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Lowest dimensions for immersions of orientable manifolds up to unoriented cobordism

Dedicated to Professor Mamoru Mimura on his 60th birthday

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ABSTRACT. We determine the lowest dimension of the Euclidean space in which all n-dimensional orientable manifolds are immersible up to unoriented cobordism. Our study is an orientable version of the work investigated by R. L. Brown.

1. Introduction

The purpose of this paper is to give a complete answer to the immersion problem of orientable manifolds up to unoriented cobordism. Let $\alpha(n)$ be the number of 1 in the dyadic expansion of an integer n, and $\nu(n)$ the integer determined by $n = 2^{\nu(n)}(2m+1)$. We set $\beta(n) = 2n - \alpha(n) - \min\{\alpha(n), \nu(n)\}$. In [10; Theorem A], we studied immersions of orientable manifolds in the Euclidean space $\mathbf{R}^f$ up to unoriented cobordism, and gave a partial answer: (a) any closed orientable manifold M^n for $n \geq 4$ is unoriented cobordant to a manifold which immerses in $\mathbf{R}^{\beta(n)}$; (b) if $\alpha(n) \leq \nu(n)$ and $n \geq 4$, then there exists an n-dimensional closed orientable manifold satisfying that any manifold unoriented cobordant to it does not immerse in $\mathbf{R}^{\beta(n)-1}$.

We always assume that a manifold is closed C^∞ differentiable, and by *cobordant* we mean unoriented cobordant between manifolds. Then, our main results are stated as follows:

THEOREM A. *Assume that $\alpha(n) > \nu(n)$ and $n \geq 4$. Then, $\beta(n) = 2n - \alpha(n) - \nu(n)$, and any orientable manifold M^n is cobordant to a manifold which immerses, respectively, in $\mathbf{R}^{\beta(n)-1}$ or $\mathbf{R}^{\beta(n)-2}$ if the following* (1) *or* (2) *holds*:

(1) $\alpha(n) + \nu(n)$ *is odd, or*

(2) $\alpha(n) + \nu(n)$ *is even and $n \equiv 0$ or $3 \pmod 4$.*

THEOREM B. *Assume that $\alpha(n) > \nu(n)$ and $n \geq 4$ with $n \neq 6, 7$. Then, there exists an n-dimensional orientable manifold satisfying that any manifold cobordant to it does not immerse, respectively, in $\mathbf{R}^{\beta(n)-2}$, $\mathbf{R}^{\beta(n)-3}$ or $\mathbf{R}^{\beta(n)-1}$ if the following* (1), (2) *or* (3) *holds*:

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(1) $\alpha(n)+\nu(n)$ *is odd,*
(2) $\alpha(n)+\nu(n)$ *is even and* $n \equiv 0$ *or* $3 \pmod 4$, *or*
(3) $\alpha(n)+\nu(n)$ *is even and* $n \equiv 1$ *or* $2 \pmod 4$.

It is well known that the class of any n-dimensional oriented manifold with $1 \le n < 4$, $n=6$ or $n=7$ is 0 in the oriented cobordism ring (cf. [11, Théorème IV. 13]), and thus, 0 in the unoriented cobordism ring. Hence, for any given n, if $h(n)$ is the minimum integer such that every n-dimensional orientable manifold immerses in $\mathbf{R}^{h(n)}$ up to cobordism, then Theorems A and B with the results in [10] completely determine the value of $h(n)$. The efficient uses of symmetric characteristic classes seem the key ingredient of success in this paper.

Theorems A and B can be compared with the original study due to Brown [1; Theorem 5.1, Proposition 5.2].

This paper is organized as follows: In §2 we fix some bases of the cobordism rings and prepare Proposition 2.3 which plays a crucial role in the proof of Theorem A. Theorem A is proved in §3 by using Proposition 2.3, the Theorem B is in §4. In §5 we prove a part of Proposition 2.3. After preparing necessary properties of the symmetric characteristic classes in §6, we complete the proof of Proposition 2.3 in §7.

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2. Bases of cobordism rings

First, we recall some generators of the unoriented cobordism ring $\mathfrak{N}_*$. Let $\mathbf{C}P^n$ be the complex projective space, and $S^m = \{(t_1,\dots,t_{m+1}) \in \mathbf{R}^{m+1} \mid \sum_{i=1}^{m+1} t_i^2 = 1\}$ the unit sphere. The Dold manifold $P(m,n)$ is defined as the orbit space $(S^m \times \mathbf{C}P^n)/J$ for the involution $J(u,z) = (-u,\bar{z})$, where $\bar{z}$ is the conjugate number of z. Consider a reflection T on S^m concerning the plane $t_{m+1}=0$. Then, the map $(u,z) \to (Tu,z)$ on $S^m \times \mathbf{C}P^n$ induces an involution A of $P(m,n)$. We define $Q(m,n)$ to be the manifold constructed from $P(m,n)\times[0,1]$ by identifying $(p,0)$ with $(Ap,\ 1)$ for each $p \in P(m,n)$. Let $x_{2^j} \in \mathfrak{N}_{2^j}$ be the cobordism class of the real projective space $X^{2^j} = \mathbf{R}P^{2^j}$. For an integer k not a power of 2, we write $k = 2^{r-1}(2s+1)$ with $s \ge 1$. We set $x_{2k-1} \in \mathfrak{N}_{2k-1}$ and $x_{2k} \in \mathfrak{N}_{2k}$ to be the cobordism classes of $X^{2k-1} = P(2^r-1, 2^r s)$ and $X^{2k} = Q(2^r-1, 2^r s)$, respectively. Then, Dold [2; Satz 3] and Wall [12; Lemma 6] have shown that each x_q is indecomposable in $\mathfrak{N}_*$, and thus $\{x_q \mid q \neq 2^i - 1\}$ is the polynomial generators of $\mathfrak{N}_*$.

Next, we recall some generators of the oriented cobordism ring Ω_* introduced by Wall [12; §9]. In order to state them, we need some notations. A *partition* ω of n is an unordered sequence $(a_1,\dots,a_k)$ of positive integers with $\sum_{j=1}^k a_j = n$. We set $|\omega| = n$, $l(\omega) = k$ and $\alpha(\omega) = \sum_{j=1}^k \alpha(a_j)$. For partitions $\omega_j = (a_{j1}, \dots, a_{jm_j})$ $(1 \le j \le k)$, we denote $(\omega_1, \dots, \omega_k) = (a_{11}, \dots, a_{1m_1}, \dots, a_{k_1}, \dots, a_{km_k})$. Let P be the set of all partitions, and consider the following subsets of P:

$P_0 = \{(a_1, \dots, a_k) \in P \mid a_j \ne 2^i - 1$ for $1 \le j \le k$ and any $i \ge 1\}$;

$P_1 = \{(2a_1, \dots, 2a_k) \in P_0 \mid a_i \ne a_j$ for $i \ne j$, and $\alpha(a_j) \ge 2\}$.

Let $I : \Omega_* \to \mathfrak{N}_*$ be the natural map obtained by ignoring orientation. We make essential use of the following result:

THEOREM 2.1 (Wall [12; §9]). *There are elements $h_{4q} \in \Omega_{4q}$ $(q \ge 1)$ and $g_\omega \in \Omega_{|\omega|-1}$ $(\omega = (2a_1, \dots, 2a_k) \in P_1)$ which satisfy the following* (1) *and* (2):

(1) *The set $\{h_{4q}, g_\omega \mid q \ge 1, \omega \in P_1\}$ generates Ω_*;*

(2) *$I(h_{4q}) = x_{2q}^2$ and $I(g_\omega) = \sum_{j=1}^k x_{2a_1} \cdots x_{2a_j-1} \cdots x_{2a_k}$.*

We say that M^n *immerses with α-efficiency k* if M^n is cobordant to a manifold which immerses in $\mathbf{R}^{2n-\alpha(n)-k}$. Concerning this terminology, we have the following:

LEMMA 2.2. (1) (Brown [1; Theorem 5.1]) *Any manifold M^n immerses with α-efficiency 0 for any $n \ge 2$.*

(2) *Let $n = \sum_{j=1}^k n_j$. If each M^{n_j} immerses with α-efficiency a_j for $1 \le j \le k$, then $\prod_{j=1}^k M^{n_j}$ immerses with α-efficiency $b + \sum_{j=1}^k a_j$, where $b = \sum_{j=1}^k \alpha(n_j) - \alpha(n)$.*

PROOF. (2) Since each M^{n_j} is cobordant to a manifold which immerses in $\mathbf{R}^{2n_j-\alpha(n_j)-a_j}$, $\prod_{j=1}^k M^{n_j}$ is cobordant to a manifold which immerses in $\mathbf{R}^f$ for $f = \sum_{j=1}^k \{2n_j - \alpha(n_j) - a_j\} = 2n - \sum_{j=1}^k \alpha(n_j) - \sum_{j=1}^k a_j = 2n - \alpha(n) - b - \sum_{j=1}^k a_j$, as required. □

In §5–7, we will prove the following proposition which plays a crucial role in the proof of Theorem A.

PROPOSITION 2.3. (1) *If $\alpha(n) \ge 3$, then X^n immerses with α-efficiency* 1.

(2) *If n satisfies one of the following conditions* (i)–(iii), *then X^n immerses with α-efficiency* 2:

(i) $\alpha(n) = 3$ *and* $n \equiv 1 \pmod 4$;

(ii) $\alpha(n) \ge 4$ *and n is odd;*

(iii) $\alpha(n) \ge 4$, $\alpha(n)$ *is even and* $n \equiv 2 \pmod 4$.

(3) *If $n_1 \equiv n_2 \equiv 2 \pmod 4$, $\alpha(n_1) = \alpha(n_2) = 2$ and $n_1 \ne n_2$, then $(X^{n_1-1} \times X^{n_2}) \coprod (X^{n_1} \times X^{n_2-1})$ immerses with α-efficiency* 2.

3. Proof of Theorem A

For elements h_{4q} $(q \ge 1)$ and g_ω $(\omega \in P_1)$ in Ω_* given in Theorem 2.1, we take orientable manifolds H^{4q} and G_ω whose cobordism classes are $I(h_{4q})$ and

$I(g_\omega)$, respectively. By Theorem 2.1 (1), any orientable manifold is cobordant to a finite disjoint union of the form $(\prod_{i=1}^k H^{4q_i}) \times (\prod_{j=1}^l G_{\omega_j})$, where $q_i \geq 1$ and $\omega_j \in P_1$. Thus, in order to establish Theorem A, it is sufficient to prove it for the following manifolds:

(i) $M^n = \prod_{i=1}^k H^{4q_i}$, where $n = \sum_{i=1}^k 4q_i$;

(ii) $M^n = (\prod_{i=1}^k H^{4q_i}) \times (\prod_{j=1}^l G_{\omega_j})$, where $n = \sum_{i=1}^k 4q_i + \sum_{j=1}^l \{|\omega_j| - 1\}$ and $k,\ l \geq 1$;

(iii) $M^n = \prod_{j=1}^l G_{\omega_j}$, where $n = \sum_{j=1}^l \{|\omega_j| - 1\}$.

For a partition $\omega = (2a_1, \dots, 2a_k) \in P_1$, we put $Y_j = X^{2a_j - 1} \times (\prod_{i \neq j} X^{2a_i})$. By Theorem 2.1 (2), G_ω is cobordant to $\coprod_{j=1}^k Y_j$.

Proposition 3.1. *Let $\omega \in P_1$ and $n = |\omega| - 1$. Then, any G_ω immerses with, respectively, α-efficiency* 1 *or* 2 *if the following* (1) *or* (2) *holds*:

(1) *$\alpha(n)$ if odd, or*

(2) *$\alpha(n)$ is even and $n \equiv 3 \pmod 4$.*

Proof. Let $\omega = (2a_1, \dots, 2a_k) \in P_1$. First, we remark that, if $\alpha(\omega)$ is odd, then there exists t, $1 \leq t \leq k$, such that X^{2a_t} and $X^{2a_t - 1}$ immerse with α-efficiency 1. In fact, since $\omega \in P_1$ and $\alpha(\omega)$ is odd, there exists t with $\alpha(2a_t - 1) \geq \alpha(2a_t) \geq 3$, and thus, X^{2a_t} and $X^{2a_t - 1}$ immerse with α-efficiency 1 by Proposition 2.3 (1). We shall prove (2) and omit the proof of (1), since the methods are similar. Thus, assume that $\alpha(n)$ is even and $n \equiv 3 \pmod 4$. Since G_ω is cobordant to $\coprod_{j=1}^k Y_j$, it is sufficient to show that each Y_j immerses with α-efficiency 2.

(a) The case $\alpha(\omega) \geq \alpha(n) + 2$: Since $\alpha(2a_j - 1) + \sum_{i \neq j} \alpha(2a_i) - \alpha(n) \geq \alpha(\omega) - \alpha(n) \geq 2$ for each j, $1 \leq j \leq k$, Y_j immerses with α-efficiency 2 by Lemma 2.2.

(b) The case $\alpha(\omega) = \alpha(n) + 1$: Since $\alpha(\omega)$ is odd, there exists t, $1 \leq t \leq k$, such that X^{2a_t} and $X^{2a_t - 1}$ immerse with α-efficiency 1 by the remark above. Similarly to (a), we have $\alpha(2a_j - 1) + \sum_{i \neq j} \alpha(2a_i) - \alpha(n) \geq 1$. Hence, by Lemma 2.2, each Y_j immerses with α-efficiency 2.

(c) The case $\alpha(\omega) \leq \alpha(n) - 2$: For each j, we have $\alpha(n) \leq \alpha(2a_j - 1) + \sum_{i \neq j} \alpha(2a_i) = \alpha(2a_j - 1) + \alpha(\omega) - \alpha(2a_j)$, and thus, $\alpha(2a_j - 1) \geq \alpha(2a_j) + \alpha(n) - \alpha(\omega) \geq 4$, since $\omega \in P_1$. Hence, $X^{2a_j - 1}$ immerses with α-efficiency 2 by Proposition 2.3 (2), and thus, Y_j immerses with α-efficiency 2.

(d) The case $\alpha(\omega) = \alpha(n) - 1$: Since $\alpha(\omega)$ is odd, there exists t, $1 \leq t \leq k$, such that X^{2a_t} immerses with α-efficiency 1. Similarly to (c), $\alpha(2a_j - 1) \geq 3$ for each j. Hence, by Proposition 2.3 (1) and Lemma 2.2, each Y_j for $j \neq t$ immerses with α-efficiency 2. Further, when there exists at least one integer s, $1 \leq s \leq k$, such that $s \neq t$ and $\alpha(2a_s) \geq 3$, Y_t immerses with α-efficiency 2 by Proposition 2.3 (1) and Lemma 2.2. When $\alpha(2a_j) = 2$ for any $j \neq t$, we have $\alpha(2a_t)$ is odd and $\alpha(2a_t) \geq 3$, since $\alpha(\omega)$ is odd. Then,

$\alpha(2a_t-1)\geq 3$, and $2a_t-1\equiv 1 \pmod 4$ if $\alpha(2a_t-1)=3$. Thus, by Proposition 2.3(2) and Lemma 2.2, Y_t immerses with α-efficiency 2, as required.

(e) The case $\alpha(\omega)=\alpha(n)$: First, assume that there exist at least two integers t and s which satisfy $\alpha(2b)\geq 3$ for $b=a_t,\ a_s$. Since $\alpha(2b-1)\geq \alpha(2b)\geq 3$, X^{2b} and X^{2b-1} ($b=a_t,a_s$) immerse with α-efficiency 1 by Proposition 2.3(1). Thus, each Y_j immerses with α-efficiency 2, as required. Hence, we may assume that $\alpha(a_1)$ is even with $\alpha(a_1)\geq 2$ and $\alpha(a_j)=2$ for $2\leq j\leq k$, since $\alpha(\omega)$ is even. Further, when $\alpha(a_1)\geq 4$ and $2a_1\equiv 2 \pmod 4$, X^{2a_1} and X^{2a_1-1} immerse with α-efficiency 2 by Proposition 2.3(2), and each Y_j immerses with α-efficiency 2. Thus, hereafter, we also assume that $\alpha(a_1)=2$ or $2a_1\equiv 0 \pmod 4$.

If $2a_j\equiv 2 \pmod 4$ for some j, we have $\alpha(a_j)=2$ by the assumptions. Then, since $n+1\equiv 0 \pmod 4$, there exists another integer $t\neq j$ with $2a_t\equiv 2 \pmod 4$ and $\alpha(a_t)=2$. Hence, by Proposition 2.3(3), $(X^{2a_j-1}\times X^{2a_t})\coprod(X^{2a_j}\times X^{2a_t-1})$ immerses with α-efficiency 2, and thus, $Y_j\coprod Y_t=\{(X^{2a_j-1}\times X^{2a_t})\coprod(X^{2a_j}\times X^{2a_t-1})\}\times(\prod_{i\neq j,t}X^{2a_i})$ immerses with α-efficiency 2.

Lastly, we consider the case $2a_j\equiv 0 \pmod 4$ for some j. Since $\alpha(b)+\nu(b)=\alpha(b-1)+1$ in general and $\nu(2a_j)\geq 2$, we have $\alpha(2a_j-1)=\alpha(2a_j)+\nu(2a_j)-1\geq 3$. Thus, by Proposition 2.3 (1), X^{2a_j-1} immerses with α-efficiency 1. Further, since $\alpha(n)=\alpha(\omega)=\sum_{i=1}^{k}\alpha(2a_i)$, we have $\alpha(2a_j-1)+\sum_{i\neq j}\alpha(2a_i)-\alpha(n)=\alpha(2a_j-1)-\alpha(2a_j)=\nu(2a_j)-1\geq 1$. Hence, Y_j immerses with α-efficiency 2, as required. □

PROPOSITION 3.2. *Let $M^n=\prod_{j=1}^{l}G_{\omega_j}$ for $l\geq 2$, where $\omega_j\in P_1$ and $n=\sum_{j=1}^{l}\{|\omega_j|-1\}$. Then, M^n immerses with, respectively, α-efficiency $\nu(n)+1$ or $\nu(n)+2$ if the following* (1) *or* (2) *holds*:

(1) *$\alpha(n)+\nu(n)$ is odd, or*

(2) *$\alpha(n)+\nu(n)$ is even and $n\equiv 0$ or $3 \pmod 4$.*

PROOF. We omit the proof of (1), since it is similar to that of (2). Thus, assume that $\alpha(n)+\nu(n)$ is even and $n\equiv 0$ or $3 \pmod 4$. We put $n_j=|\omega_j|-1$ and $c_j=(n_j-1)/2$ for each j, $1\leq j\leq l$. Notice that $\alpha(a)+\alpha(b)\geq\alpha(a+b)$ and $\alpha(a)+\nu(a)=\alpha(a-1)+1$ for any positive integers a and b. Hence, we have $\sum_{j=1}^{l}\alpha(n_j)-\alpha(n)\geq\alpha(n-n_1)+\alpha(n_1)-\alpha(n)=\alpha(n-2c_1-1)+\alpha(2c_1)+1-\alpha(n)\geq\alpha(n-1)+1-\alpha(n)=\nu(n)$.

When there exists at least one integer t such that $\alpha(n_t)$ is even and $n_t\equiv 3 \pmod 4$, G_{ω_t} immerses with α-efficiency 2 by Proposition 3.1(2). Hence, M^n immerses with α-efficiency $\nu(n)+2$ by Lemma 2.2, as required. Thus, in the remaining of the proof, we assume that $\alpha(n_j)$ is odd or $n_j\equiv 1 \pmod 4$ for each j, $1\leq j\leq l$.

When there exist at least two integers t, s such that each $\alpha(n_i)$ is odd for $i=t,\ s$, both G_{ω_i} immerse with α-efficiency 1 by Proposition 3.1(1), and M^n

immerses with α-efficiency $\nu(n)+2$. When only one $\alpha(n_t)$ is odd for $1 \le t \le l$, G_{ω_t} immerses with α-efficiency 1 by Proposition 3.1 (1). If $\sum_{j=1}^{l} \alpha(n_j) - \alpha(n) = \nu(n)$, then we have that $\alpha(n) + \nu(n) = \sum_{j=1}^{l} \alpha(n_j)$ is odd, which contradicts the assumption that $\alpha(n) + \nu(n)$ is even. Hence, $\sum_{j=1}^{l} \alpha(n_j) - \alpha(n) \ge \nu(n) + 1$, and M^n immerses with α-efficiency $\nu(n)+2$, as required.

Lastly, we assume that all $\alpha(n_j)$ are even for $1 \le j \le l$. Then, we notice that $n_j \equiv 1 \pmod 4$ by the above assumption. When $l = 2$, $n = n_1 + n_2 \equiv 2 \pmod 4$, which contradicts the assumption that $n \equiv 0$ or $3 \pmod 4$. When $l \ge 3$, we have $\sum_{j=1}^{l} \alpha(n_j) - \alpha(n) = \alpha(2c_1) + \alpha(2c_2) + \alpha(2c_3) + 3 + \sum_{j=4}^{l} \alpha(n_j) - \alpha(n) \ge \alpha(n-3) + 3 - \alpha(n) = \nu(n-2) + \alpha(n-2) + 2 - \alpha(n) = \nu(n-2) + \nu(n-1) + \alpha(n-1) + 1 - \alpha(n) = \nu(n-2) + \nu(n-1) + \nu(n) \ge \nu(n) + 1$. If $\sum_{j=1}^{l} \alpha(n_j) - \alpha(n) = \nu(n) + 1$, then $\alpha(n) + \nu(n) = \sum_{j=1}^{l} \alpha(n_j) - 1$ is odd by the assumption, which contradicts the assumption that $\alpha(n) + \nu(n)$ is even. Hence, $\sum_{j=1}^{l} \alpha(n_j) - \alpha(n) \ge \nu(n) + 2$, and M^n immerses with α-efficiency $\nu(n)+2$, as required. $\square$

PROOF OF THEOREM A. Propositions 3.1 and 3.2 establish Theorem A for the case (iii): $M^n = \prod_{j=1}^{l} G_{\omega_j}$. In order to show the remaining cases, we first remark that H^{4q} immerses with α-efficiency $\alpha(q)$ for any $q \ge 1$. In face, since H^{4q} is cobordant to $(X^{2q})^2$ by Theorem 2.1 (2), and $\alpha(2q) + \alpha(2q) - \alpha(4q) = \alpha(q)$, H^{4q} immerses with α-efficiency $\alpha(q)$ by Lemma 2.2.

(i) The case $M^n = \prod_{i=1}^{k} H^{4q_i}$: Since $\alpha(n) > \nu(n)$ and each H^{4q_i} immerses with α-efficiency $\alpha(q_i)$ by the remark above, M^n immerses with α-efficiency $\sum_{i=1}^{k} \alpha(4q_i) - \alpha(n) + \sum_{i=1}^{k} \alpha(q_i) \ge \sum_{i=1}^{k} \alpha(q_i) = \sum_{i=1}^{k} \alpha(4q_i) \ge \alpha(n) \ge \nu(n) + 1$ by Lemma 2.2. If $\alpha(n) + \nu(n)$ is even and $\alpha(n) > \nu(n)$, then $\alpha(n) \ge \nu(n) + 2$. Thus, similarly, we have the required result in this case.

(ii) The case $M^n = (\prod_{i=1}^{k} H^{4q_i}) \times (\prod_{j=1}^{l} G_{\omega_j})$ for $k,\ l \ge 1$: Similarly to the proof of Proposition 3.2, we have $\sum_{i=1}^{k} \alpha(4q_i) + \sum_{j=1}^{l} \alpha(|\omega_j| - 1) - \alpha(n) \ge \nu(n)$. Let $G' = \prod_{j=1}^{l} G_{w_j}$ and $n' = \sum_{j=1}^{l} \{|\omega_j| - 1\}$, then $M^n = (\prod_{i=1}^{k} H^{4q_i}) \times G'$ and $n = \sum_{i=1}^{k} 4q_i + n'$. Since H^{4q_1} immerses with α-efficiency $\alpha(q_1) \ge 1$, M^n immerses with α-efficiency $\nu(n)+1$ by Lemma 2.2. If $n \equiv 0$ or $3 \pmod 4$, then $n' = n - \sum_{i=1}^{k} 4q_i \equiv 0$ or $3 \pmod 4$, thus G' immerses with α-efficiency $\nu(n')+1$ by Propositions 3.1 and 3.2. Since H^{4q_1} also immerses with α-efficiency 1, M^n immerses with α-efficiency $\nu(n)+2$, as required. $\square$

4. Proof of Theorem B

Let $w_i(M^n) \in H^i(M^n)$ for $i \ge 0$ be the Stiefel-Whitney class of $\tau(M^n)$, and $\overline{w}_i(M^n)$ its dual class. That is, they satisfy $(\sum_{i \ge 0} w_i(M^n)) \times (\sum_{i \ge 0} \overline{w}_i(M^n)) = 1$. Throughout the paper, the cohomology and the homology are always assumed to be with coefficient Z_2. Since the manifolds treated in this paper may not be connected, we have to distinguish the Stiefel-Whitney class $\bar{w}_{n_1} \cdots \bar{w}_{n_k}(M^n)$ and the Stiefel-Whitney number $\bar{w}_{n_1} \cdots \bar{w}_{n_k}[M^n]$ for $n = \sum_{i=1}^k n_i$. Then, we recall the following:

Lemma 4.1 (cf. [10; Lemma 4.1]). *Let L^n and $L_i^{n_i}$ $(1 \le i \le k)$ be manifolds with $L^n = \prod_{i=1}^k L_i^{n_i}$. If each $L_i^{n_i}$ satisfies the following*

(i) *the Stiefel-Whitney number $\bar{w}_{\sigma_i}\bar{w}_{n_i-\sigma_i}[L_i^{n_i}] \neq 0$,*

(ii) *any Stiefel-Whitney number which contains $\bar{w}_j(L_i^{n_i})$ $(j > n_i - \sigma_i)$ as a factor vanishes, for some $\sigma_i < n_i$, then $\bar{w}_\sigma \bar{w}_{n-\sigma}[L^n] \neq 0$ and any Stiefel-Whitney number which contains $\bar{w}_j(L^n)$ $(j > n - \sigma)$ as a factor vanishes, where $\sigma = \sum_{i=1}^k \sigma_i$.*

Since two manifolds are cobordant if and only if they have the same Stiefel-Whitney numbers (see [9; Chapter VI]), if a manifold L^n satisfies $\bar{w}_\sigma \bar{w}_{n-\sigma}[L^n] \neq 0$, then any manifold M^n cobordant to L^n satisfies $\bar{w}_\sigma \bar{w}_{n-\sigma}[M^n] \neq 0$, and thus, $\bar{w}_{n-\sigma}(M^n) \neq 0$. Since a necessary condition for M^n to immerse in $\mathbf{R}^{2n-\sigma-1}$ is that $\bar{w}_j(M^n) = 0$ for any $j \ge n - \sigma$, the following proposition establishes Theorem B, and this section is devoted to proving it.

Proposition 4.2. *Assume that $\alpha(n) > \nu(n)$ and $n \ge 4$ with $n \neq 6, 7$. If n and $\sigma(n)$ satisfy one of the following* (i)–(iii), *then there exists an orientable manifold L^n which satisfies $\bar{w}_{\sigma(n)}\bar{w}_{n-\sigma(n)}[L^n] \neq 0$ and any Stiefel-Whitney number which contains $\bar{w}_j(L^n)$ $(j > n - \sigma(n))$ as a factor vanishes:*

(i) *$\alpha(n) + \nu(n)$ is odd and $\sigma(n) = \alpha(n) + \nu(n) + 1$;*

(ii) *$\alpha(n) + \nu(n)$ is even, $n \equiv 0$ or $3 \pmod 4$ and $\sigma(n) = \alpha(n) + \nu(n) + 2$;*

(iii) *$\alpha(n) + \nu(n)$ is even, $n \equiv 1$ or $2 \pmod 4$ and $\sigma(n) = \alpha(n) + \nu(n)$.*

We prepare some lemmas for the proof of Proposition 4.2. In [10; Lemma 4.3], we have shown the following:

Lemma 4.3. (i) *Let $n = 2r$, where $r \ge 2$ and r is a power of 2. Then, $\bar{w}_j(\mathbf{C}P^r) = 0$ for any $j > n - 2$, and $\bar{w}_2\bar{w}_{n-2}[\mathbf{C}P^r] \neq 0$.*

(ii) *Let $n = 2t + s - 1$, where $t \ge s \ge 2$ and t, s are both powers of 2. Then, $\bar{w}_j(P(s-1,t)) = 0$ for any $j > n - s$, and $\bar{w}_s\bar{w}_{n-s}[P(s-1,t)] \neq 0$.*

Wall [12; Lemmas 4, 5] has shown that the total Stiefel-Whitney class of $Q(m,n)$ is

$$w(Q(m,n)) = (1 + c + x)(1 + c)^{m-1}(1 + c + d)^{n+1}, \tag{4.4}$$

where $c, x \in H^1(Q(m,n))$ and $d \in H^2(Q(m,n))$ which are bound by the relations $x^2 = 0$, $c^{m+1} = c^m x$ and $d^{n+1} = 0$.

Lemma 4.5. *Let $n = 2t + s$, where $t \ge s \ge 2$ and t, s are both powers of 2. Then, $\bar{w}_j(Q(s-1,t)) = 0$ for any $j > n - 2$, and $\bar{w}_2\bar{w}_{n-2}[Q(s-1,t)] \neq 0$.*

Proof. By (4.4), $w(Q(s-1,t)) = (1+c+x)(1+c)^{s-2}\ (1+c+d)^{t+1}$. Since $x^2 = 0$, $c^i = c^{i-1}x = c^{i-2}x^2 = 0$ for $i \geq s+1$ and $d^{t+1} = 0$, we have $\bar{w}(Q(s-1,t)) = (1+c+x)^{-1}(1+c)^{-s+2}(1+c+d)^{-t-1} = (1+c+x)^{2s-1} \cdot (1+c)^{s+2}(1+c+d)^{t-1}$. Here, $(1+c+x)^{2s-1}(1+c)^{s+2} = \{(1+c)^{2s-1} + (1+c)^{2s-2}x\}(1+c)^{s+2} = (1+c+x)(1+c)^{3s} = (1+c+x)(1+c^s)^3 = (1+c+x)(1+c^s) = 1+c+x+c^s$, and hence, $\bar{w}(Q(s-1,t)) = (1+c+x+c^s)(1+c+d)^{t-1}$. Thus, we have $\bar{w}_j(Q(s-1,t)) = 0$ for any $j > s+2t-2 = n-2$ and $\bar{w}_{n-2}(Q(s-1,t)) = c^s d^{t-1}$. Because $\bar{w}_2(Q(s-1,t)) = (t-1)d$ up to terms which contain c or x, we have $\bar{w}_2\bar{w}_{n-2}\ (Q(s-1,t)) = (t-1)c^s d^t = c^s d^t \neq 0$, which completes the proof. $\square$

When we denote by $m = \sum_{i=1}^{t} s_i$ a dyadic expansion of m, we assume that each s_i is a power of 2 and $s_1 > \cdots > s_t \geq 1$.

Lemma 4.6. *Assume that n is odd and $\alpha(n+1)$ is even. Let $n+1 = \sum_{i=1}^{2k} 2r_i$ be a dyadic expansion of $n+1$, and $\omega = (2r_1 + 2r_{2k}, \dots, 2r_k + 2r_{k+1}) \in P_1$. Then, any Stiefel-Whitney number which contains $\bar{w}_j(G_\omega)$ $(j > n - 2r_{2k} - 2k + 2)$ as a factor vanishes, and the Stiefel-Whitney number $\bar{w}_{2r_{2k}+2k-2} \cdot \bar{w}_{n-2r_{2k}-2k+2}[G_\omega] \neq 0$.*

Proof. We put $b_t = 2r_t + 2r_{2k-t+1}$ $(1 \leq t \leq k)$. By Theorem 2.1 (2), G_ω is cobordant to $\coprod_{i=1}^{k} Y_i$ where $Y_i = X^{b_i-1} \times (\prod_{t\neq i} X^{b_t})$. Here, $X^{b_i-1} = P(2r_{2k-i+1} - 1, r_i)$ and $X^{b_t} = Q(2r_{2k-t+1} - 1, r_t)$ by definition. By Lemmas 4.1, 4.3 (ii) and 4.5, we have $\bar{w}_j(Y_i) = 0$ for each $i \geq 2$ and any $j \geq n - 2r_{2k} - 2k + 2$. Similarly, for $i = 1$, $\bar{w}_j(Y_1) = 0$ for any $j > n - 2r_{2k} - 2k + 2$, and $\bar{w}_{2r_{2k}+2k-2}\bar{w}_{n-2r_{2k}-2k+2}[Y_1] \neq 0$. Hence, we have the required result. $\square$

Now, we shall complete the proof of Proposition 4.2, which establishes Theorem B.

Proof of Proposition 4.2. In the below, each r_j $(j \geq 1)$ is always a power of 2, and $r_i > r_j$ for $i < j$. We first consider the case n is odd, namely $v(n) = 0$.

(i) In this case, $\alpha(n)$ is odd. When $n \equiv 1 \pmod 4$, we put $n = \sum_{i=1}^{2k} 2r_i + 1$ for $r_{2k} \geq 2$, and $\omega = (2r_2 + 2, 2r_3 + 2r_{2k}, \dots, 2r_{k+1} + 2r_{k+2})$. Then, by Lemmas 4.1, 4.3 (i) and 4.6, $L^n = \mathbf{C}P^{r_1} \times G_\omega$ satisfies the conditions of Proposition 4.2 for $\sigma(n) = 2k+2 = \alpha(n)+1$, as required. When $n \equiv 3 \pmod 8$, we put $n = \sum_{i=1}^{2k-1} 2r_i + 3$ for $r_{2k-1} \geq 4$, and $\omega = (2r_1 + 4, 2r_2 + 2r_{2k-1}, \dots, 2r_k + 2r_{k+1})$. By Lemma 4.6, $L^n = G_\omega$ satisfies the conditions for $\sigma(n) = 2k+2 = \alpha(n)+1$. When $n \equiv 7 \pmod 8$, we put $n = \sum_{i=1}^{2k} 2r_i + 7$ for $r_{2k} \geq 4$, $\omega_1 = (6)$, $\omega_2 = (2r_{2k} + 2)$ and $\omega_3 = (2r_1 + 2, 2r_2 + 2r_{2k-1}, \dots, 2r_k + 2r_{k+1})$. By Lemmas 4.1 and 4.6, $L^n = G_{\omega_1} \times G_{\omega_2} \times G_{\omega_3}$ satisfies the conditions for $\sigma(n) = 2k+4 = \alpha(n)+1$.

(ii) Since $n \equiv 3 \pmod 4$ and $\alpha(n)$ is even in this case, we put $n = \sum_{i=1}^{2k} 2r_i + 3$ for $r_{2k} \geq 2$, and $\omega = (2r_1 + 4,\ 2r_2 + 2r_{2k-1}, \dots, 2r_k + 2r_{k+1})$. By Lemmas 4.1, 4.3 (i) and 4.6, $L^n = \mathbf{C}P^{r_{2k}} \times G_\omega$ satisfies the conditions for $\sigma(n) = 2k + 4 = \alpha(n) + 2$.

(iii) Since $n \equiv 1 \pmod 4$ and $\alpha(n)$ is even in this case, we put $n = \sum_{i=1}^{2k-1} 2r_i + 1$ for $r_{2k-1} \geq 2$, and $\omega = (2r_1 + 2, 2r_2 + 2r_{2k-1}, \dots, 2r_k + 2r_{k+1})$. By Lemma 4.6, $L^n = G_\omega$ satisfies the conditions for $\sigma(n) = 2k = \alpha(n)$.

Next, we consider the case n is even.

(i) When $\nu(n) = 1$ and $\alpha(n) = 2$, we put $n = 4r_1 + 2$ $(r_1 \geq 2)$, since $n \neq 6$ by the assumption. By Lemmas 4.1 and 4.3 (ii), $L^n = P(1, r_1) \times P(1, r_1)$ satisfies the conditions for $\sigma(n) = 4 = \alpha(n) + \nu(n) + 1$. When $\nu(n) = 1$ and $\alpha(n) \geq 4$ is even, we put $n = \sum_{i=1}^{2k-1} 2r_i + 2$ for $r_{2k-1} \geq 2$, and $\omega = (2r_3 + 2, 2r_4 + 2r_{2k-1}, \dots, 2r_{k+1} + 2r_{k+2})$. By Lemmas 4.1, 4.3 and 4.6, $L^n = P(1, r_1) \times \mathbf{C}P^{r_2} \times G_\omega$ satisfies the conditions for $\sigma(n) = 2k + 2 = \alpha(n) + \nu(n) + 1$. When $\nu(n) \geq 2$, by the assumption $\alpha(n) > \nu(n)$, we put $n = \sum_{i=1}^{k} 2r_i$ for $r_k \geq 2$ and $k \geq 3$, and $m = \sum_{i=2}^{k} 2r_i - 3$. Here, we notice that $m \equiv 1 \pmod 4$ and $\alpha(m) = \alpha(n) + \nu(n) - 3$ is even. By the above case (iii) for odd n, there exists an orientable manifold N^m which satisfies the conditions of Proposition 4.2 for $\sigma(m) = \alpha(m)$. Hence, by Lemmas 4.1 and 4.3 (ii), $L^n = P(3, r_1) \times N^m$ satisfies the conditions for $\sigma(n) = 4 + \sigma(m) = \alpha(n) + \nu(n) + 1$.

(ii) Since $n \equiv 0 \pmod 4$ and $\alpha(n) > \nu(n) \geq 2$, we put $n = \sum_{i=1}^{k} 2r_i$ for $r_k \geq 2$ and $k \geq 3$, and $m = \sum_{i=2}^{k} 2r_i - 3$. Then, m and $\alpha(m) = \alpha(n) + \nu(n) - 3$ are odd. By the above case (i) for odd n, there exists an orientable manifold N^m which satisfies the conditions of Proposition 4.2 for $\sigma(m) = \alpha(m) + 1$. Hence, by Lemmas 4.1 and 4.3 (ii), $L^n = P(3, r_1) \times N^m$ satisfies the conditions for $\sigma(n) = 4 + \sigma(m) = \alpha(m) + 5 = \alpha(n) + \nu(n) + 2$.

(iii) Since $n \equiv 2 \pmod 4$ and $\alpha(n) + \nu(n)$ is even, we put $n = \sum_{i=1}^{2k} 2r_i + 2$ for $r_{2k} \geq 2$, and $\omega = (2r_2 + 2, 2r_3 + 2r_{2k}, \dots,\ 2r_{k+1} + 2r_{k+2})$. By Lemmas 4.1, 4.3 (ii) and 4.6, $L^n = P(1, r_1) \times G_\omega$ satisfies the conditions for $\sigma(n) = 2k + 2 = \alpha(n) + \nu(n)$. □

5. Immersions of X^n for $\alpha(n) \geq 3$ or n is odd

The remaining of this paper is devoted to proving Proposition 2.3. For a space Y and a positive integer m, let $P(m, Y)$ be the space constructed from $S^m \times Y \times Y$ by identifying (u, x, y) with $(-u, y, x)$. For odd n not of the form $2^i - 1$, we write $n = 2^r(2s + 1) - 1$ with r, $s \geq 1$, and set $V^n = P(2^r - 1, \mathbf{R}P^{2^r s})$. Brown [1; Corollary 7.5] has shown that V^n is cobordant to the Dold manifold $X^n = P(2^r - 1, 2^r s)$. In this section, we prove the following proposition, which establishes Proposition 2.3 (1) and (2) (i), (ii).

PROPOSITION 5.1. (1) *Let n be odd and not of the form $2^i - 1$ for any i. If $\alpha(n) \geq 3$, then V^n immerses with α-efficiency* 1. *Moreover, if n satisfies one of the following conditions* (i) *and* (ii), *then V^n immerses with α-efficiency* 2:

(i) $\alpha(n) = 3$ *and* $n \equiv 1 \pmod 4$;

(ii) $\alpha(n) \geq 4$.

(2) *If n is even with $\alpha(n) \geq 3$, then X^n immerses with α-efficiency* 1.

We need the following results.

THEOREM 5.2 (Brown [1; Proposition 4.3, Theorem 6.3]). (1) *$P(m, \mathbf{R}^k)$ is the total space of the bundle $k(\gamma_m \oplus \varepsilon_m)$, where γ_m and ε_m are the canonical line bundle and trivial line bundle over $\mathbf{R}P^m$, respectively.*

(2) *For even integer n, if the Stiefel-Whitney number $\bar{w}_{\alpha(n)}\bar{w}_{n-\alpha(n)}[M^n]$ vanishes, then M^n immerses with α-efficiency* 1.

THEOREM 5.3 (Mahowald and Milgram [6; Theorem 4.1]). *Let p and q be odd and $m = p + q + 1$. Then, the total space of $(p+1)\gamma_q$ immerses in $\mathbf{R}^f$ for $f = 2q + p + 1 - \alpha(m) + \alpha(p+1) - k(p, m)$. Here, $k(p, m) = \min(k(p), k(m))$ and $k(t) = 0$, 1 and 4 if $t \equiv 1$ or 5, 3 and $7 \pmod 8$, respectively.*

PROOF OF PROPOSITION 5.1. (1) We set $n = 2^r(2s+1) - 1$ for r, $s \geq 1$, $a = 2^r - 1$ and $b = 2^r s$. Then $n = a + 2b$. Milgram [7; Theorem 1] and Lam [5; Theorem (6.2)] have proved that $\mathbf{R}P^l$ immerses in $\mathbf{R}^{2l-\alpha(l)}$ for $l > 7$. We remark that, for $2 \leq l \leq 7$, $\mathbf{R}P^l$ also immerses in $\mathbf{R}^{2l-\alpha(l)}$. Hence, by Theorem 5.2(1), $V^n = P(a, \mathbf{R}P^b)$ immerses in the total space of the bundle $(2b - \alpha(b))(\gamma_a \oplus \varepsilon_a)$.

First, assume that $r \geq 3$. We apply Theorem 5.3 to $2b\gamma_a$ with $p + 1 = 2b$, $q = a$ and $m = a + 2b = n$. Since $\alpha(m) - \alpha(p+1) = r$ and $k(p, m) = k(2b-1, n) = 4$, we obtain an immersion of $2b\gamma_a$ in $\mathbf{R}^{2a+2b-r-4}$. Hence, $2b\gamma_a \oplus (2b - \alpha(b))\varepsilon_a$ immerses in $\mathbf{R}^{2a+4b-\alpha(b)-r-4} = \mathbf{R}^{2n-\alpha(n)-4}$. Since $(2b - \alpha(b)) \cdot (\gamma_a \oplus \varepsilon_a)$ is a subbundle of $2b\gamma_a \oplus (2b - \alpha(b))\varepsilon_a$, V^n immerses in $\mathbf{R}^{2n-\alpha(n)-4}$, as required.

Next, assume that $r = 2$. Then, we have $a = 3$, $n = 2b + 3$ and $\alpha(n) = \alpha(b) + 2$. Since $\varepsilon_3 \approx \mathbf{R}P^3 \times \mathbf{R}$ immerses in $\mathbf{R}^4$, we have an immersion of $4\gamma_3 \approx \tau(\mathbf{R}P^3) \oplus \varepsilon_3 \approx 4\varepsilon_3$ in $\mathbf{R}^7$, and $4k\gamma_3 \approx 4k\varepsilon_3$ in $\mathbf{R}^{4k+3}$ for any positive integer k. We further assume $\alpha(b) \geq 4$. We put $\alpha(b) \equiv l \pmod 4$ where $0 \leq l \leq 3$. Then we notice that $\alpha(b) - l \geq 4$. Since $2b - \alpha(b) + l \equiv 0 \pmod 4$, we have an immersion of $(2b - \alpha(b) + l)\gamma_3$ in $\mathbf{R}^{2b-\alpha(b)+l+3}$ by the above remark. Similarly to the proof of the case $r \geq 3$, by taking the product with $\mathbf{R}^{2b-\alpha(b)}$, we obtain an immersion of V^n in $\mathbf{R}^f$ for $f = 4b - 2\alpha(b) + l + 3 = 2n - \alpha(n) - (\alpha(b) - l) - 1 \leq 2n - \alpha(n) - 5$, as required.

We omit the proof of the remaining cases, since they are shown similarly, except that we use the following result by Gitler-Mahowald [3; Theorem E]: $\mathbf{R}P^l$ immerses in $\mathbf{R}^{2l-5}$ for $l \equiv 0 \pmod 4$ with $\alpha(l) \geq 2$.

(2) We set $n = 2^r(2s+1)$ for $r \geq 1$ and $\alpha(s) \geq 2$, $a = 2^r$ and $b = 2^r s$. Let $b = \sum_{i=1}^k r_i$ be a dyadic expansion of b for $r_k \geq 2^r \geq 2$. Then, we have $n = a + 2b$, $\alpha(n) = k+1$ and $X^n = Q(a-1, b)$. By (4.4), $w(X^n) = (1+c+x)(1+c)^{a-2}(1+c+d)^{b+1}$, where $c, x \in H^1(X^n)$ and $d \in H^2(X^n)$ with $x^2 = 0$, $c^a = c^{a-1}x$ and $d^{b+1} = 0$. Since $c^j = c^{j-2}x^2 = 0$ for $j \geq a+1$, we have $\bar{w}(X^n) = (1+c+x)^{-1}(1+c)^{-a+2}(1+c+d)^{-b-1} = (1+c+x)^{2a-1}(1+c)^{a+2} \cdot (1+c+d)^{2r_1-b-1}$. Here, $(1+c+x)^{2a-1}(1+c)^{a+2} = 1+c+x+c^a$, and $\bar{w}(X^n) = (1+c+x+c^a)(1+c+d)^{2r_1-b-1}$. Thus, $\bar{w}_j(X^n) = 0$ for any $j > a + 4r_1 - 2b - 2$. Since $(n - \alpha(n)) - (a + 4r_1 - 2b - 2) = 4b - 4r_1 - k + 1 = \sum_{i=2}^k (4r_i - 1) > 0$, we have $\bar{w}_{n-\alpha(n)}(X^n) = 0$, which completes the proof by Theorem 5.2(2) □

6. Symmetric characteristic classes

In this section, we prepare some results about the symmetric characteristic classes, which will be used in the next section. Let $s_\omega \in \mathbf{Z}[t_1, \ldots, t_l]$, a polynomial ring over $\mathbf{Z}$, be the smallest symmetric function which contains the monomial $t_1^{a_1} \cdots t_k^{a_k}$ for any partition $\omega = (a_1, \ldots, a_k) \in P$ with $l \geq |\omega|$. Then, for the partition $\zeta_i = (\underbrace{1, \ldots, 1}_{i})$, s_{ζ_i} is the elementary symmetric function θ_i, and s_ω is expressible as a polynomial $s_\omega = P_\omega(\theta_1, \ldots, \theta_{|\omega|})$ with integral coefficients. P_ω is uniquely determined by ω if we take $l \geq |\omega|$. We define $s_\omega(M^n) \in H^{|\omega|}(M^n)$ to be $s_\omega(M^n) = P_\omega(w_1, \ldots, w_{|\omega|})$ for the Stiefel-Whitney classes $w_i = w_i(M^n)$ of M^n. Then, when $M = M_1 \times M_2$, $s_\omega(M) = \sum_{(\omega_1, \omega_2) = \omega} s_{\omega_1}(M_1) \otimes s_{\omega_2}(M_2)$, and $[M^n]$ is indecomposable in $\mathfrak{N}_*$ if and only if the Stiefel-Whitney number $s_{(n)}[M^n] \neq 0$ (cf. [9; Chapters V, VI]). We remark that M^n is cobordant to N^n if and only if $s_\omega[M^n \coprod N^n] = 0$ for any partition $\omega \in P$ with $|\omega| = n$. For the manifolds X^n defined in §2, we denote $X^\omega = \prod_{j=1}^k X^{a_j}$ for $\omega = (a_1, \ldots, a_k) \in P_0$. Since $\mathfrak{N}_*$ is the polynomial algebra with $[X^n]$ as generators, any manifold M^n is cobordant to a finite disjoint union of X^ω for $|\omega| = n$ and $\omega \in P_0$. We denote by $\overline{M}^n$ such a finite disjoint union of X^ω for M^n. Then, we have the following lemma.

Lemma 6.1. *Assume that* $s_\eta[M^n] = 0$ *for any* $\eta \in P$ *with* $l(\eta) < m$ *and* $|\eta| = n$. *Then,*

(1) $s_\eta[M^n] = 0$ *for any* $\eta \in P - P_0$ *with* $l(\eta) = m$ *and* $|\eta| = n$.

(2) *For* $\omega \in P_0$ *with* $l(\omega) = m$ *and* $|\omega| = n$, $\overline{M}^n$ *contains* X^ω *if and only if* $s_\omega[M^n] \neq 0$.

PROOF. For any $\eta \in P$ and $\omega = (a_1, \ldots, a_k) \in P_0$ with $|\eta| = |\omega|$, we have $s_\eta[X^\omega] = \sum_{(\eta_1,\ldots,\eta_k)=\eta} s_{\eta_1}[X^{a_1}] \cdots s_{\eta_k}[X^{a_k}]$. If $l(\eta) < l(\omega)$, or if $l(\eta) = l(\omega)$ and $\eta \neq \omega$, then there exists at least one integer j which satisfies $|\eta_j| > a_j$, and hence $s_\eta[X^\omega] = 0$. If $\eta = \omega$, then $s_\eta[X^\omega] = s_{(a_1)}[X^{a_1}] \cdots s_{(a_k)}[X^{a_k}] \neq 0$. Thus, by the assumption, $\overline{M}^n$ contains only X^ω with $\omega \in P_0$, $l(\omega) \geq m$ and $|\omega| = n$. Hence, for any $\eta \in P - P_0$ with $l(\eta) = m$ and $|\eta| = n$, we have $s_\eta[\overline{M}^n] = 0$, and thus $s_\eta[M^n] = 0$. For $\omega \in P_0$ with $l(\omega) = m$ and $|\omega| = n$, $\overline{M}^n$ contains X^ω if and only if $s_\omega[\overline{M}^n] \neq 0$, namely, $s_\omega[M^n] \neq 0$. □

By Lemma 6.1, we remark that M^n is cobordant to N^n if and only if $s_\omega[M^n \coprod N^n] = 0$ for any partition $\omega \in P_0$ with $|\omega| = n$. Thus, hereafter in this paper, we always assume that any partition is in P_0. For a manifold N^n and a partition ω with $|\omega| = n$, if N^n satisfies $s_\omega[N^n] \neq 0$ and $s_\eta[N^n] = 0$ for any partition $\eta \neq \omega$ with $|\eta| = n$, we say *ω is realized by N^n* or *N^n realizes ω*. We define $Rd_\sigma = \{\omega \in P_0 \mid \omega$ is realized by a manifold which immerses with α-efficiency $\sigma\}$. We remark that $Rd_0 \supset Rd_1 \supset Rd_2 \supset \cdots$, and $\omega \in Rd_0$ for any partition ω by Lemma 2.2.

LEMMA 6.2. *Let σ be a non-negative integer. If $s_\omega[M^n] = 0$ for any partition ω with $|\omega| = n$ and $\omega \notin Rd_\sigma$, then M^n immerses with α-efficiency σ.*

PROOF. By the assumption, for any partition ω with $s_\omega[M^n] \neq 0$ and $|\omega| = n$, we have $\omega \in Rd_\sigma$. Then, there exists a manifold N_ω which realizes ω and immerses with α-efficiency σ. Since M^n is cobordant to a manifold which is a disjoint union of such manifolds N_ω, we have the required result. □

COROLLARY 6.3. *Let σ be a non-negative integer. If N^n immerses with α-efficiency σ and $s_\omega[M^n \coprod N^n] = 0$ for any partition ω with $|\omega| = n$ and $\omega \notin Rd_\sigma$, then M^n immerses with α-efficiency σ.*

LEMMA 6.4. *Let $\omega = (a_1, \ldots, a_k)$ be a partition with $|\omega| = n$, and $\sigma = \sum_{j=1}^k \sigma_j$ for non-negative integers σ_j. If $(a_j) \in Rd_{\sigma_j}$ for each j, $1 \leq j \leq k$, then $\omega \in Rd_{\alpha(\omega)-\alpha(n)+\sigma}$.*

PROOF. We denote by $N_{(a_j)}$ a manifold which realizes (a_j) and immerses with α-efficiency σ_j for each j. Then, clearly, $\prod_{j=1}^k N_{(a_j)}$ realizes ω, and immerses with α-efficiency $\alpha(\omega) - \alpha(n) + \sigma$ by Lemma 2.2 (2), as required. □

By a similar proof of Lemma 6.2, we have the following:

LEMMA 6.5. *Let $(n) \in P_0$ and σ be a positive integer. If there exists a manifold L^n which immerses with α-efficiency σ and satisfies that $s_{(n)}[L^n] \neq 0$ and $s_\omega[L^n] = 0$ for any partition ω with $|\omega| = n$, $l(\omega) \geq 2$ and $\omega \notin Rd_\sigma$, then $(n) \in Rd_\sigma$.*

In the rest of this section, we show the next proposition which plays a crucial role to complete the proof of Proposition 2.3.

PROPOSITION 6.6. *Let n be even. If $\alpha(n) \geq 2$, then $(n) \in Rd_1$. Moreover, if n satisfies one of the following* (i)–(iv), *then* $(n) \in Rd_2$:

(i) $\alpha(n) = 2$ *and* $n \equiv 0 \pmod 4$;

(ii) $\alpha(n) = 3$;

(iii) $\alpha(n) = 4$ *and* $n \equiv 2 \pmod 4$;

(iv) $\alpha(n) \geq 5$.

For even integer n with $1 \leq \alpha(n) \leq 3$, we set $W^n = \mathbf{R}P^n$ the real projective space. For even n with $\alpha(n) \geq 4$, let $n = \sum_{j=1}^{k} r_j$ be a dyadic expansion of n. Then, for odd k, we put $b_1 = r_1 + r_2, \dots, b_{m-1} = r_{k-2} + r_{k-1}$, $b_m = r_k + 1$ where $m = (k+1)/2$, and for even k, we put $b_1 = r_1$, $b_2 = r_2 + r_3, \dots,$ $b_{m-1} = r_{k-2} + r_{k-1}$, $b_m = r_k + 1$ where $m = (k+2)/2$. Let $K^{n+1} = \prod_{j=1}^{m} \mathbf{R}P^{b_j}$. Then, $H^*(K^{n+1}) = \mathbf{Z}_2[c_1, \dots, c_m]/(c_1^{b_1+1}, \dots, c_m^{b_m+1})$ for $c_j \in H^1(\mathbf{R}P^{b_j})$. Consider the submanifold $W^n \subset K^{n+1}$ dual to the cohomology class $\mu = \sum_{j=1}^{m} c_j$. That is, the inclusion $\imath : W^n \to K^{n+1}$ sends the fundamental homology class $(W^n) \in H_n(W^n)$ to the Poincaré dual of μ (cf. [9; Chapter V]). For $n = r_1 + r_2 + 3$ ($r_1 > r_2 \geq 4$) where each r_j is a power of 2, similarly, we define W^n to be the submanifold of $K^{n+1} = \mathbf{R}P^{r_1+2} \times \mathbf{R}P^{r_2+2}$ dual to $c_1 + c_2$, where $H^*(K^{n+1}) = \mathbf{Z}_2[c_1, c_2]/(c_1^{r_1+3}, c_2^{r_2+3})$. Further, for even n with $\alpha(n) = 4$, similarly to the above definitions of W^n, we define $\tilde{W}^n$ to be the submanifold of $\tilde{K}^{n+1} = \mathbf{R}P^{r_1+r_2} \times \mathbf{R}P^{r_3} \times \mathbf{R}P^{r_4+1}$ dual to $\sum_{j=1}^{3} c_j$, where $n = \sum_{j=1}^{4} r_j$ is a dyadic expansion of n and $H^*(\tilde{K}^{n+1}) = \mathbf{Z}_2[c_1, c_2, c_3]/(c_1^{r_1+r_2+1}, c_2^{r_3+1}, c_3^{r_4+2})$.

LEMMA 6.7. (1) *When n is even with $\alpha(n) \geq 2$, W^n immerses with α-efficiency* 2.

(2) *When $n \equiv 0 \pmod 4$ with $\alpha(n) = 4$, $\tilde{W}^n$ immerses with α-efficiency* 2.

(3) *When $n \equiv 3 \pmod 4$ with $\alpha(n) = 4$, W^n immerses with α-efficiency* 2.

PROOF. (1) Sanderson [8; Theorem (5.3)] has proved that $\mathbf{R}P^l$ immerses in $\mathbf{R}^{2l-6}$ for $l \equiv 3 \pmod 4$ with $l > 8$. We remark that $\mathbf{R}P^7$ immerses in $\mathbf{R}^8$. Hence, when $n \equiv 2 \pmod 4$ with $\alpha(n) = 2$, $W^n \subset \mathbf{R}P^{n+1}$ immerses in $\mathbf{R}^{2n-\alpha(n)-2}$. Gitler-Mahowald [3; Theorem E] has proved that $\mathbf{R}P^l$ immerses in $\mathbf{R}^{2l-5}$ for $l \equiv 0 \pmod 4$ with $\alpha(l) \geq 2$. Hence, when $n \equiv 0 \pmod 4$ with $2 \leq \alpha(n) \leq 3$, W^n immerses in $\mathbf{R}^{2n-\alpha(n)-2}$. Sanderson [8] has also proved that $\mathbf{R}P^l$ immerses in $\mathbf{R}^{2l-8}$ for $l \equiv 3 \pmod 4$ with $\alpha(l) \geq 4$. Hence, when $n \equiv 2 \pmod 4$ with $\alpha(n) = 3$, $W^n \subset \mathbf{R}P^{n+1}$ immerses in $\mathbf{R}^{2n-\alpha(n)-3}$. Further, Sanderson [8] has proved that $\mathbf{R}P^l$ immerses in $\mathbf{R}^{2l-3}$ for odd integer $l > 8$. We remark that $\mathbf{R}P^5$ (resp. $\mathbf{R} \times \mathbf{R}P^3$) immerses in $\mathbf{R}^7$ [4; Theorem 7.1] (resp. $\mathbf{R}^4$). Hence, for even integer n with odd $\alpha(n) = k \geq 5$, $W^n \subset K^{n+1}$ immerses in $\mathbf{R}^f$ for $f = \sum_{j=1}^{m-1}(2b_j - 5) + 2b_m - 3 = 2n - 5m + 4 = 2n -$

$5(k+1)/2+4 \leq 2n-\alpha(n)-6$. For even integer n with even $\alpha(n)=k \geq 4$, $W^n \subset K^{n+1}$ immerses in $\mathbf{R}^f$ for $f=2b_1-1+\sum_{j=2}^{m-1}(2b_j-5)+2b_m-3=2n-5m+8=2n-5(k+2)/2+8 \leq 2n-\alpha(n)-3$, as required. (2) is similar. (3) By the result of Sanderson [8], we have the immersion of W^n in $\mathbf{R}^{2r_1} \times \mathbf{R}^{2r_2}=\mathbf{R}^{2n-6}=\mathbf{R}^{2n-\alpha(n)-2}$, as required. $\square$

Since $w(\mathbf{R}P^n)=(1+c)^{n+1}$ where $c \in H^1(\mathbf{R}P^n)$ which satisfies $c^{n+1}=0$, for a partition $\omega=(\underbrace{a_1,\ldots,a_1}_{n_1},\ldots,\underbrace{a_k,\ldots,a_k}_{n_k})$ with $|\omega|=n$, we have $s_\omega(\mathbf{R}P^n)=\{n_1,\ldots,n_k,n'\}c^n$, where $\{s_1,\ldots,s_l\}$ denotes the multinomial coefficient $(s_1+\cdots+s_l)!/((s_1!)\cdots(s_l!))$, and $n'=n+1-\sum_{j=1}^k n_j$.

PROPOSITION 6.8. *Let n be even.*

(1) *If $\alpha(n)=2$, then $(n) \in Rd_1$. If $n \equiv 0 \pmod 4$ furthermore, then $(n) \in Rd_2$.*

(2) *If $\alpha(n)=3$, then $(n) \in Rd_2$.*

PROOF. (1) We notice that $s_{(n)}(W^n)=s_{(n)}(\mathbf{R}P^n)=\{1,n\}c^n \neq 0$, and W^n immerses with α-efficiency 2 by Lemma 6.7. When $|\omega|=n$, $\alpha(\omega)=2$ and $l(\omega) \geq 2$, ω is a form $\omega=(r_1,r_2)$ $(r_1>r_2)$ where each r_j is a power of 2. Then, by the above calculation, $s_\omega[W^n]=0$. When $|\omega|=n$ and $\alpha(\omega) \geq 3$, we have $\omega \in Rd_1$ by Lemma 6.4. Hence, by Lemma 6.5, $(n) \in Rd_1$. When $|\omega|=n$, $\alpha(\omega)=3$ and $\omega \notin Rd_2$, since $(n') \in Rd_1$ for even integer n' with $\alpha(n')=2$ and by Lemma 6.4, ω is a form $\omega=(r_1,r_2,2r_2)$ $(r_1 \neq r_2)$ where each r_j is a power of 2. Thus, by the above calculation, if $n \equiv 0 \pmod 4$ then $s_\omega[W^n]=0$. Further, when $|\omega|=n$ and $\alpha(\omega) \geq 4$, we have $\omega \in Rd_2$ by Lemma 6.4. Hence, if $n \equiv 0 \pmod 4$, then we have $(n) \in Rd_2$ by Lemma 6.5, as required. The proof of (2) is similar, and we omit it. $\square$

Let $\alpha(n) \geq 4$ and ν be the normal line bundle of W^n in K^{n+1}. Then, $w(\nu)=\iota^*(1+\mu)$. Since $\iota^{-1}\tau(K^{n+1})=\tau(W^n) \oplus \nu$, we have $w(W^n)\iota^*(1+\mu)=\iota^*w(K^{n+1})$ and $w(W^n)=\iota^*((1+\mu)^{-1}w(K^{n+1}))$. Here, $w(K^{n+1})=\prod_{j=1}^m(1+c_j)^{b_j+1}$ with $c_j^{b_j+1}=0$, and if $r>n+1$ is a power of 2, then $(1+\mu)^{r-1}=(1+\mu^r)(1+\mu)^{-1}=(1+\mu)^{-1}$. We set $\tilde{w}=1+\tilde{w}_1+\cdots+\tilde{w}_{n+1}=(1+\mu)^{r-1}w(K^{n+1}) \in H^*(K^{n+1})$ where $\tilde{w}_j \in H^j(K^{n+1})$, and $\tilde{s}_\omega(K^{n+1})=P_\omega(\tilde{w}_1,\ldots,\tilde{w}_{|\omega|}) \in H^{|\omega|}(K^{n+1})$. Then, for a partition ω with $|\omega|=n$, we have $\langle s_\omega(W^n),(W^n)\rangle=\langle \iota^*\tilde{s}_\omega(K^{n+1}),(W^n)\rangle=\langle \tilde{s}_\omega(K^{n+1}),\ \iota_*(W^n)\rangle=\langle \tilde{s}_\omega(K^{n+1}),\ \mu \cap (K^{n+1})\rangle=\langle \mu\tilde{s}_\omega(K^{n+1}),(K^{n+1})\rangle$. Hence, for $|\omega|=n$, $s_\omega[W^n]=0$ if and only if $\mu\tilde{s}_\omega[K^{n+1}]=0$.

LEMMA 6.9. *Let n be even with $\alpha(n) \geq 4$. Then, $s_{(n)}[W^n] \neq 0$.*

PROOF. Since $\mu\tilde{s}_{(n)}(K^{n+1})=\mu^{n+1}=(c_1+\cdots+c_m)^{n+1}=\{b_1,\ldots,b_m\} \cdot c_1^{b_1}\cdots c_m^{b_m} \neq 0$, we have $s_{(n)}[W^n] \neq 0$, as required. $\square$

LEMMA 6.10. *Let n be even with $\alpha(n) \geq 4$, and $m = (\alpha(n)+1)/2$ or $(\alpha(n)+2)/2$ according as $\alpha(n)$ is odd or even. If ω contains more than m numbers each of which appears odd times in ω, then $\tilde{s}_\omega[K^{n+1}] = 0$, and hence $s_\omega[W^n] = 0$.*

PROOF. We remark that $\tilde{w} = (1+\mu)^{r-1}\prod_{j=1}^{m}(1+c_j)^{b_j+1}$ and b_m+1 is even. By the assumption of ω, each monomial of $\tilde{s}_\omega(K^{n+1})$ contains $(r-1)(r-2)$ or $(b_j+1)b_j$ $(1 \leq j \leq m-1)$ as a factor of its coefficient. Since $(r-1)(r-2) \equiv (b_j+1)b_j \equiv 0 \pmod 2$, we have $\tilde{s}_\omega[K^{n+1}] = 0$ as required.

PROPOSITION 6.11. *If n is even with $\alpha(n) = 4$, then $(n) \in Rd_1$.*

PROOF. By Lemma 6.9, we have $s_{(n)}[W^n] \neq 0$. When ω satisfies $|\omega| = n$, $\alpha(\omega) = 4$, $l(\omega) > 2$ and $\omega \notin Rd_1$, ω is a form $\omega = (r_1, \dots, r_4)$ where $r_i \neq r_j$ for $i \neq j$ and each r_j is a power of 2 by Proposition 6.8 and Lemma 6.4, and hence we have $s_\omega[W^n] = 0$ by Lemma 6.10. When $|\omega| = n$ and $\alpha(\omega) \geq 5$, $\omega \in Rd_1$ by Lemma 6.4, and thus, we have the required result by Lemma 6.5. □

Let n be even with $\alpha(n) \geq 4$. For $\omega = (a_1, \dots, a_l)$ with $a_i \neq a_j$ $(i \neq j)$, $|\omega| = n$ and $l \leq m$, we have $\mu\tilde{s}_\omega(K^{n+1}) = \sum \mu^{a'_1+1}c_1^{a'_2}\cdots c_{m-1}^{a'_m} = \sum\{b_1 - a'_2, \dots, b_{m-1} - a'_m, b_m\}c_1^{b_1}\cdots c_m^{b_m}$. Here, the summation is taken for all series $\{a'_1, \dots, a'_m\}$ in which $m-l$ elements are zero and the rest l elements are $a_1, \dots, a_l$. We remark that, in the case of $\alpha(\omega) = \alpha(n)$, $\{b_1 - a'_2, \dots, b_{m-1} - a'_m, b_m\} \equiv 1 \pmod 2$ if and only if $\{b_j - a'_{j+1}, a'_{j+1}\} \equiv 1 \pmod 2$ for any j with $1 \leq j \leq m-1$.

LEMMA 6.12. *Let n be even with $\alpha(n) \geq 4$, and $\omega = (a_1, \dots, a_l)$ with $a_i \neq a_j$ for $i \neq j$ and $|\omega| = n$. If ω contains an odd number, then $\tilde{s}_\omega[K^{n+1}] = 0$, and hence $s_\omega[W^n] = 0$.*

PROOF. We remark that b_m is odd, and there exists j $(1 \leq j \leq m-1)$ such that $b_j - a'_{j+1}$ is odd by the assumption. Hence, each $\{b_1 - a'_2, \dots, b_{m-1} - a'_m, b_m\} \equiv 0 \pmod 2$, and so, we have $\tilde{s}_\omega[K^{n+1}] = 0$ as required. □

PROPOSITION 6.13. *If n is even with $\alpha(n) = 4$ and $n \equiv 2 \pmod 4$, then $(n) \in Rd_2$.*

PROOF. Since $\alpha(n) = 4$, we have only to show the case that $\alpha(\omega) = 4$ or 5, by Lemma 6.4. Let $n = \sum_{j=1}^{3} 2r_j + 2$ be a dyadic expansion of n for $r_3 \geq 2$, and ω satisfy $|\omega| = n$ and $\omega \notin Rd_2$. We put $Y_1^n = W^{2r_1} \times W^{2r_2} \times W^{2r_3+2}$ and $Y_2^n = W^{2r_1} \times W^{2r_2+2} \times W^{2r_3}$.

(a) Applying Lemma 6.7 (1) to W^{2r_3+2} and W^{2r_2+2}, we see that Y_1^n and Y_2^n immerse with α-efficiency 2 by Lemma 2.2.

(b) When ω satisfies $\alpha(\omega) = 4$ and $l(\omega) \geq 2$, by Proposition 6.8 and Lemma 6.4, we have $\omega = (2r_1, 2r_2, 2r_3+2)$, $(2r_1, 2r_2+2, 2r_3)$, $(2r_1+2, 2r_2, 2r_3)$

or $(2r_1, 2r_2, 2r_3, 2)$. Here, if $\omega = (2r_1, 2r_2, 2r_3 + 2)$, by the remark above, then $\mu\tilde{s}_\omega(K^{n+1}) = \{0, 2r_3, 3\}c_1^{b_1}c_2^{b_2}c_3^{b_3} \neq 0$, where $b_1 = 2r_1$, $b_2 = 2r_2 + 2r_3$ and $b_3 = 3$. Similarly, if $\omega = (2r_1, 2r_2 + 2, 2r_3)$, then $\mu\tilde{s}_\omega[K^{n+1}] \neq 0$. If $\omega = (2r_1 + 2, 2r_2, 2r_3)$, then, by a similar proof of Lemma 6.12, $\mu\tilde{s}_\omega[K^{n+1}] = 0$. If $\omega = (2r_1, 2r_2, 2r_3, 2)$, then $\mu\tilde{s}_\omega[K^{n+1}] = 0$ by Lemma 6.10.

(c) When $\alpha(\omega) = 5$, similarly to the above and by Lemma 6.12, we have $\mu\tilde{s}_\omega[K^{n+1}] \neq 0$ if and only if $\omega = (2r_1, r_2, r_2, 2r_3, 2)$ or $(2r_1, 2r_2, r_3, r_3, 2)$.

(d) Let η satisfy $\eta \notin Rd_2$. By a similar proof of Proposition 6.8, when $|\eta| = 2r_j$ for $j = 1$ or 2, we see that $s_\eta[W^{2r_j}] \neq 0$ if and only if $\eta = (2r_j)$. Also, when $|\eta| = 2r_3 + 2$, $s_\eta[W^{2r_3+2}] \neq 0$ if and only if $\eta = (2r_3 + 2)$ or $(r_3, r_3, 2)$. Hence, $s_\omega[Y_1^n] \neq 0$ if and only if $\omega = (2r_1, 2r_2, 2r_3 + 2)$ or $(2r_1, 2r_2, r_3, r_3, 2)$. Similarly, $s_\omega[Y_2^n] \neq 0$ if and only if $\omega = (2r_1, 2r_2 + 2, 2r_3)$ or $(2r_1, r_2, r_2, 2r_3, 2)$.

By (a)–(d) and Lemma 6.9, $s_\omega[W^n \coprod Y_1^n \coprod Y_2^n] \neq 0$ if and only if $\omega = (n)$, and hence, $(n) \in Rd_2$ by Lemma 6.5, as required. □

Similar methods as in the proof of Proposition 6.13 show the following lemma.

Lemma 6.14. (1) *Let $n = \sum_{j=1}^{4} 2r_j$ be a dyadic expansion of n for $r_4 \geq 2$, and ω satisfy $|\omega| = n$ and $\omega \notin Rd_2$. Then,*

(i) *$s_\omega[W^n] \neq 0$ if and only if $\omega = (n)$, $(2r_1, r_2, r_2, 2r_3, 2r_4)$ or $(2r_1, 2r_2, r_3, r_3, 2r_4)$,*

(ii) *$s_\omega[\tilde{W}^n] \neq 0$ if and only if $\omega = (n)$, $(r_1, r_1, 2r_2, 2r_3, 2r_4)$ or $(2r_1, r_2, r_2, 2r_3, 2r_4)$.*

(2) *Let $n = 2r_1 + 2r_2 + 3$ $(r_1 > r_2 \geq 2)$ where each r_j is a power of 2, and ω satisfy $|\omega| = n$ and $\omega \notin Rd_2$. Then, $s_\omega[W^n] \neq 0$ if and only if $\omega = (2r_1 + 2, 2r_2 + 1)$, $(2r_1 + 1, 2r_2 + 2)$, $(2r_1, 2r_2 + 1, 2)$, $(2r_1 + 1, 2r_2, 2)$, $(r_2, 2r_1 + r_2 + 1, 2)$ or $(r_1, r_1 + 2r_2 + 1, 2)$.*

Proposition 6.15. *If n is even with $\alpha(n) \geq 5$, then $(n) \in Rd_2$.*

Proof. We only show the case $\alpha(n) = 5$, since the cases $\alpha(n) \geq 6$ are similarly proved. Let $n = \sum_{j=1}^{5} 2r_j$ be a dyadic expansion of n for $r_5 \geq 1$, and ω satisfy $|\omega| = n$ and $\omega \notin Rd_2$. By a similar proof of Proposition 6.13, when $n \equiv 2 (\mathrm{mod}\ 4)$, we have that $s_\omega[W^n] \neq 0$ if and only if $\omega = (n)$, and hence, $(n) \in Rd_2$ by Lemmas 6.5 and 6.7.

When $n \equiv 0 (\mathrm{mod}\ 4)$, similarly to the proof of Proposition 6.13, we have $s_\omega[W^n] \neq 0$ if and only if $\omega = (n)$, $(2r_1, n - 2r_1)$, $(2r_2, n - 2r_2)$, $(2r_3, n - 2r_3)$ or $(2r_4, n - 2r_4)$. By Lemma 6.14(1), we have the following equivalences: $s_\omega[W^{2r_1} \times W^{n-2r_1}] \neq 0$ if and only if $\omega = (2r_1, n - 2r_1)$, $(2r_1, 2r_2, r_3, r_3, 2r_4, 2r_5)$ or $(2r_1, 2r_2, 2r_3, r_4, r_4, 2r_5)$; $s_\omega[W^{2r_2} \times W^{n-2r_2}] \neq 0$ if and only if $\omega = (2r_2, n - 2r_2)$, $(2r_1, 2r_2, r_3, r_3, 2r_4, 2r_5)$ or $(2r_1, 2r_2, 2r_3, r_4, r_4, 2r_5)$; $s_\omega[W^{2r_3} \times \tilde{W}^{n-2r_3}] \neq 0$ if and only if $\omega = (2r_3, n - 2r_3)$, $(r_1, r_1, 2r_2, 2r_3, 2r_4, 2r_5)$ or

$(2r_1, 2r_2, r_2, 2r_3, 2r_4, 2r_5)$; $s_\omega[W^{2r_4} \times \tilde{W}^{n-2r_4}] \neq 0$ if and only if $\omega = (2r_4, n-2r_4)$, $(r_1, r_1, 2r_2, 2r_3, 2r_4, 2r_5)$ or $(2r_1, r_2, r_2, 2r_3, 2r_4, 2r_5)$. Thus, $s_\omega[W^n \coprod (W^{2r_1} \times W^{n-2r_1}) \coprod (W^{2r_2} \times W^{n-2r_2}) \coprod (W^{2r_3} \times \tilde{W}^{n-2r_3}) \coprod (W^{2r_4} \times \tilde{W}^{n-2r_4})] \neq 0$ if and only if $\omega = (n)$, and hence, $(n) \in Rd_2$ by Lemmas 6.5, 6.7 and 2.2. Thus, we have completed the proof. □

Propositions 6.8, 6.11, 6.13 and 6.15 establish Proposition 6.6.

7. Proof of Proposition 2.3

In this section, we prove Proposition 2.3(2) (iii) and (3), which together with Proposition 5.1 establish Proposition 2.3. First, assume that $n \equiv 2 \pmod 4$ and $\alpha(n)$ is even with $\alpha(n) \geq 4$, and set $m = (n-2)/2$. Then, $X^n = Q(1, m)$, and by (4.4), $w(X^n) = (1 + c + x)(1 + c + d)^{m+1}$ where c, $x \in H^1(X^n)$ and $d \in H^2(X^n)$ which are bound by the relations $x^2 = 0$, $c^2 = cx$ and $d^{m+1} = 0$.

Lemma 7.1. *Let $n \equiv 2 \pmod 4$ and $\alpha(n)$ be even with $\alpha(n) \geq 4$. If ω satisfies one of the following* (i) *and* (ii), *then $s_\omega[X^n] = 0$:*

(i) $\alpha(\omega) = \alpha(n)$ *and* $l(\omega) \geq 3$;

(ii) $\alpha(\omega) = \alpha(n) + 1$ *and* $l(\omega) \geq 4$.

Proof. According to the splitting principle as usual, we may assume that $1 + c + d = (1 + u)(1 + v)$, and thus $u + v = c$ and $uv = d$. Then, $w(X^n) = (1 + c + x)(1 + u)^{m+1}(1 + v)^{m+1}$. Since $x^2 = 0$, we have $(c + x)^{2a} = c^{2a}$ and $(c + x)^{2a+1} = c^{2a}(c + x)$ for any positive integer a. Moreover, since $c^3 = cx^2 = 0$, we have $(c + x)^j = 0$ for any $j \geq 4$. We shall only show the case (i) and omit the case (ii), since (ii) follows by a similar methods, and thus, assume $\alpha(\omega) = \alpha(n)$ and $l(\omega) \geq 3$.

Let $\omega = (a_1, \ldots, a_l)$. When $a_j \geq 4$ for any j, each monomial of $s_\omega(X^n)$ contains $(c + x)^4$, and hence, $s_\omega[X^n] = 0$. Also, when $l \geq 4$, each monomial of $s_\omega(X^n)$ contains $(m + 1)m \equiv 0 \pmod 2$ as a factor of its coefficient, and hence, $s_\omega[X^n] = 0$. Thus, to complete the proof of the case (i), we may assume that $\omega = (2, a, b)$ where $a > b \geq 4$. Then, $s_\omega(X^n) = (c + x)^2(u^a v^b + u^b v^a) = (c + x)^2 u^b v^b (u^{a-b} + v^{a-b}) = c^2 u^b v^b \sum_{i+2j=a-b} \{i - 1, j\}(u + v)^i (uv)^j = \sum_{i+2j=a-b} \{i - 1, j\} c^{i+2} d^{b+j} = 0$, since $c^3 = 0$, and thus, we obtain the required result. □

Proof of Proposition 2.3(2)(iii). Assume $n \equiv 2 \pmod 4$ and $\alpha(n)$ is even with $\alpha(n) \geq 4$. If ω satisfies $\alpha(\omega) \geq \alpha(n) + 2$, then $\omega \in Rd_2$ by Lemma 6.4. If ω satisfies $\alpha(\omega) = \alpha(n)$ with $l(\omega) \leq 2$, or $\alpha(\omega) = \alpha(n) + 1$ with $l(\omega) \leq 3$, then ω contains a number which satisfies one of the conditions (i)–(iv) of Proposition 6.6, and thus, $\omega \in Rd_2$ by Lemma 6.4. If ω satisfies $\alpha(\omega) = \alpha(n)$

with $l(\omega) \geq 3$, or $\alpha(\omega) = \alpha(n) + 1$ with $l(\omega) \geq 4$, then $s_\omega[X^n] = 0$ by Lemma 7.1. Hence, by Lemma 6.2, X^n immerses with α-efficiency 2, as required. $\square$

Dold [2; Satz 1, 2] has shown that the total Stiefel-Whitney class of $P(m,n)$ is

$$w(P(m,n)) = (1+c)^m(1+c+d)^{n+1},$$

where $c \in H^1(P(m,n))$ and $d \in H^2(P(m,n))$ which are bound by the relations $c^{m+1} = 0$ and $d^{n+1} = 0$. By making use of this fact and a similar proof as in Lemma 7.1, we have the following:

Lemma 7.2. (1) *Let $n = 2r + 2$ where r is a power of 2 with $r \geq 2$, and ω be a partition with $|\omega| = n$ and $\omega \notin Rd_2$. Then, $s_\omega[X^n] \neq 0$ if and only if $\omega = (n)$, $(2r, 2)$ or $(r, r, 2)$.*

(2) *Let $n = 2r + 1$ where r is a power of 2 with $r \geq 2$,* and *ω be a partition with $|\omega| = n$ and $\omega \notin Rd_2$. Then,*

(i) *when $r > 2$, $s_\omega[X^n] \neq 0$ if and only if $\omega = (n)$ or $(r+1, r)$.*

(ii) *when $r = 2$, $s_\omega[X^n] \neq 0$ if and only if $\omega = (n)$.*

(3) *Let $n = r_1 + r_2 + 1$ $(r_1 > r_2 \geq 2)$ with $n \neq 7$, where each r_j is a power of 2, and ω be a partition with $|\omega| = n$ and $\omega \notin Rd_1$. Then,*

(i) *when $r_2 > 2$, $s_\omega[X^n] \neq 0$ if and only if $\omega = (n)$, (r_1+1, r_2) or (r_1, r_2+1).*

(ii) *when $r_2 = 2$, $s_\omega[X^n] \neq 0$ if and only if $\omega = (n)$ or (r_1+1, r_2).*

Proof of Proposition 2.3(3). Assume that $n_1 = 2r_1 + 2$ and $n_2 = 2r_2 + 2$ for $r_1 > r_2 \geq 2$, where each r_j is a power of 2, and set $n = 2r_1 + 2r_2 + 3$. In the following, we also assume that ω satisfies $|\omega| = n$ and $\omega \notin Rd_2$.

(a) By Lemmas 6.4 and 7.2(1),(2), when $r_2 > 2$, $s_\omega[X^{n_1} \times X^{n_2-1}] \neq 0$ if and only if $w = (2r_1+2, 2r_2+1)$, $(2r_1, 2r_2+1, 2)$, $(r_1, r_1, 2r_2+1, 2)$ or $(2r_1, r_2+1, r_2, 2)$. When $r_2 = 2$, $s_\omega[X^{n_1} \times X^{n_2-1}] \neq 0$ if and only if $\omega = (2r_1+2, 2r_2+1)$, $(2r_1, 2r_2+1, 2)$ or $(r_1, r_1, 2r_2+1, 2)$.

(b) Similarly to (a), $s_\omega[X^{n_1-1} \times X^{n_2}] \neq 0$ if and only if $\omega = (2r_1+1, 2r_2+2)$, $(2r_1+1, 2r_2, 2)$, $(2r_1+1, r_2, r_2, 2)$ or $(r_1+1, r_1, 2r_2, 2)$.

(c) By Lemma 6.14(2), $s_\omega[W^n] \neq 0$ if and only if $\omega = (2r_1+2, 2r_2+1)$, $(2r_1+1, 2r_2+2)$, $(2r_1, 2r_2+1, 2)$, $(2r_1+1, 2r_2, 2)$, $(r_2, 2r_1+r_2+1, 2)$ or $(r_1, r_1+2r_2+1, 2)$. Here, we notice that, when $r_1 = 2r_2$, $(2r_1+1, 2r_2, 2) = (r_1, r_1+2r_2+1, 2)$.

(d) By Lemmas 6.4 and 7.2(3), when $r_2 > 2$, $s_\omega[X^{r_2} \times X^{2r_1+r_2+1} \times X^2] \neq 0$ if and only if $\omega = (r_2, 2r_1+r_2+1, 2)$, $(2r_1+1, r_2, r_2, 2)$ or $(2r_1, r_2+1, r_2, 2)$. When $r_2 = 2$, $s_\omega[X^{r_2} \times X^{2r_1+r_2+1} \times X^2] \neq 0$ if and only if $\omega = (r_2, 2r_1+r_2+1, 2)$ or $(2r_1+1, r_2, r_2, 2)$.

(e) Similarly to (d), $s_\omega[X^{r_1} \times X^{r_1+2r_2+1} \times X^2] \neq 0$ if and only if $\omega = (r_1, r_1+2r_2+1, 2)$, $(r_1, r_1, 2r_2+1, 2)$ or $(r_1+1, r_1, 2r_2, 2)$.

By (a)–(e), when $r_1 > 2r_2$, $s_\omega[(X^{n_1} \times X^{n_2-1}) \coprod (X^{n_1-1} \times X^{n_2}) \coprod W^n \coprod (X^{r_2} \times X^{2r_1+r_2+1} \times X^2) \coprod (X^{r_1} \times X^{r_1+2r_2+1} \times X^2)] = 0$ for any ω, and when $r_1 = 2r_2$, $s_\omega[(X^{n_1} \times X^{n_2-1}) \coprod (X^{n_1-1} \times X^{n_2}) \coprod W^n \coprod (X^{r_2} \times X^{2r_1+r_2+1} \times X^2)] = 0$ for any ω. Hence, by Lemmas 2.2 and 6.7(3), Proposition 2.3(1) and Corollary 6.3, $(X^{n_1} \times X^{n_2-1}) \coprod (X^{n_1-1} \times X^{n_2})$ immerses with α-efficiency 2, as required. □

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