always (C,k)-summability to $\mathcal{M}(f,x,0+)$ if $k > (n-3)/2$, that is, also with values of k lower than the "critical index" $(n-1)/2$. 2. If f is the indicator function of the unit ball, $l = \lfloor (n-3)/2 \rfloor$ at $x \neq 0$ and $l = -1$ at $x = 0$. 3. The particular case h piecewise C^∞ with bounded support and $k = 0$ is Theorem 1.a of [7].

6. Fourier transformation in hyperbolic space

We recall first some elementary facts about hyperbolic spaces (see [5] [9] [10]).

The hyperbolic space of dimension $n \geq 2$ can be defined as $\mathbf{H}^n = \{x \in \mathbf{R}^{n+1} | [x|x] = 1, x_0 > 0\}$ where $[x|y] = x_0y_0 - x_1y_1 - \cdots - x_ny_n$ if $x, y \in \mathbf{R}^{n+1}$ (so $\mathbf{H}^n$ is one sheet of a two-sheeted hyperboloid). If $x, y \in \mathbf{H}^n$, then $x_0 \geq 1$ and $[x|y] \geq 1$, with $[x|y] = 1$ if and only if $x = y$; we can also define $d(x, y) = \operatorname{Argch}[x|y]$: this is a distance on $\mathbf{H}^n$ which makes it a Riemannian manifold with constant curvature -1.

Let $SO(1,n)$ be the set of linear transformations of $\mathbf{R}^{n+1}$ with positive determinant which preserve the bilinear form $[\,|\,]$; it is a subgroup of $GL_{n+1}(\mathbf{R})$ that acts by isometries and transitively on $\mathbf{H}^n$. We choose an 'origin' in $\mathbf{H}^n : e = (1, 0, \ldots, 0)$. Given a point x in $\mathbf{H}^n$, let $t = d(x, e)$; we have $x_0 = \operatorname{ch} t$ and $x_1^2 + \cdots + x_n^2 = x_0^2 - 1 = \operatorname{sh}^2 t$, so the n-tuple $(x_1, \ldots, x_n)$ is on the sphere $\{u \in \mathbf{R}^n \mid \|u\| = \operatorname{sh} t\}$. With this parametrisation we can define an $SO(1,n)$-invariant integral on $\mathbf{H}^n$:

$$\int_{\mathbf{H}^n} f(x)\,dx = \int_0^{+\infty} \int_{S^{n-1}} f(\operatorname{ch} t, \operatorname{sh} t \cdot u) \operatorname{sh}^{n-1} t \, d\sigma(u)\,dt$$

where $d\sigma(u)$ is the area element on $S^{n-1} = \{u \in \mathbf{R}^n \mid \|u\| = 1\}$.

For a well behaved function f on $\mathbf{H}^n$ (e.g. $f \in C_c^\infty(\mathbf{H}^n)$), the Fourier inversion formula can be written ([9] p. 79)

$$f(x) = \int_0^{+\infty} \mathscr{P}_\lambda f(x)\,d\lambda$$

where $\mathscr{P}_\lambda f(x) = \int_{\mathbf{H}^n} {}_n\varphi_\lambda(d(x,y)) f(y)\,dy$ and the function ${}_n\varphi_\lambda$ is given by

$${}_n\varphi_\lambda(d(x,y)) = \gamma_n(\lambda) \int_{S^{n-1}} [x \mid \xi(u)]^{-\varsigma + i\lambda} [y|\xi(u)]^{-\varsigma - i\lambda}\,d\sigma(u)$$

with $\varsigma = (n-1)/2$, $\xi(u) = (1, u) \in \mathbf{R}^{n+1}$ and

$$\gamma_n(\lambda) = \begin{cases} (2\pi)^{-n}\left(\prod_{j=0}^{\varsigma-1}(\lambda^2 + j^2)\right) & \text{if } n \text{ is odd,} \\ (2\pi)^{-n}\left(\lambda \operatorname{th} \pi\lambda \prod_{j=0}^{\varsigma-3/2}(\lambda^2 + (j+1/2)^2)\right) & \text{if } n \text{ is even.} \end{cases}$$

Since $d(x, e) = \operatorname{Argch} x_0$ does not depend on $x_1, \ldots, x_n$, we can suppose, to calculate ${}_n\varphi_\lambda(d(x,e))$, that $x = (\operatorname{ch} t, -\operatorname{sh} t, 0, \ldots, 0)$. We then have $[x|\xi(u)] = \operatorname{ch} t + \operatorname{sh} t \cdot \xi(u)_1$ and $[e|\xi(u)] = \xi(u)_0 = 1$. If we parametrise $u = (u_1, \ldots, u_n) \in S^{n-1} : u_1 = \cos\theta_1, u_2 = \sin\theta_1 \cos\theta_2, \ldots, u_n = \sin\theta_1 \cdot \ldots \cdot \sin\theta_{n-2} \sin\theta_{n-1}$ with $0 \leq \theta_j \leq \pi$ for $j = 1, \ldots, n-2$ and $0 \leq \theta_{n-1} \leq 2\pi$, then $\xi(u)_1 = u_1 = \cos\theta_1$ and

$$ {}_n\varphi_\lambda(d(x,e)) = {}_n\varphi_\lambda(t) = \gamma_n(\lambda)\omega_{n-1}\int_0^\pi (\operatorname{ch} t + \operatorname{sh} t \cos\theta_1)^{-\varsigma+i\lambda} \sin^{n-2}\theta_1\, d\theta_1 $$

Hence, using formula 3.7(7) p. 156 of [2] and denoting by P_ν^μ the Legendre function of the first kind, order μ and degree ν, we obtain

$$ {}_n\varphi_\lambda(t) = \gamma_n(\lambda)(2\pi)^{n/2}\operatorname{sh}^{1-n/2} t \cdot P^{1-n/2}_{i\lambda-1/2}(\operatorname{ch} t). \tag{2} $$

From the formula 3.8(9) p. 161 of [2] follows:

$$ \left(\frac{-1}{\operatorname{sh} t}\frac{d}{dt}\right)[\operatorname{sh}^\mu t \cdot P^\mu_{i\lambda-1/2}(\operatorname{ch} t)] = \left[\left(\frac{1}{2}-\mu\right)^2 + \lambda^2\right]\operatorname{sh}^{\mu-1} t \cdot P^{\mu-1}_{i\lambda-1/2}(\operatorname{ch} t) $$

and so, if $n \geq 3$,

$$ {}_n\varphi_\lambda(t) = \frac{1}{2\pi}\left(\frac{-1}{\operatorname{sh} t}\frac{d}{dt}\right)_{n-2}\varphi_\lambda(t). \tag{3} $$

By iteration we obtain, noting that $P_\nu^{1/2}(\operatorname{ch}\zeta) = \sqrt{\frac{2}{\pi \operatorname{sh}\zeta}}\operatorname{ch}((\nu+1/2)\zeta)$ ([2] 3.6.1(12)),: When n is odd,

$$ {}_n\varphi_\lambda(t) = (2\pi)^{-n/2}\sqrt{\frac{2}{\pi}}\left(\frac{-1}{\operatorname{sh} t}\frac{d}{dt}\right)^{(n-1)/2}\cos\lambda t. \tag{4} $$

When n is even,

$$ {}_n\varphi_\lambda(t) = (2\pi)^{-n/2}\lambda \operatorname{th}\pi\lambda\left(\frac{-1}{\operatorname{sh} t}\frac{d}{dt}\right)^{(n-2)/2} P_{i\lambda-1/2}(\operatorname{ch} t). \tag{5} $$

PROPOSITION 4. *Let $t > 0$ and $k \geq 0$; then $\int_0^{+\infty} {}_n\varphi_\lambda(t)\,d\lambda = 0$ (C,k) if and only if $k > (n-1)/2$.*

PROOF: When n is odd one easily calculates using formula (4) that

$$ {}_{2m+1}\varphi_\lambda(t) = \frac{1}{2^m\pi^{m+1}}\frac{1}{\operatorname{sh}^{2m-1} t}\cdot\sum_{j=0}^m \lambda^j\cos(\lambda t - j\pi/2)R_{m,j}(\operatorname{ch} t, \operatorname{sh} t) $$

where the $R_{m,j}$'s are polynomials in two variables of degree $\leq m-1$, and $R_{m,m}(\operatorname{ch} t, \operatorname{sh} t) = \operatorname{sh}^{m-1} t$. The proposition follows in this case from proposition 1.

When n is even, ${}_n\varphi_\lambda(t)$ is (C,k)-summable to 0 for $k = (n+2)/2$ by formula (5) and proposition 3. To show that ${}_n\varphi_\lambda(t)$ is in fact (C,k)-summable if and only if $k > (n-1)/2$, we use an asymptotic expansion in [8] II p. 232:

$$P^{-m}_{i\lambda-1/2}(\operatorname{ch}\zeta) \sim \frac{\Gamma(i\lambda+1/2)}{\Gamma(i\lambda+m+1/2)} \frac{e^{\zeta/2}}{\sqrt{\pi(i\lambda-1/2)(e^{2\zeta}-1)}}$$

$$\times \left[e^{i\lambda\zeta} \sum_{p=0}^{\infty} \frac{c_p \Gamma(p+1/2)}{\Gamma(1/2)(i\lambda-1/2)^p} \right.$$

$$\left. + e^{(m+1/2)\pi i} e^{-i\lambda\zeta} \sum_{p=0}^{\infty} \frac{c'_p \Gamma(p+1/2)}{\Gamma(1/2)(i\lambda-1/2)^p} \right]$$

as $\lambda \to +\infty$. Therefore

$${}_n\varphi_\lambda(t) \sim \lambda^{(n-1)/2} \left[\cos\lambda t \sum_{p=0}^{\infty} \frac{d_p}{\lambda^p} + \sin\lambda t \sum_{p=0}^{\infty} \frac{d'_p}{\lambda^p} \right]$$

(the d_p and d'_p being constants) as $\lambda \to +\infty$; and we can conclude by a similar argument to that used in the proof of proposition 2.

REMARKS. 1. Identities (2) and (3) are incorrectly stated in [7] (pp. 664, 665 resp.), with a sign error for (2) and a constant depending on λ and n, instead of $1/2\pi$, for (3). 2. According to [8] (II p. 223 and III p. 153), for every $N > 0$ there exists a constant $C_1 > 0$ such that $|P^{1-n/2}_{i\lambda-1/2}(\operatorname{ch} t)| \leq C_1 t(\operatorname{ch} t)^{-1/2}$ for all $t > 1$ and $0 \leq \lambda \leq N$; so there exists a constant $C_2 > 0$ such that $|{}_n\varphi_\lambda(t)| \leq C_2 t(\operatorname{sh} t)^{(1-n)/2}$ for all $t > 1$ and $0 \leq \lambda \leq N$.

7. Inversion formula in hyperbolic space

For $f \in L^1(\mathbf{H}^n)$, $n \geq 2$, the mean value of f on the sphere with centre e and radius t is: $\mathcal{M}(f, e, t) = \omega_n^{-1} \int_{S^{n-1}} f(\operatorname{ch} t, \operatorname{sh} t \cdot u)\, d\sigma(u)$.

THEOREM 2. *Let $f \in L^1(\mathbf{H}^n)$ and x in $\mathbf{H}^n$ such that $h : t \mapsto \mathcal{M}(f, x, t)$ is piecewise $C^{\lceil n/2 \rceil}$ and $h^{(j)}(t) = O(\operatorname{sh}^{(1-n-\varepsilon)/2} t)$ as $t \to +\infty$, for all $0 \leq j < \lceil n/2 \rceil$ ($\varepsilon > 0$ arbitrary). Define $l = \max\{0 \leq j \leq (n-3)/2 \mid h^{(j)}$ is continuous$\}$ if h is continuous and $l = -1$ if it is not, and take $k \geq 0$. Then*

$$\lim_{N \to +\infty} \int_0^N \left(1 - \frac{\lambda}{N}\right)^k \mathcal{P}_\lambda f(x)\, d\lambda = \mathcal{M}(f, x, 0+)$$

if and only if $k > \dfrac{n-3}{2} - l - 1$.

PROOF: Let $g \in SO(1, n)$ such that $x = ge$; then $f \circ g \in L^1(\mathbf{H}^n)$, $\mathcal{P}_\lambda f(x) = \mathcal{P}_\lambda(f \circ g)(e)$ and $\mathcal{M}(f, x, t) = \mathcal{M}(f \circ g, e, t)$. So we can suppose

$x = e$. We have

$$\int_0^N \left(1-\frac{\lambda}{N}\right)^k \mathscr{P}_\lambda f(e)\,d\lambda$$

$$= \int_0^N \left(1-\frac{\lambda}{N}\right)^k d\lambda \int_{\mathbf{H}^n} {}_n\varphi_\lambda(d(e,y)) f(y)\,dy$$

$$= \int_0^N \left(1-\frac{\lambda}{N}\right)^k d\lambda \int_0^{+\infty} {}_n\varphi_\lambda(t) \left[\int_{S^{n-1}} f(\operatorname{ch} t, \operatorname{sh} t\cdot u)\,d\sigma(u)\right] \operatorname{sh}^{n-1} t\,dt$$

$$= \int_0^N \left(1-\frac{\lambda}{N}\right)^k \omega_n \int_0^{+\infty} \mathscr{M}(f,e,t)_n\varphi_\lambda(t)\operatorname{sh}^{n-1} t\,dt\,d\lambda$$

With the points $0 = a_0 < a_1 < \cdots < a_{K+1} = +\infty$ defined as in paragraph 5, we have then, by formula (3) and an integration by parts on each interval $[a_{j-1}, a_j]$,

$$\omega_n \int_0^{+\infty} h(t)_n\varphi_\lambda(t)\operatorname{sh}^{n-1} t\,dt = \sum_{j=1}^{K+1} \frac{\omega_n}{2\pi} \int_{a_{j-1}}^{a_j} h(t)\operatorname{sh}^{n-2} t\left(\frac{-d}{dt}{}_{n-2}\varphi_\lambda(t)\right) dt$$

$$= \sum_{j=1}^{K+1} \frac{\omega_n}{2\pi}\left[- h(t)\operatorname{sh}^{n-2} t\,{}_{n-2}\varphi_\lambda(t)\Big|_{a_{j-1}}^{a_j}\right.$$

$$\left. + \int_{a_{j-1}}^{a_j} \{h'(t)\operatorname{sh}^{n-2} t + (n-2)h(t)\operatorname{sh}^{n-3} t\}_{n-2}\varphi_\lambda(t)\,dt\right]$$

$$= \sum_{j=1}^{K} \frac{\omega_n}{2\pi} \operatorname{sh}^{n-2}(a_j)_{n-2}\varphi_\lambda(a_j)\{h(a_j+) - h(a_j-)\}$$

$$+ \omega_{n-2} \int_0^{+\infty} \left\{\frac{h'(t)\operatorname{sh} t}{n-2} + h(t)\right\}_{n-2}\varphi_\lambda(t)\operatorname{sh}^{n-3} t\,dt$$

using the fact that $a_0 = 0$ and remark 2 of the preceding paragraph.

We define for every integer $0 \le j < \lceil n/2 \rceil$ a piecewise $C^{\lceil n/2\rceil - j}$ function $h^{\{j\}}$ on $\mathbf{R}_+^*$ by

$$h^{\{j\}}(t) = \begin{cases} h(t) & \text{if } j = 0,\\ \dfrac{\operatorname{sh} t}{n-2j}(h^{\{j-1\}})'(t) + h^{\{j-1\}}(t) & \text{if } 1 \le j < \lceil n/2 \rceil. \end{cases}$$

One has $h^{\{j\}}(0+) = h^{\{j-1\}}(0+)$ and $h^{\{j\}}(t) = O(\operatorname{sh}^{j+(1-n-\varepsilon)/2} t)$ as $t \to +\infty$; $h^{\{j\}}$ can be written $h^{\{j\}}(t) = c\operatorname{sh}^j t \cdot h^{(j)}(t) + \sum_{i=0}^{j-1} p_i(\operatorname{ch} t, \operatorname{sh} t) h^{(i)}(t)$ where c is a non zero constant and the p_i's polynomials in two variables of degree $\le i$.

By iteration we obtain for $n = 2m+1$ or $n = 2m+2$:

$$\omega_n \int_0^{+\infty} h(t)_n \varphi_\lambda(t) \mathrm{sh}^{n-1} t \, dt$$

$$= \sum_{i=l+1}^{m-1} \left[\frac{\omega_{n-2i}}{2\pi} \sum_{j=1}^{K} \mathrm{sh}^{n-2i-2}(a_j)[h^{\{i\}}(a_j+) - h^{\{i\}}(a_j-)]_{n-2i-2}\varphi_\lambda(a_j) \right]$$

$$+ \omega_{n-2m} \int_0^{+\infty} h^{\{m\}}(t)_{n-2m} \varphi_\lambda(t) \mathrm{sh}^{n-2m-1} t \, dt.$$

The last term is $(C,0)$-summable to $h^{\{m\}}(0+) = h(0+)$ as a function of λ : for $n = 2m+1$, we have ${}_1\varphi_\lambda(t) = \frac{1}{\pi} \cos \lambda t$ and it is the same integral as in the euclidean case; for $n = 2m+2$ it is proved in [7] (lemma 1.8).

The proof then follows exactly the same lines as that of theorem 1, using proposition 4 in place of proposition 2.

REMARKS. 1. As mentioned in the introduction, we see that Theorems 1 and 2 are parallel; in particular remarks 1 and 2 in paragraph 5 also hold here. 2. If $f \in C_c^\infty(\mathbf{H}^n)$, then $\frac{n-5}{2} < l$ and we have proved in this way the inversion formula $f(x) = \int_0^{+\infty} \mathscr{P}_\lambda f(x) \, d\lambda$. 3. The particular case h piecewise C^∞ with bounded support and $k = 0$ is Theorem 1.c of [7].

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Institut für Angewandte Mathematik
Universität Heidelberg
Im Neuenheimer Feld 294
69120 Heidelberg
Germany