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On transversely flat conformal foliations with good measures II

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ABSTRACT. Transversely flat conformal foliations with good transverse invariant measures are Riemannian in the usual sense, namely, in the C^∞ sense.

Introduction

In the previous paper [1] we have shown that transversely flat conformal foliations with good measures are transversely Riemannian in the $C^{1+\mathrm{Lip}}$ sense, that is, we can find a holonomy invariant transverse Riemannian metric of class $C^{1+\mathrm{Lip}}$. Recently, we found that this is still true even if we replace $C^{1+\mathrm{Lip}}$ with C^∞. Namely, we have the following.

THEOREM A. *Let $(M, \mathscr{F})$ be a transversely flat conformal foliation of a closed manifold M. Assume that there is a good measure on M. Then there is a transverse invariant Riemannian metric of $(M, \mathscr{F})$ which is of class C^∞, namely, $(M, \mathscr{F})$ is Riemannian in the usual sense.*

Thus the theory for Riemannian foliations, which can be found in Molino [3] for instance, applies for such foliations. The proof of Theorem A can be done if we simply replace the metric in the previous paper [1] with one constructed in Ferrand [2]. The paper [2] is informed by H. Izeki, and the author would like to express his gratitude to him.

1. Proof of Theorem A

We recall the definitions, notations and some facts appeared in [1]. First of all, we recall the notion of good measures.

DEFINITION 1.1. A transverse invariant measure μ of $(M, \mathscr{F})$ is said to be good if μ has the following properties:

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a) $\operatorname{supp} \mu = M$, and
b) μ is non-atomic, namely, μ does not concentrate on the union of finite number of compact leaves.

We also assume that μ is a Borel measure, and that μ is locally finite.

We denote by $\tilde{M}$ the universal covering of M, and by $\tilde{\mathscr{F}}$ the lift of $\mathscr{F}$ to $\tilde{M}$, respectively. We denote by $\hat{M}$ the quotient space $\tilde{M}/\tilde{\mathscr{F}}$. Since the natural action of $\pi_1(M)$ on $\tilde{M}$ preserves the lifted foliation $\tilde{\mathscr{F}}$ of $\tilde{M}$, $\pi_1(M)$ naturally acts also on $\hat{M}$.

On the other hand we have the developing map $D : \tilde{M} \to S^q$. The developing map D obviously projects down to the mapping $\Delta : \hat{M} \to S^q$, which is a local homeomorphism.

The key lemma is as follows:

Lemma 1.2 [1]. *The leaf space $\hat{M}$ is a Hausdorff space. Thus $\hat{M}$ is a conformally flat manifold of dimension q.* □

If we find a Riemannian metric on $\hat{M}$ which is invariant under the action of $\pi_1(M)$, then we have a transverse invariant metric on $\tilde{M}$ which is invariant under the action of $\pi_1(M)$. Finally projecting it down to M, we obtain a transverse invariant Riemannian metric on M. So our goal is to find out a metric on $\hat{M}$ which is of class C^∞ and invariant under the action of $\pi_1(M)$.

If the manifold $\hat{M}$ is conformally equivalent to either S^q or E^q via Δ, then we have already shown in [1] that the foliation $(M, \mathscr{F})$ is Riemannian even in the C^ω sense. Thus we may consider the remainder cases.

Now we make use of the following theorem by Ferrand [2]. We quote his theorem slightly modified to fit our aim.

Theorem 1.3 [2]. *Let M be a conformally flat manifold of dimesnion q, which we assume to be isomorphic to neither the sphere S^q nor the Euclidean space E^q. Then there is a Riemannian metric of class C^∞ which is invariant under the action of full group of conformal transformations of M.* □

Now we apply the above theorem to the manifold $\hat{M}$. Since the action of $\pi_1(M)$ is contained in the full group of conformal transformations, we can now find the desired metric on $\hat{M}$ and the proof of Theorem A is completed.

As a corollary we have the following.

Corollary B. *If $(M, \mathscr{F})$ is a transversely flat conformal flow of a closed manifold with dense orbits, then $(M, \mathscr{F})$ is transversely Riemannian.*

Proof. The proof is completely identical as in [1]. For any flow on a closed manifold, there is a non-trivial transverse invariant ergodic measure μ. The support of μ must be the whole manifold because all orbits are dense, and μ is of course non-atomic. Thus μ is good and Theorem A now applies. □

References

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