HIROSHIMA MATH. J.
28 (1998), 507–521

Oscillation criteria for half-linear second order differential equations

Ondřej DOŠLÝ

(Received April 30, 1997)
(Revised October 13, 1997)

ABSTRACT. We study oscillatory properties of half-linear second order differential equation

$$(|y'|^{p-2}y')' + c(x)|y|^{p-2}y = 0, \quad p > 1,$$

viewed as a perturbation of the generalized Euler equation

$$(|y'|^{p-2}y')' + \frac{\gamma_0}{x^p}|y|^{p-2}y = 0,$$

where $\gamma_0 = \left(\frac{p-1}{p}\right)^p$.

1. Introduction

In this paper we deal with oscillatory properties of the half-linear second order differential equation

$$[\Phi(y')]' + c(x)\Phi(y) = 0, \tag{1.1}$$

where $c : [0, \infty) \to \mathbb{R}$ is a continuous function and $\Phi(s) := |s|^{p-1} \operatorname{sgn} s = |s|^{p-2}s$ with $p > 1$. It is known, see. e.g. [4, 10], that basic oscillatory properties of (1.1) are essentially the same as those of the linear differential equation

$$y'' + c(x)y = 0 \tag{1.2}$$

which is a special case $p = 2$ of (1.1). In particular, if x_1, x_2 are consecutive zeros of a nontrivial solution y of (1.1) then any other solution which is not proportional to y has exactly one zero in (x_1, x_2). Consequently, all solutions of (1.1) are either oscillatory or nonoscillatory.

In the last years, several papers appeared showing that oscillation criteria of Hartman, Wintner, Kamenev, Philos and others for (1.2) may be extended to

---

1991 *Mathematics Subject Classification:* 34C10.

Key words and phrases: Half-linear equation, oscillation criteria, generalized Riccati equation, generalized Euler equation.

Supported by the Grant No. 201/96/0410 of the Czech Grant Agency (Prague)

(1.1), see [3, 7, 9, 11, 12]. These criteria are mostly based on the Riccati transformation consisting in the fact that if y is a nonzero solution of (1.1) then

$$w(x) = \frac{\Phi(y'(x))}{\Phi(y(x))} \tag{1.3}$$

solves the generalized Riccati equation

$$w' + c(x) + (p-1)|w|^q = 0, \tag{1.4}$$

where q is the conjugate number of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. In these criteria, equation (1.1) is essentially viewed as a perturbation of the (nonoscillatory) equation

$$(\Phi(y'))' = 0. \tag{1.5}$$

Here we use a somewhat different approach which is based on the relationship between positivity of the "p-degree" functional

$$\mathscr{F}(y; a, b) = \int_a^b [|y'|^p - c(x)|y|^p]\, dx$$

in the class of functions satisfying $y(a) = 0 = y(b)$ and disconjugacy of (1.1) in $[a, b]$ (for the precise statement see the next section). Moreover, we investigate equation (1.1) not as a perturbation of (1.5), but as a perturbation of the generalized Euler equation

$$(\Phi(y'))' + \frac{\gamma_0}{x^p}\Phi(y) = 0, \tag{1.6}$$

where $\gamma_0 = \left(\frac{p-1}{p}\right)^p$ is the so-called *critical constant* in this equation. Like in the linear case, if we replace γ_0 by a constant $\gamma > \gamma_0$ ($\gamma < \gamma_0$) then (1.6) becomes oscillatory (remains nonoscillatory), see [3].

The paper is organized as follows. In the next section we recall basic properties of solutions of (1.1) and we also formulate, for the sake of comparison, some results of "linear" oscillation theory. The main result of the paper, an oscillation criterion for (1.1), is presented in Section 3 and the last section is devoted to remarks and comments concerning possible extensions of our results and to related topics.

2. Preliminary results

First recall some results and methods of oscillation theory of linear equations. The well known variational principle, see e.g. [6], states that the equation

$$(r(x)y')' + c(x)y = 0, \tag{2.1}$$

where $c, r : [a,b] \to \mathbb{R}$, $r(x) > 0$, is *disconjugate* in $[a,b]$ (i.e., any nontrivial solution has at most one zero in $[a,b]$ if and only if

$$\mathcal{F}(y; a, b) = \int_a^b [r(x)y'^2 - c(x)y^2]\, dx > 0$$

for every nontrivial, piecewise C^1 function y for which $y(a) = 0 = y(b)$. Another important concept of oscillation theory of linear equations is the concept of *principal solution*. A solution y_0 of (2.1) is said to be principal if

$$\lim_{x\to\infty} \frac{y_0(x)}{y(x)} = 0$$

for any nonzero solution y of (2.1) which is linearly independent of y_0 (such solution is said to be *nonprincipal*). Principal solution of (2.1) exists (uniquely up to multiplication by a nonzero real constant) if and only if (2.1) is nonoscillatory.

Now, consider equation (2.1) as a perturbation of the nonoscillatory equation

$$(r(x)y')' + c_0(x)y = 0, \tag{2.2}$$

where c_0 is a continuous real-valued function and let y_0, y_1 be principal and nonprincipal solutions of (2.2), respectively. If

$$\lim_{x\to\infty} \frac{y_1(x)}{y_0(x)} \int_x^\infty (c(x) - c_0(s))y_0^2(s)\, ds > 1 \tag{2.3}$$

then (2.1) is oscillatory and if

$$\lim_{x\to\infty} \frac{y_1(x)}{y_0(x)} \int_x^\infty (c(s) - c_0(s))_+ y_0^2(s)\, ds < \frac{1}{4} \tag{2.4}$$

then (2.1) is nonoscillatory, here the subscript "+" denotes the positive part of a function in brackets. In particular, if $r = 1$, $c_0(x) = \frac{1}{4x^2}$, the condition

$$\lim_{x\to\infty} \lg x \int_x^\infty \left(c(s) - \frac{1}{4s^2}\right) s\, ds > 1, \tag{2.5$_1$}$$

is sufficient for (1.2) to be oscillatory and the condition

$$\lim_{x\to\infty} \lg x \int_x^\infty \left(c(s) - \frac{1}{4s^2}\right)_+ s\, ds < \frac{1}{4} \tag{2.5$_2$}$$

is sufficient for nonoscillation of (1.2).

In this paper we extend (2.5_1) to (1.1) and we also explain why we were not successful in extending (2.5_2) to half-linear equation. Our investigation is based on the following statement proved recently by Li and Yeh in [8].

Lemma 2.1. *Let y be a solution of the equation*

$$(r(x)\Phi(y'))' + c(x)\Phi(y) = 0 \tag{2.6}$$

on $[a,b]$ satisfying $y(x) \neq 0$ on (a,b). Denote by Ω the family

$$\Omega = \{\omega \in C^1[a,b] \,|\, \omega(a) = 0 = \omega(b) \text{ and } \omega(x) \neq 0 \text{ on } (a,b)\}.$$

Then for every $\omega \in \Omega$

$$\mathcal{T}(\omega; a, b) = \int_a^b [r(x)|\omega'(x)|^p - c(x)|\omega(x)|^p]\, dx \geq 0, \tag{2.7}$$

where equality holds if and only if ω and y are proportional.

A closer examination of the proof of this lemma shows that it applies without changes also to the case when the condition $\omega \in C^1[a,b]$ in definition of Ω is replaced by a weaker condition: ω is piecewise of the class C^1 in $[a,b]$ and at any discontinuity point $\bar{x} \in [a,b]$ of ω' there exist finite limits $\omega'(\bar{x}+)$, $\omega'(\bar{x}-)$. This larger class of function we denote by Ω'.

Consequently, if we find a nontrivial function $y \in \Omega'$ such that $\mathcal{T}(y; a, b) \leq 0$, then equation (2.6) is *conjugate* in $[a,b]$, i.e., there exists a nontrivial solution with at least two zeros in $[a,b]$. Conversely, if $\mathcal{T}(y; a, b) > 0$ for every nontrivial $y \in \Omega'$ then (2.6) is disconjugate in $[a,b]$ since if y is a nontrivial solution with consecutive zeros $x_1, x_2 \in [a,b]$, then for

$$\omega(x) = \begin{cases} y(x) & x \in [x_1, x_2], \\ 0 & x \in [a,b] \backslash [x_1, x_2] \end{cases}$$

we have

$$\begin{aligned} \mathcal{J}(\omega; a, b) &= \int_{x_1}^{x_2} (r(x)|\omega'|^p - c(x)|\omega|^p)\, dx \\ &= r(x)\omega(x)\Phi(\omega'(x))\big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \omega(x)[(r(x)\Phi(\omega'))' + c(x)\Phi(\omega)]\, dx = 0. \end{aligned}$$

3. Oscillation criterion

In this section we prove our main result.

THEOREM 3.1. *Suppose that*

$$\lim_{x\to\infty} \lg x \int_x^\infty \left[c(s) - \frac{\gamma_0}{s^p}\right] s^{p-1}\, ds > 2\left(\frac{p-1}{p}\right)^{p-1} \tag{3.1}$$

then (1.1) is oscillatory.

PROOF. According to Lemma 2.1 it is sufficient to find for any $d > 0$ a piecewise differentiable function y, with compact support in (d, ∞), say $[x_0, x_3]$, such that

$$\mathcal{J}(y; x_0, x_3) = \int_{x_0}^{x_3} [|y'|^p - c(x)|y|^p]\, dx < 0.$$

Let $x_3 > x_2 > x_1 > x_0$ and let f, g be solutions of (1.6) satisfying boundary conditions

$$f(x_0) = 0, \quad f(x_1) = x_1^{(p-1)/p}, \quad g(x_2) = x_2^{(p-1)/p}, \quad g(x_3) = 0.$$

Define a test function y as follows

$$y(x) = \begin{cases} 0 & \text{if } x \le x_0, \\ f(x) & \text{if } x_0 \le x \le x_1, \\ x^{(p-1)/p} & \text{if } x_1 \le x \le x_2, \\ g(x) & \text{if } x_2 \le x \le x_3, \\ 0 & \text{if } x \ge x_3. \end{cases}$$

In the next computation, for the convenience of notation, denote

$$G(x_0, x_1) = x_1^{(p-1)/p}[\Phi(f'(x_1)) - v_0 x_1^{-(p-1)/p}],$$

$$H(x_2, x_3) = x_2^{(p-1)/p}[v_0 x_2^{-(p-1)/p} - \Phi(g'(x_2))],$$

$$v_0 = \left(\frac{p-1}{p}\right)^{p-1}, \quad y_0(x) = x^{(p-1)/p}$$

Using this notation and integration by parts, we have

$$\begin{aligned}
\mathcal{F}(y;x_0,x_3) &= \int_{x_0}^{x_3} [|y'|^p - c(x)|y|^p]\,dx \\
&= \int_{x_0}^{x_3} \left[|y'|^p - \frac{\gamma_0}{x^p}|y|^p\right] dx - \int_{x_0}^{x_3} \left[c(x) - \frac{\gamma_0}{x^p}\right] |y|^p\,dx \\
&= \int_{x_0}^{x_1} \left[|f'|^p - \frac{\gamma_0}{x^p}|f|^p\right] dx + \int_{x_1}^{x_2} \left[|y_0'|^p - \frac{\gamma_0}{x^p}|y_0|^p\right] dx \\
&\quad + \int_{x_2}^{x_3} \left[|g'|^p - \frac{\gamma_0}{x^p}|g|^p\right] dx - \int_{x_0}^{x_1} \left[c(x) - \frac{\gamma_0}{x^p}\right] |f|^p\,dx \\
&\quad - \int_{x_1}^{x_2} \left[c(x) - \frac{\gamma_0}{x^p}\right] |y_0|^p\,dx - \int_{x_2}^{x_3} \left[c(x) - \frac{\gamma_0}{x^p}\right] |g|^p\,dx \\
&= f(x)\Phi(f'(x))|_{x_0}^{x_1} + y_0(x)\Phi(y_0'(x))|_{x_1}^{x_2} + g(x)\Phi(g'(x))|_{x_2}^{x_3} \\
&\quad - \int_{x_0}^{x_1} \left[c(x) - \frac{\gamma_0}{x^p}\right] |f|^p\,dx - \int_{x_1}^{x_2} \left[c(x) - \frac{\gamma_0}{x^p}\right] |y_0|^p\,dx \\
&\quad - \int_{x_2}^{x_3} \left[c(x) - \frac{\gamma_0}{x^p}\right] |g|^p\,dx \\
&= G(x_0,x_1) + H(x_2,x_3) - \int_{x_0}^{x_1} \left[c(x) - \frac{\gamma_0}{x^p}\right] |f|^p\,dx \\
&\quad - \int_{x_1}^{x_2} \left[c(x) - \frac{\gamma_0}{x^p}\right] |y_0|^p\,dx - \int_{x_2}^{x_3} \left[c(x) - \frac{\gamma_0}{x^p}\right] |g|^p\,dx.
\end{aligned}$$

Next we prove that the functions $\frac{f}{y_0}$ and $\frac{g}{y_0}$ are strictly monotonic on (x_0, x_1) and (x_2, x_3), respectively. If $(f/y_0)'(\bar{x}) = 0$ at some $\bar{x} \in (x_0, x_1)$, then we have $(f'/f)(\bar{x}) = (y_0'/y_0)(\bar{x})$. Denote by $w_1 = \Phi(f')/\Phi(f)$, $w_2 = \Phi(y_0')/\Phi(y_0)$. Both these functions satisfy the generalized Riccati equation

$$w' + \frac{\gamma_0}{x^p} + (p-1)|w|^q = 0, \tag{3.2}$$

where q is the conjugate number of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$, with the same initial condition at $x = \bar{x}$. Hence these solutions coincide over the whole interval of their existence, which is the interval (x_0, x_1). But this is the obvious contradiction (since $f(x_0) = 0$), i.e., $(f/y_0)' > 0$ on this interval. The same argument applies to prove monotonicity of g/y_0 over (x_2, x_3). Now, the second mean value theorem of integral calculus implies the existence of $\xi_1 = (x_0, x_1)$ such that

$$\int_{x_0}^{x_1}\left(c(x)-\frac{\gamma_0}{x^p}\right)|f(x)|^p\,dx=\int_{x_0}^{x_1}\left(c(x)-\frac{\gamma_0}{x^p}\right)\frac{|f(x)|^p}{|y_0(x)|^p}|y_0(x)|^p\,dx$$
$$=\int_{\xi_1}^{x_1}\left(c(x)-\frac{\gamma_0}{x^p}\right)x^{p-1}\,dx.$$

Similarly,

$$\int_{x_2}^{x_3}\left(c(x)-\frac{\gamma_0}{x^p}\right)|g(x)|^p\,dx=\int_{x_2}^{\xi_2}\left(c(x)-\frac{\gamma_0}{x^p}\right)x^{p-1}\,dx,$$

for some $\xi_2\in(x_2,x_3)$. Consequently,

$$\int_{x_0}^{x_3}\left(c(x)-\frac{\gamma_0}{x^p}\right)|y(x)|^p\,dx=\int_{\xi_1}^{\xi_2}\left(c(x)-\frac{\gamma_0}{x^p}\right)x^{p-1}\,dx.$$

In the following part we derive the asymptotic formula for $G(x_0,x_1)$ as $x_1\to\infty$ and for $H(x_2,x_3)$ as $x_3\to\infty$. To this end, we use the transformation of the independent variable $x=e^t$. Put $z(t)=y(e^t)=y(x)$, $v(t)=x^{p-1}w(x)$, then this transformation transforms equations (1.6) and (3.2) into the equations (with constant coefficients)

$$[\Phi(z')]'-(p-1)\Phi(z')+\gamma_0\Phi(z)=0 \tag{3.3}$$

and

$$v'=(p-1)v-\gamma_0-(p-1)|v|^q=:F(v), \tag{3.4}$$

respectively. Concerning the last equation, this equation possesses the constant solution $v=v_0=\left(\frac{p-1}{p}\right)^{p-1}$ and any solution v tends to v_0 as $t\to\infty$. Indeed, $F(v_0)=0$ and if $v=v(t)$ exists on some interval $[T,\infty)$, then

$$\int_{v(T)}^{v(t)}\frac{d\xi}{F(\xi)}=t-T. \tag{3.5}$$

Now, if $t\to\infty$, the integral on the left-hand-side must diverge and this occurs only if $v(\infty)=v_0$.

The function $F(v)$ may be expressed in the form (since also $F'(v_0)=0$)

$$F(v)=\tfrac{1}{2}F''(v_0)(v-v_0)^2+O((v-v_0)^3),\quad \text{as } v\to v_0,$$

hence

$$\frac{1}{F(v)}=\frac{2}{F''(v_0)(v-v_0)^2}+O\left(\frac{1}{v-v_0}\right)\quad \text{as } v\to v_0,$$

this implies (compare (3.5))

$$\frac{2}{F''(v_0)(v_0 - v)} + O(\lg|v - v_0|) = t - T \quad \text{as } t \to \infty,$$

hence

$$\frac{2}{F''(v_0)} + (v_0 - v)O(\lg|v - v_0|) = (v_0 - v)(t - T) \quad \text{as } t \to \infty.$$

Since $\lim_{t\to\infty}(v_0 - v(t))O(\lg|v_0 - v(t)|) = 0$ we have

$$\lim_{t\to\infty}(v_0 - v(t))(t - T) = \lim_{t\to\infty} t(v_0 - v(t)) = \frac{2}{F''(v_0)}.$$

Consequently,

$$O(\lg|v_0 - v(t)|) = O\left(\lg\frac{1}{t}\right) = O(\lg t) \quad \text{as } t \to \infty$$

and thus

$$\frac{1}{v_0 - v} = \frac{1}{2}F''(v_0)t + O(\lg t) \quad \text{as } t \to \infty,$$

which means

$$v - v_0 = -\frac{1}{\frac{1}{2}F''(v_0)t + O(\lg t)} = -\frac{1}{\frac{1}{2}F''(v_0)t\left(1 + O\left(\frac{\lg t}{t}\right)\right)}$$

$$= -\frac{2}{F''(v_0)t}\left(1 - O\left(\frac{\lg t}{t}\right)\right) = -\frac{2}{F''(v_0)t} + O\left(\frac{\lg t}{t^2}\right)$$

as $t \to \infty$. Taking into account the relation between solutions w and v of (3.2) and (3.4) we have

$$x^{p-1}w(x) - v_0 = \frac{-2}{F''(v_0)\lg x} + O\left(\frac{\lg(\lg x)}{\lg^2 x}\right) \quad \text{as } x \to \infty.$$

Let $w(x, x_0) := \frac{\Phi(f'(x))}{\Phi(f(x))}$. Then this function solves (3.2) and hence

$$G(x_0, x_1) = x_1^{(p-1)/p}\left\{\Phi(f'(x_1)) - \left(\frac{p-1}{p}\right)^{p-1} x_1^{-(p-1)/p}\right\} = x_1^{p-1}w(x_1, x_0) - v_0$$

$$= -\frac{2}{F''(v_0)\lg x_1} + O\left(\frac{\lg(\lg x_1)}{\lg^2 x_1}\right) = \frac{2v_0}{\lg x_1} + O\left(\frac{\lg(\lg x_1)}{\lg^2 x_1}\right),$$

as $x_1 \to \infty$ since by a direct computation we have $F''(v_0) = -\frac{1}{v_0}$. Moreover, observe that $G(x_0, x_1)$ is positive and decreasing. Indeed, if $t_1 = \lg x_1$ and $\tilde{v}(t) = e^{(p-1)t} w(e^t, x_0)$ then $G(x_0, x_1) = \tilde{v}(t_1) - v_0 > 0$ for every $t_1 > t_0$ (since $f'(x_1) > y_0'(x_1)$) and $\tilde{v}'(t) = F(\tilde{v}(t)) < H(v_0) = 0$.

Concerning the asymptotic behaviour of $H(x_2, x_3)$ as $x_3 \to \infty$, we proceed similarly as for $G(x_0, x_1)$. We have

$$H(x_2, x_3) = x_2^{(p-1)/p}[v_0 x_2^{-(p-1)/p} - x_2^{(p-1)^2/p} w(x_2, x_3)] = v_0 - x_2^{p-1} w(x_2, x_3),$$

where $w(x, x_3) = \dfrac{\Phi(g'(x)))}{\Phi(g(x))}$ is the solution of (1.4) generated by the solution g of (1.6), i.e., the solution for which $w(x_3-, x_3) = -\infty$. We will show that $v_0 - x_2^{p-1} w(x_2, x_3) \to 0$ as $x_3 \to \infty$. To this end, we use again the transformation $x = e^t$, $v(t) = x^{p-1} w(x)$ which transforms (1.4) into (3.4). Let $\bar{t} \in \mathbb{R}$ be arbitrary and denote by $w(t, \bar{t})$ the solution of (3.4) determined by the solution of (3.3) satisfying $z(\bar{t}) = 0$, i.e., $\lim_{t \to \bar{t}-} w(t, \bar{t}) = -\infty$. Similarly as for $t \to \infty$ in the previous part of the proof we have $\lim_{t \to -\infty}(v_0 - v(t, \bar{t})) = 0$. Using the fact that (3.4) is autonomous, i.e. $v(t - \bar{t}, \bar{t})$ solves this equation too, we have $v_0 - v(t, \bar{t}) \to 0$ as $|t - \bar{t}| \to \infty$, regardless whether $t \to -\infty$ and $\bar{t}$ is fixed or t is fixed and $\bar{t} \to \infty$. Consequently, if $t_2 = \lg x_2$, $t_3 = \lg x_3$ and $v(t, t_3) = x^{p-1} w(x, x_3)$, $x = e^t$, we have

$$0 = \lim_{t_3 \to \infty} (v_0 - v(t_2, t_3)) = \lim_{x_3 \to \infty} (v_0 - x_2^{p-1} w(x_2, x_3)),$$

i.e. $v_0 - x_2^{p-1} w(x_2, x_3) \to 0$ as $x_3 \to \infty$.

Now, let us return to the computation of the functional $\mathcal{T}$. Let $d \le x_0$ be fixed, we have

$$\begin{aligned}\mathcal{T}(y; x_0, x_3) &= G(x_0, x_1) + H(x_2, x_3) - \int_{\xi_1}^{\xi_2} \left(c(x) - \frac{\gamma_0}{x^p}\right) x^{p-1}\, dx \\ &= G(d, x_1)\left[\frac{G(x_0, x_1)}{G(d, x_1)} + \frac{H(x_2, x_3)}{G(d, x_1)} - \frac{\int_{\xi_1}^{\xi_2} (c(x) - \frac{\gamma_0}{x^p}) x^{p-1}\, dx}{G(d, x_1)}\right] \\ &\le G(d, x_1)\left[\frac{G(x_0, x_1)}{G(d, x_1)} + \frac{H(x_2, x_3)}{G(d, x_1)} - \frac{\int_{\xi_1}^{\xi_2} (c(x) - \frac{\gamma_0}{x^p}) x^{p-1}\, dx}{G(d, \xi_1)}\right]. \qquad (3.6)\end{aligned}$$

In the last inequality we have used monotonicity of G and the fact that the numerator of the third term in brackets is positive (compare (3.1)).

Now, let $\varepsilon > 0$ be (sufficiently small) such that the limit in (3.1) is greater than $2\left(\frac{p-1}{p}\right)^{p-1}(1 + 6\varepsilon) = 2v_0(1 + 6\varepsilon)$. We have $\lim_{x \to \infty} \lg x G(d, x) = 2v_0$, i.e., $\lg x G(d,x) < 2v_0(1+\varepsilon)$ if x is sufficiently large, and in view of (3.1) $x_0 > d$ can be chosen in such a way that

$$\frac{\int_{\xi_1}^{\infty}\left(c(x)-\frac{\gamma_0}{x^p}\right)x^{p-1}\,dx}{G(d,\xi_1)} = \lg\xi_1\int_{\xi_1}^{\infty}\left(c(x)-\frac{\gamma_0}{x^p}\right)x^{p-1}\,dx\frac{1}{\lg\xi_1 G(d,\xi_1)}$$

$$> \frac{2v_0(1+6\varepsilon)}{2v_0(1+\varepsilon)} > 1+4\varepsilon \tag{3.7}$$

whenever $\xi_1 > x_0$ and ε is sufficiently small.

Since

$$\lim_{x\to\infty}\frac{G(x_0,x)}{G(d,x)} = \lim_{x\to\infty}\frac{v_0 - x^{p-1}w(x,x_0)}{v_0 - x^{p-1}w(x,d)}$$

$$= \lim_{x\to\infty}\frac{-\frac{2}{F''(v_0)\lg x}+O\left(\frac{\lg(\lg x)}{\lg^2 x}\right)}{-\frac{2}{F''(v_0)\lg x}+O\left(\frac{\lg(\lg x)}{\lg^2 x}\right)} = 1,$$

there exists $x_1 > x_0$ such that

$$\frac{G(x_0,x_1)}{G(d,x_1)} < 1+\varepsilon.$$

Further, (3.7) implies the existence of $x_2 > x_1$ such that

$$\frac{\int_{\xi_1}^{\xi_2}\left(c(x)-\frac{\gamma_0}{x^p}\right)x^{p-1}\,dx}{G(d,\xi_1)} > 1+3\varepsilon$$

whenever $\xi_2 > x_2$.

Finally, since $H(x_2,x_3)\to 0$ as $x_3\to\infty$, we have $H(x_2,x_3)G^{-1}(d,x_1) < \varepsilon$ if x_3 is sufficiently large.

Combining all previous computations we see that the expression in the brackets in (3.6) is $< -\varepsilon$, hence $\mathscr{T}(y;x_0,x_3) < 0$, and by Lemma 2.1 equation (1.1) is oscillatory. □

4. Remarks and comments

(i) Observe that the transformation of the independent variable

$$t = \int_0^x [r(s)]^{1-q}\,ds \tag{4.1}$$

where q is the conjugate number of p, i.e., $\frac{1}{p}+\frac{1}{q}=1$, transforms (2.7) into the equation

$$\frac{d}{dt}\left[\Phi\left(\frac{d}{dt}y\right)\right] + [r(x(t))]^{q-1}c(x(t)))\Phi(y) = 0$$

which is the equation of the form (1.1), here $x = x(t)$ is the inverse of $t = t(x)$ given by (4.1). Using this transformation, a criterion similar to that given in Theorem 3.1 for (1.1) can be also formulated for a more general equation (2.6).

(ii) In [1] we have shown that the $2n$ order equation

$$y^{(2n)} + p(x)y = 0 \tag{4.2}$$

is conjugate in $(0, \infty)$ (i.e., there exists a nontrivial solution of (4.2) with at least two different zero points of multiplicity n in $(0, \infty)$) provided

$$\int_0^\infty \left(p(x) - \frac{\mu_n}{x^{2n}}\right)x^{2n-1}\,dx \geq 0 \quad \text{and} \quad p(x) \not\equiv \frac{\mu_n}{x^{2n}} \quad \text{in } (0, \infty),$$

where $\mu_n = (-1)^{n-1}\dfrac{[(2n-1)!!]^2}{4^n}$ is the critical constant in the $2n$ order Euler equation $y^{(2n)} + \frac{\mu_n}{x^{2n}}y = 0$. In particular, if $n = 1$ then (1.2) is conjugate in $(0, \infty)$ provided

$$\int_0^\infty \left(c(x) - \frac{1}{4x^2}\right)x\,dx \geq 0 \quad \text{and} \quad c(x) \not\equiv \frac{1}{4x^2} \quad \text{in } (0, \infty).$$

A similar conjugacy criterion may be proved also for half-linear equation (1.1), using essentially the same test function as in the proof of Theorem 3.1.

THEOREM 4.1. *Suppose that*

$$\int_0^\infty \left(c(x) - \frac{\gamma_0}{x^p}\right)x^{p-1}\,dx \geq 0 \quad \textit{and} \quad c(x) \not\equiv \frac{\gamma_0}{x^p} \quad \textit{in } (0, \infty) \tag{4.3}$$

then (1.1) *is conjugate in* $(0, \infty)$.

As we mentioned above, the proof of this theorem is essentially the same as that of Theorem 3.1. Let y be the same test function as in the proof of this theorem, with the following exception. By virtue of (4.3) there exists $\bar{x} \in (0, \infty)$ such that $c(\bar{x}) > \frac{\gamma_0}{\bar{x}^p}$ and we construct a "small hill" on the graph of the function $y_0(x) = x^{(p-1)/p}$ in the neighbourhood the point $x = \bar{x}$ in such a way that $\mathcal{F}(y; x_0, x_3) < 0$ if we let $x_0 \to 0$ and $x_3 \to \infty$. Construction of this hill is exactly the same as in linear case.

(iii) In Theorem 3.1 we suppose that the integral $\int^\infty \left(c(x) - \frac{\gamma_0}{x^p}\right)x^{p-1}\,dx$ is convergent. If this integral diverges to ∞ then (1.1) is oscillatory as it is shown in [5, Theorem 7].

(iv) In Theorem 3.1 and also in previous remarks equation (1.1) is viewed as a perturbation of generalized Euler equation (1.6). Of course, one may also consider (2.6) as a perturbation of the one-term (nonoscillatory) equation $(r(x)\Phi(y'))' = 0$. Using essentially the same test function as in the linear case one can prove the following oscillation criterion.

Theorem 4.2. *Suppose that* $\int^\infty r^{1-q}(x)\,dx = \infty$ *(q is the conjugate number of p) and*

$$\lim_{x\to\infty}\left(\int^x r^{1-q}(s)\,ds\right)^{p-1}\left(\int_x^\infty c(s)\,ds\right) > 1$$

then (2.6) is oscillatory.

Concerning the exact construction of the test function y for which the functional $\mathscr{T}$ given by (2.7) is negative, we define

$$y(x) = \begin{cases} 0 & x \le x_0, \\ \dfrac{\int_{x_0}^x r^{1-q}(s)\,ds}{\int_{x_0}^{x_1} r^{1-q}(s)\,ds} & x_0 \le x \le x_1, \\ 1 & x_1 \le x \le x_2, \\ \dfrac{\int_x^{x_3} r^{1-q}(s)\,ds}{\int_{x_2}^{x_3} r^{1-q}(s)\,ds} & x_2 \le x \le x_3, \\ 0 & x \ge x_3. \end{cases}$$

Now, if $x_0 < x_1 < x_2 < x_3$ are sufficiently large, we have $\mathscr{T}(y; x_0, x_3) < 0$.

Similarly as in the linear case, one may also prove a "nonoscillatory supplement" of the previous theorem.

Theorem 4.3. *Suppose that* $\int^\infty r^{1-q}(x)\,dx = \infty$ *and*

$$\lim_{x\to\infty}\left(\int^x r^{1-q}(s)\,ds\right)^{p-1}\left(\int_x^\infty c_+(s)\,ds\right) < \frac{(p-1)^{p-1}}{p^p} \tag{4.4}$$

then (2.6) is nonoscillatory. Here $c_+(s) = \max\{0, c(s)\}$.

Proof. We will show that conditions of theorem imply the existence of $N \in \mathbb{R}$ such that

$$\mathscr{F}_N(y) = \int_N^\infty (r(x)|y'|^p - c(x)|y|^p)\,dx > 0,$$

for any nontrivial C^1 function y with compact support in (N, ∞) (see Lemma 2.1).

First we establish the following useful inequality which is proved in case $p = 2$ e.g. in [6].

Let M be a positive differentiable function for which $M'(x) \neq 0$ in $[a, b]$ and let $z \in \Omega$ (Ω is defined in Lemma 2.1). Then

$$\int_a^b |M'|\,|z|^p\,dx \le p^p \int_a^b \frac{|M|^p}{|M'|^{p-1}} |z'|^p\,dx. \tag{4.5}$$

Indeed, using integration by parts and the Hölder inequality we have

$$\begin{aligned}\int_a^b |M'|\,|z|^p\,dx &\le p\int_a^b M|z|^{p-1}|z'|\,dx\\ &\le p\left(\int_a^b |M'|\,|z|^{(p-1)q}\,dx\right)^{1/q}\left(\int_a^b \frac{M^p}{|M'|^{p-1}}|z'|^p\,dx\right)^{1/p}\\ &= \left(\int_a^b |M'|\,|z|^p\,dx\right)^{1/q}\left(\int_a^b \frac{M^p}{|M'|^{p-1}}|z'|^p\,dx\right)^{1/p},\end{aligned}$$

hence the required inequality follows. Now, denote

$$\nu := \frac{(p-1)^{p-1}}{p^p}, \quad M(t) := \left(\int^x r^{1-q}(s)\,ds\right)^{1-p}$$

and let $N \in \mathbb{R}$ be such that the expression in (4.4) is less than ν for $x > N$. Using the previous inequality and (4.4) we have for any differentiable y with compact support in (N, ∞)

$$\begin{aligned}\int_N^\infty c(x)|y|^p\,dx &\le \int_N^\infty c_+(x)|y|^p\,dx = p\int_N^\infty c_+(x)\left(\int_N^x y'\Phi(y)\,dt\right)dx\\ &\le p\int_N^\infty |y'|\Phi(y)M(x)\frac{\int_x^\infty c_+(t)\,dt}{M(x)}\,dx < p\nu\int_N^\infty M(x)|y'|\Phi(y)\,dx\\ &\le p\nu\left(\int_N^\infty |M'|\,|y|^p\,dx\right)^{1/q}\left(\int_N^\infty \frac{|M|^p}{|M'|^{p-1}}|y'|^p\,dx\right)^{1/p}\\ &\le p^p\nu\int_N^\infty \frac{|M|^p}{|M'|^{p-1}}|y'|^p\,dx \le \int_N^\infty r(x)|y'|^p\,dx\end{aligned}$$

since directly one may verify that

$$\frac{|M(x)|^p}{|M'(x)|^{p-1}} = (p-1)^{1-p}r(x).$$

Hence we have $\int_N^\infty [r(x)|y'|^p - c(x)|y|^p]\,dx > 0$. $\square$

(v) The method used in Theorems 3.1, 4.2 as well as in (2.3) and (2.4) suggests the following general approach to the investigation of oscillatory properties of (2.6). In this method, this equation would be viewed as a perturbation of the general nonoscillatory equation

$$(4.5)\qquad (r(x)\Phi(y'))' + c_0(x)\Phi(y) = 0.$$

However, except for the special case treated in Theorems 3.1, 4.1, no "half-linear" analogy of (2.3), (2.4) is known till now. The reason is not only the absence of equivalents of principal and nonprincipal solutions, but also (and first of all) one has in disposal no transformation theory similar to that for linear equations. Using the linear transformation theory, e.g. criterion (2.3) may be proved as follows. If y_0 is the principal solution of (2.2) then the transformation $y = y_0 u$ transforms this equation into the one-term equation

$$(4.6)\qquad (r(x)y_0^2(x)u')' = 0,$$

hence this transformation converts (2.1) into the equation

$$(r(x)y_0^2(x)u')' + (c(x) - c_0(x))y_0^2(x)u = 0.$$

Now, the last equation is treated as a perturbation of (4.6) in a way as suggested in remark (iv) (applied to the linear case) and the obtained results are then transformed "back" into (2.1). This approach cannot be directly extended to half-linear equations just because of absence of a "half-linear" transformation theory. For this reason we also failed in extending (2.5_2) to half-linear equation (1.1).

(vi) Consider the partial differential equation

$$(4.7)\qquad \operatorname{div}\,(|\nabla u(x)|^{p-2}\nabla u(x)) + q(x)|u|^{p-2}u = 0, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

This so-called p-Laplace equation has been treated in many recent papers, see e.g. [2] and the references given therein. Oscillatory properties of the usual Laplace equation

$$\Delta u + q(x)u = 0$$

was studied e.g. in [13] using the Riccati-type transform $w = \frac{\nabla u}{u}$. Then w satisfies the equation

$$\operatorname{div} w + q(x) + \|w(x)\|^2 = 0$$

and methods typical for ordinary differential equations can be used also in this case. In case of p-Laplace equation, the function $w = \frac{\|\nabla u\|^{p-2}\nabla u}{|u|^{p-2}u}$ verifies the equation

$$\operatorname{div} w + q(x) + (p-1)\|w(x)\|^q = 0, \quad q := \frac{p}{p-1}$$

and one may try to extend the idea used in [13] to p-Laplace equation (4.7). We hope to follow this idea in a subsequent paper.

References

[1] O. Došlý, J. Komenda, *Conjugacy criteria and principal solutions of self-adjoint linear differential equations,* Arch. Math. **31** (1995), 217–238.

[2] P. Drábek, A. Kufner, F. Nicolosi, Nonlinear Elliptic Equations—Singular and Degenerate Case, West Bohemia Univ. Press, Pilsen, 1996.

[3] Á. Elbert, *Asymptotic behaviour of autonomous half-linear differential systems on the plane,* Studia Sci. Math. Hungar. **19** (1984), 447–464.

[4] Á. Elbert, *A half-linear second order differential equation,* In *Theory of Differential Equation,* Colloq. Math. Soc. János Bolyai **30** (1979), 158–180.

[5] Á. Elbert, *Oscillation and nonoscillation theorems for some non-linear ordinary differential equations,* Lectures Notes in Math. No. **964** (1982), 187–212.

[6] I. M. Glazman, Direct Methods of Qualitative Analysis of Singular Differential Operators, Jerusalem 1965.

[7] H. J. Li, *Oscillation criteria for half-linear second order differential equations,* Hiroshima Math. J. **25** (1995), 571–583.

[8] H. J. Li, C. C. Yeh, *Sturmian comparison theorem for half-linear second order differential equations,* Proc. Roy. Soc. Edinburgh **125A** (1996), 1193–1204.

[9] A. Lomtatidze, *Oscillation and nonoscillation of Emden-Fowler type equation of second order,* Arch. Math. **32** (1996), 181–193.

[10] J. D. Mirzov, *On some analogs of Sturm's and Kneser's theorems for nonlinear systems,* J. Math. Anal. Appl. **53** (1976), 418–426.

[11] M. Del Pino, R. Manásevich, *Oscillation and nonoscillation for* $(|u'|^{p-2}u')' + a(t)|u|^{p-2}u = 0$, $p > 1$, Houston J. Math. **14** (1988), 173–177.

[12] M. Del Pino, M. Elgueta, R. Manásevich, *Generalizing Hartman's oscillation result for* $(|x'|^{p-2}x')' + c(t)|x|^{p-2}x = 0$, $p > 1$, Houston J. Math. **17** (1991), 63–70.

[13] U. W. Schminke, *The lower spectrum of Schrödinger operators,* Arch. Rat. Mech. Anal. **75** (1981), 147–155.

Department of Mathematics
Faculty of Science, Masaryk University
Janáčkovo nám. 2a, 662 95 Brno
Czech Republic
e-mail: dosly@math.muni.cz