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On localized weak precompactness in Banach spaces II

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ABSTRACT. This is a supplement and continuation of our previous paper [9], in which we have made a study of K-weakly precompact sets in Banach spaces. For a bounded subset A of dual Banach spaces, we introduce notions of A-separated δ-trees, A-midpoint-Bocce-dentability, A-strong regularity and A-weak*-strong regularity. Making use of these notions and arguments analogous to that of [9], we give some more characterizations of K-weakly precompact sets.

1. Introduction

Throughout this paper, X denotes an arbitrary real Banach space, X^* and X^{**} its topological dual space and bidual space, respectively, and $B(X)$ (resp. $S(X)$) the closed unit ball (resp. sphere) of X. The triple (I, Λ, λ) refers to the Lebesgue measure space on I $(= [0, 1])$, Λ^+ to the sets in Λ with positive measure, L_1 to $L_1(I, \Lambda, \lambda)$ and L_∞ to $L_\infty(I, \Lambda, \lambda)$. For each $B \in \Lambda^+$, denote $\Delta(B) = \{\chi_F/\lambda(F) : F \subset B, F \in \Lambda^+\}$. For each $g \in L_\infty$ and $B \in \Lambda^+$, ess-O $(g|B)$ denotes the essential oscillation of g (as a function) on B. We always understand that I is endowed with Λ and λ. If C is a subset of X^{**}, a function $f : I \to X^*$ is said to be C-measurable if the real-valued function $(x^{**}, f(t))$ is λ-measurable for each $x^{**} \in C$. If $C = X$, we say that f is weak*-measurable. If $f : I \to X^*$ is a bounded weak*-measurable function, we obtain a bounded linear operator $T_f : X \to L_1$ given by $T_f(x) = x \circ f$ for every $x \in X$, where $(x \circ f)(t) = (x, f(t))$ for every $t \in I$. The dual operator of T_f is denoted by T_f^* $(: L_\infty \to X^*)$. Furthermore, if we define a vector measure $\alpha_f : \Lambda \to X^*$ by $\alpha_f(B) = T_f^*(\chi_B)$ for every $B \in \Lambda$, we then have that

$$(x, \alpha_f(B)) = \int_B (x, f(t))\, d\lambda(t)$$

for every $x \in X$ and every $B \in \Lambda$. Let $\{I(n, i) : n = 0, 1, \ldots; i = 0, \ldots, 2^n - 1\}$ be a system of intervals in I given by $I(n, i) = [i/2^n, (i + 1)/2^n)$ if $n \geqq 1$, $0 \leqq i \leqq 2^n - 2$ and $I(n, 2^n - 1) = [(2^n - 1)/2^n, 1]$ if $n \geqq 0$. If Λ_n denotes the

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σ-algebra generated by $\Pi_n = \{I(n-1,i) : i = 0,\dots,2^{n-1}-1\}$ $(n \geqq 1)$, define $f_n : I \to X^*$ by

$$f_n(t) = \sum_{B \in \Pi_n} (T_f^*(\chi_B)/\lambda(B))\chi_B(t)$$

$$\left(= 2^{n-1} \cdot \sum_{i=0}^{2^{n-1}-1} \alpha_f(I(n-1,i))\,\chi_{I(n-1,i)}(t) \right)$$

for every $t \in I$. Then we have an X^*-valued martingale $(f_n, \mathcal{A}_n)_{n \geq 1}$.

In the following, all notations and terminology, unless otherwise stated, are as in [8], [9] and [10].

Now let us define the notion of localized weak precompactness in Banach spaces as follows (cf. [7] and [2]).

DEFINITION 1. Let A be a bounded subset of X and K a weak*-compact subset of X^*. Then we say that A is *K-weakly precompact* (or, A is *weakly precompact with respect to K*) if every sequence $\{x_n\}_{n \geq 1}$ in A has a pointwise convergent subsequence $\{x_{n(k)}\}_{k \geq 1}$ on K.

Note that if $K = B(X^*)$, A is simply said to be weakly precompact, which has been considered in [12] to characterize Banach spaces not containing a copy of l_1. We also know from the characterization of Pettis sets (cf. [11]) that for each weak*-compact subset K of X^*, K is a Pettis set if and only if $B(X)$ is K-weakly precompact. In what follows, we always understand that for every weak*-compact subset K of X^*, K is endowed with the weak*-topology $\sigma(X^*, X)$.

In [2], Bator and Lewis have made a systematic study of K-weakly precompact sets A in Banach spaces and they have obtained various characterizations of such sets which are analogous to the results of Fitzpatrick [3], Saab [13] and Saab and Saab [14]. Succeedingly, in [9] and [10], we have made some attempts to analyze K-weakly precompact sets A by means of the lifting theory, weak*-$\bar{A}^*$-dentability and a K-valued weak*-measurable function constructed in the case where A is not K-weakly precompact and we have obtained various characterizations, which we sum up in the following Theorem A. We note that these characterizations can be regarded as generalizations of corresponding ones of Pettis sets, weak Radon-Nikodym sets and weakly precompact sets.

THEOREM A. *Let A be a bounded subset of X and K a weak*-compact subset of X^*. Then the following statements about A and K are equivalent.*

(1) *The set A is K-weakly precompact.*

(2) *The set $\overline{\mathrm{co}}^*(K)$ has the $\bar{A}^*$-PCP.*

(3) *For every weak*-measurable function* $f : I \to K$ *and every* $B \in \Lambda^+$, *the set* $\overline{\mathrm{co}}^*(T_f^*(\Delta(B)))$ *is weak*-*$\bar{A}^*$*-dentable.*

(4) *Every weak*-measurable function* $f : I \to K$ *is* $\bar{A}^*$*-Pettis decomposable.*

(5) *Every weak*-measurable function* $f : I \to K$ *is weak*-equivalent to a* $\bar{A}^*$*-measurable function* $g : I \to \overline{\mathrm{co}}^*(K)$.

(6) *For every weak*-measurable function* $f : I \to K$, *the set* $\{x \circ f_n : x \in A, n \geqq 1\}$ *has the Bourgain property.*

(7) *For every weak*-measurable function* $f : I \to K$, *it holds that*

$$\inf_{n \geqq 1} \left\{ \sup_{x \in A} \|x \circ f_n - x \circ f_{n+1}\|_1 \right\} = 0.$$

(8) *For every weak*-measurable function* $f : I \to K$, *the set* $T_f(A)$ *is relatively norm compact.*

(9) *For every weak*-measurable function* $f : I \to K$, *it holds that*

$$\inf_{n \geqq 1} \left\{ \sup_{x \in A} |\langle x, T_f^*(r_n)\rangle| \right\} = 0$$

(*Here* r_n *denotes the nth Rademacher function on* I).

(10) *For every weak*-measurable function* $f : I \to K$, *the set* $T_f^*(\Delta(I))$ *contains no* A-δ-*Rademacher tree.*

It is remarkable that when A is not K-weakly precompact the construction of a K-valued weak*-measurable function in [9] plays a very important role in the establishment of these characterizations in Theorem A, especially, implications such as (7) $\Rightarrow$ (1), (8) $\Rightarrow$ (1), (9) $\Rightarrow$ (1) and (10) $\Rightarrow$ (1).

In this paper as well, by following the same ideas of the best use of such K-valued weak*-measurable functions, various properties of K-weakly precompact sets can be analyzed in terms of some new notions, and some other characterizations of such sets are presented. This is the aim of our paper. Of course, these characterizations also are generalizations of corresponding ones of Pettis sets, weak Radon-Nikodym sets and weakly precompact sets.

In §2, we restate the construction of K-valued weak*-measurable functions, which plays the central role in this paper, for the sake of necessity and importance. In §3, the notion of A-separated δ-trees is introduced and the equivalence among the statements (1), (7), (8), (9) and (10) in Theorem A is reconsidered by the effective use of the function constructed in §2 and the notion of A-separated δ-trees. In §4, the notion of A-midpoint-Bocce-dentability is introduced and a relation between A-separated δ-trees and A-midpoint-Bocce-dentability is clarified. In consequence, characterizations of K-weakly precompact sets in terms of A-midpoint-Bocce-dentability are given. In §5, notions of A-strong regularity and A-weak*-strong regularity are introduced. Some characterizations of K-weakly precompact sets in terms of these notions are given. These characterizations may be regarded as generalizations of corresponding parts of Theorem VI.16 in [4]. Finally, we list various characterizations of K-weakly precompact sets established in [9], [10] and this paper as a convenient summary.

2. A brief on the construction of K-valued weak*-measurable functions

In order to proceed our argument concerning the construction of K-valued weak*-measurable functions, we first need the following:

DEFINITION 2 ([12]). A sequence $(A_n, B_n)_{n\geq 1}$ of pairs of subsets of some set is called *independent* provided $A_n \cap B_n$ is empty for every n and for every $\{\varepsilon_j\}_{1\leq j\leq k}$ with $\varepsilon_j = 1$ or -1, $\bigcap\{\varepsilon_j A_j : 1 \leqq j \leqq k\}$ is a nonempty set, where $\varepsilon_j A_j = A_j$ if $\varepsilon_j = 1$ and $\varepsilon_j A_j = B_j$ if $\varepsilon_j = -1$.

Then, as is seen in §3 of [9], we should note the following fact. Let K be a weak*-compact subset of X^*. Suppose that there exists an independent sequence $(A_n, B_n)_{n\geq 1}$ of pairs of closed subsets of K. Then, $\Gamma = \bigcap_{n\geq 1}(A_n \cup B_n)$ is a nonempty compact subset of K, since $(A_n, B_n)_{n\geq 1}$ is independent. Now, define $\phi : \Gamma \to \mathscr{P}(N)$ (Cantor space, with its usual compact metric topology) by $\phi(x^*) = \{j : A_j \ni x^*\} \in \mathscr{P}(N)$. Then ϕ is a continuous surjection and so we have a Radon probability measure γ on Γ such that $\phi(\gamma) = \nu$ (the normalized Haar measure if we identify $\mathscr{P}(N)$ with $\{0,1\}^N$) and $\{f \circ \phi : f \in L_1(\mathscr{P}(N), \Sigma_\nu, \nu)\} = L_1(\Gamma, \Sigma_\gamma, \gamma)$, where Σ_ν (resp. Σ_γ) is the family of all ν (resp. γ)-measurable subsets of $\mathscr{P}(N)$ (resp. Γ). Further, consider a function $\tau : \mathscr{P}(N) \to I$ defined by $\tau(J) = \sum_{j\in J} 1/2^j$ for every $J \in \mathscr{P}(N)$. Then τ is a continuous surjection such that $\tau(\nu) = \lambda$ and $\{u \circ \tau : u \in L_1\} = L_1(\mathscr{P}(N), \Sigma_\nu, \nu)$. Then, making use of the lifting theory, we have a weak*-measurable function $h : I \to \Gamma$ $(\subset K)$ such that

$$\int_E f(h(t))\, d\lambda(t) = \int_{\phi^{-1}(\tau^{-1}(E))} f(x^*)\, d\gamma(x^*)$$

for every $E \in \Lambda$ and every $f \in C(\Gamma)$. Further we should remark here that $\tau(\phi(\gamma)) = \lambda$, $\bigcup\{\phi^{-1}(\tau^{-1}(I(n, 2i))) : 0 \leqq i \leqq 2^{n-1} - 1\} \equiv \Gamma \cap B_n$, and $\bigcup\{\phi^{-1}(\tau^{-1}(I(n, 2i+1))) : 0 \leqq i \leqq 2^{n-1} - 1\} \equiv \Gamma \cap A_n$ (with respect to γ) for $n = 1, 2, \ldots$.

For convenient reference, we wish to indicate here two typical cases where the existence of such functions h can be assured. These two cases are very important for us later on.

PROPOSITION 1. *Let K be a weak*-compact subset of X^* and A a bounded subset of X. Suppose that A is not K-weakly precompact. Then there exist a sequence $\{x_n\}_{n\geq 1}$ in A and real numbers r and δ with $\delta > 0$ such that putting $A_n = \{x^* \in K : (x_n, x^*) \leqq r\}$ and $B_n = \{x^* \in K : (x_n, x^*) \geqq r + \delta\}$, then $(A_n, B_n)_{n\geq 1}$ is an independent sequence of pairs of closed subsets of K.*

PROOF. See §3 in [9].

In this case, we easily get that the weak*-measurable function $h : I \to K$ constructed as above satisfies the following:

$$\int_{I(n,2i)} (x_n, h(t))\, d\lambda(t) = \int_{\phi^{-1}(\tau^{-1}(I(n,2i)))} (x_n, x^*)\, d\gamma(x^*)$$
$$\geqq (r+\delta)/2^n,$$

and

$$\int_{I(n,2i+1)} (x_n, h(t))\, d\lambda(t) = \int_{\phi^{-1}(\tau^{-1}(I(n,2i+1)))} (x_n, x^*)\, d\gamma(x^*)$$
$$\leqq r/2^n$$

for $n = 1, 2, \ldots$ and $i = 0, \ldots, 2^{n-1} - 1$.

PROPOSITION 2. *Let K be a weak*-compact subset of X^*. Suppose that there exists a system $\{V(n,i) : n = 0, 1, \ldots; i = 0, \ldots, 2^n - 1\}$ of nonempty closed subsets of K such that $V(n+1, 2i) \cup V(n+1, 2i+1) \subset V(n,i)$ and $V(n+1, 2i) \cap V(n+1, 2i+1)$ is empty for $n = 0, 1, \ldots$ and $i = 0, \ldots, 2^n - 1$. Then there is an independent sequence $(A_n, B_n)_{n\geq 1}$ of pairs of closed subsets of K.*

PROOF. Let $A_n = \bigcup\{V(n, 2i+1) : 0 \leqq i \leqq 2^{n-1} - 1\}$ and $B_n = \bigcup\{V(n, 2i) : 0 \leqq i \leqq 2^{n-1} - 1\}$ for $n = 1, 2, \ldots$. Then it is easily seen that $(A_n, B_n)_{n\geq 1}$ is a desired sequence.

In this case as well, we easily know that the weak*-measurable function $g : I \to K$ constructed as above satisfies the following:

$$\int_{I(n,2i)} f(g(t))\, d\lambda(t) = \int_{\phi^{-1}(\tau^{-1}(I(n,2i)))} f(x^*)\, d\gamma(x^*)$$
$$= \int_{\Gamma \cap V(n,2i)} f(x^*)\, d\gamma(x^*),$$

and

$$\int_{I(n,2i+1)} f(g(t))\,d\lambda(t) = \int_{\phi^{-1}(\tau^{-1}(I(n,2i+1)))} f(x^*)\,d\gamma(x^*)$$
$$= \int_{\Gamma\cap V(n,2i+1)} f(x^*)\,d\gamma(x^*)$$

for $f \in C(\Gamma)$, $n = 1, 2, \ldots$ and $i = 0, \ldots, 2^{n-1} - 1$.

3. A-separated δ-trees

In this section, let us reconsider the equivalence among the statements (1), (7), (8), (9) and (10) in Theorem A by the effective use of a K-valued weak*-measurable function h guaranteed by Proposition 1 and the notion of A-separated δ-trees defined as follows. It is a simple generalization of the notion of separated δ-trees introduced in [5].

DEFINITION 3. A system $\{x(n,i) : n = 0, 1, \ldots; i = 0, \ldots, 2^n - 1\}$ in X is called a *tree* if $x(n,i) = \{x(n+1, 2i) + x(n+1, 2i+1)\}/2$ for $n = 0, 1, \ldots;$ $i = 0, \ldots, 2^n - 1$.

Let A be a nonempty bounded subset of X^*. A tree $\{x(n,i) : n = 0, 1, \ldots; i = 0, \ldots, 2^n - 1\}$ in X is called an *A-separated δ-tree* if there exists a sequence $\{x_n^*\}_{n\geq 1}$ in A such that for $n = 1, 2, \ldots$ and $i = 0, \ldots, 2^{n-1} - 1$

$$(x_n^*, x(n,2i) - x(n,2i+1)) > 2\delta.$$

In this case, we say that the tree is *separated* by $\{x_n^*\}_{n\geq 1}$.

If $A = B(X^*)$, then this tree is simply called a *separated δ-tree*.

Note that a tree in X^* is a separated δ-tree if and only if it is a $B(X)$-separated δ-tree, since $B(X)$ is dense in $B(X^{**})$ with respect to the weak* topology $\sigma(X^{**}, X^*)$.

Now, consider the following statement $(*)$ about a bounded subset A of X and a weak*-compact subset K of X^*, which may be regarded as the starting point for our argument in §2. Lemma 3 mentioned below concerns $(*)$.

$(*)$ For every weak*-measurable function $f : I \to K$, the tree $\{2^n\alpha_f(I(n,i)) : n = 0, 1, \ldots; i = 0, \ldots, 2^n - 1\}$ is not an A-separated δ-tree.

In the sequel, the tree $\{2^n\alpha_f(I(n,i)) : n = 0, 1, \ldots; i = 0, \ldots, 2^n - 1\}$ is called a tree associated with f.

The cornerstone for our argument in this section is the following Lemmas 1, 2 and 3. Lemma 1 is a very simple fact, which is suggested by Girardi [5] and can be proved by a straightforward calculation. It is useful for us. Lemma 2 is essentially the same as the implication (i) $\Rightarrow$ (ii) of Proposition 2 in

[1]. So its proof is omitted. Lemma 3 has been obtained implicitly in the proof of Theorem in [9]. But we here wish to state its proof in an explicit form.

LEMMA 1. *Let $f : I \to X^*$ be a bounded weak*-measurable function. Then it holds that for every $n \geqq 1$,*

$$\begin{aligned}
&\sup_{x \in A}\left|\left(x, \sum_{i=0}^{2^{n-1}-1}\{\alpha_f(I(n,2i)) - \alpha_f(I(n,2i+1))\}\right)\right| \\
&\quad = \sup_{x \in A}|(x, T_f^*(r_n))| \left(= \sup_{x \in A}|(T_f(x), r_n)|\right) \\
&\quad \leqq \sup_{x \in A}\|x \circ f_{n+1} - x \circ f_n\|_1.
\end{aligned}$$

PROOF. For every $x \in X$, we have that

$$\begin{aligned}
\|x \circ f_{n+1} - x \circ f_n\|_1 &= \int_I |(x, f_{n+1}(t) - f_n(t))|\, d\lambda(t) \\
&= \sum_{i=0}^{2^n-1} \int_{I(n,i)} |(x, f_{n+1}(t) - f_n(t))|\, d\lambda(t) \\
&= \sum_{i=0}^{2^{n-1}-1} \int_{I(n,2i)} |(x, f_{n+1}(t) - f_n(t))|\, d\lambda(t) \\
&\quad + \sum_{i=0}^{2^{n-1}-1} \int_{I(n,2i+1)} |(x, f_{n+1}(t) - f_n(t))|\, d\lambda(t) \\
&= \sum_{i=0}^{2^{n-1}-1} |(x, \alpha_f(I(n,2i)) - \alpha_f(I(n,2i+1)))|,
\end{aligned}$$

since $f_{n+1}(t) = 2^n \cdot \sum_{i=0}^{2^n-1} \alpha_f(I(n,i))\, \chi_{I(n,i)}(t)$ (cf. [5]).

On the other hand, for every $x \in X$, we have that

$$\begin{aligned}
&(x, T_f^*(r_n)) \;(= (T_f(x), r_n)) \\
&\quad = \int_I (x, f(t)) \left\{\sum_{i=0}^{2^n-1}(-1)^i \chi_{I(n,i)}(t)\right\} d\lambda(t) \\
&\quad = \sum_{i=0}^{2^n-1}(-1)^i (x, \alpha_f(I(n,i))) = \sum_{i=0}^{2^{n-1}-1} (x, \alpha_f(I(n,2i)) - \alpha_f(I(n,2i+1)))
\end{aligned}$$

(cf. [9]). Hence Lemma 1 follows.

LEMMA 2. *Let $f : I \to X^*$ be a bounded weak*-measurable function and A a bounded subset of X. If the set $T_f(A)$ is relatively norm compact, we have that*

$$\inf_{n \geqq 1}\left\{\sup_{x \in A} \|x \circ f_n - x \circ f_{n+1}\|_1\right\} = 0.$$

LEMMA 3. *Let A be a bounded subset of X and K a weak*-compact subset of X^*. If A is not K-weakly precompact, then there exists a weak*-measurable function $h : I \to K$ such that the tree associated with h is an A-separated η-tree for an appropriate positive number η.*

PROOF. Let h be the function assured by Proposition 1. In order to prove that the tree associated with h is an A-separated η-tree, take a sequence $\{x_n\}_{n \ge 1}$ in A and a positive number δ obtained in Proposition 1. Let η be a positive number with $\eta < \delta/2$. Then we have by the remark after Proposition 1 that

$$\begin{aligned}
&(x_n, 2^n\{\alpha_h(I(n,2i)) - \alpha_h(I(n,2i+1))\}) \\
&\quad = 2^n \cdot \left\{\int_{I(n,2i)} (x_n, h(t))\, d\lambda(t) - \int_{I(n,2i+1)} (x_n, h(t))\, d\lambda(t)\right\} \\
&\quad = 2^n \cdot \left\{\int_{\phi^{-1}(\tau^{-1}(I(n,2i)))} (x_n, x^*)\, d\gamma(x^*) - \int_{\phi^{-1}(\tau^{-1}(I(n,2i+1)))} (x_n, x^*)\, d\gamma(x^*)\right\} \\
&\quad \geqq (r+\delta) - r = \delta > 2\eta,
\end{aligned}$$

which implies that the tree $\{2^n\alpha_h(I(n,i)) : n = 0, 1, \ldots; i = 0, \ldots, 2^n - 1\}$ is an A-separated η-tree separated by $\{x_n\}_{n \ge 1}$. Hence the proof is completed.

Combining Lemmas 1, 2 and 3, we easily have:

PROPOSITION 3. *Let A be a bounded subset of X and K a weak*-compact subset of X^*. Then the following statements about A and K are equivalent.*

(1) *The set A is K-weakly precompact.*

(2) *For every weak*-measurable function $f : I \to K$, the set $T_f(A)$ is relatively norm compact.*

(3) *For every weak*-measurable function $f : I \to K$, it holds that*

$$\inf_{n \geqq 1}\left\{\sup_{x \in A} \|x \circ f_n - x \circ f_{n+1}\|_1\right\} = 0.$$

(4) *For every weak*-measurable function $f : I \to K$, it holds that*

$$\inf_{n \geqq 1}\left\{\sup_{x \in A} |(x, T_f^*(r_n))|\right\} = 0.$$

(5) *For every weak*-measurable function* $f : I \to K$, *the tree associated with* f *is not an* A-δ-*Rademacher tree.*

(6) *For every weak*-measurable function* $f : I \to K$, *the set* $T_f^*(\Delta(I))$ *contains no* A-δ-*Rademacher tree.*

(7) *For every weak*-measurable function* $f : I \to K$, *the tree associated with* f *is not an* A-*separated* δ-*tree.*

(8) *For every weak*-measurable function* $f : I \to K$, *the set* $T_f^*(\Delta(I))$ *contains no* A-*separated* δ-*tree.*

Proof. (1) $\Rightarrow$ (2). This follows immediately from the bounded convergence theorem.

(2) $\Rightarrow$ (3). This follows from Lemma 2.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5). This follows from Lemma 1.

(5) $\Rightarrow$ (6). This has already been shown in the proof of the part (viii) of Theorem in [9].

(6) $\Rightarrow$ (8). This follows from the fact that every A-separated δ-tree is an A-δ-Rademacher tree. Its proof is very easy and it has essentially been given in [5].

(8) $\Rightarrow$ (7). This follows from the fact that $T_f^*(\Delta(I)) \supset \{2^n \alpha_f(I(n,i)) : n = 0, 1, \ldots; i = 0, \ldots, 2^n - 1\}$.

(7) $\Rightarrow$ (1). This follows from Lemma 3. Hence the proof is completed.

These observations indicate that this notion of A-separated δ-trees is one of fundamental and important tools to analyze the property of K-weakly precompact sets A, when it is combined with K-valued weak*-measurable functions.

4. A-midpoint-Bocce-dentability

Our main result in this section clarifies the relation between A-separated δ-trees and bounded subsets without a geometric property (called the A-midpoint-Bocce-dentability). Applying this, we get characterizations of K-weakly precompact sets A in terms of A-midpoint-Bocce-dentability. Now let us define the notion of A-midpoint-Bocce-dentability of bounded sets in X as follows. This is a simple generalization of the notion of midpoint-Bocce-dentability of such sets introduced by Girardi [5].

Definition 4. Let D and A be nonempty bounded subsets of X and X^*, respectively. We say that the set D is *A-midpoint-Bocce-dentable* if for every positive number ε there exists a finite subset C of D that satisfies the following property $(**)$.

($**$) For every $x^* \in A$ there is an element x of C such that if $x = (z_1 + z_2)/2$ $(z_1, z_2 \in D)$, then $|(x^*, x - z_1)| = |(x^*, x - z_2)| < \varepsilon$.

If $A = S(X^*)$, then the set D is simply said to be *midpoint-Bocce-dentable.*

Then, concerning the relation between bounded A-separated δ-trees and bounded subsets without the A-midpoint-Bocce-dentability in X, we have the following Proposition 4. Its main part is the latter statement, and the point to be emphasized is how to make up a non-A-midpoint-Bocce-dentable bounded set from a given bounded A-separated δ-tree, which is significant. The former one has essentially been shown by Girardi [5]. Hence we omit its proof.

PROPOSITION 4. *Non A-midpoint-Bocce-dentable bounded set contains an A-separated δ-tree for an appropriate positive number δ. Conversely, every bounded A-separated δ-tree can be contained in a non-A-midpoint-Bocce-dentable bounded subset.*

PROOF. We give a proof of the latter statement. Let $\{x(n,i) : n = 0, 1, \dots; i = 0, \dots, 2^n - 1\}$ be a bounded A-separated δ-tree separated by $\{x_n^*\}_{n \geq 1}$ in A. For every $n \geqq 0$, define a subset D_n by

$$D_n = \left\{ \left(\sum_{j \in E} x(n,j) \right) \Big/ \|E\| : E \subset \{0, 1, \dots, 2^n - 1\}, E \neq \phi \right\}.$$

Here $\|E\|$ denotes the number of elements of E. Then the set $D = \bigcup_{n \geq 0} D_n$ is a non-A-midpoint-Bocce-dentable bounded subset containing this tree, which can be shown in the following. Since it is clear that the set D contains this tree, let us show the non-A-midpoint-Bocce-dentability of D. For this purpose, take any finite subset $C = \{z_1, \dots, z_p\}$ of D. Then there exists a finite set $\{n(1), \dots, n(p)\}$ of non-negative integers such that $z_i \in D_{n(i)}$ for every i with $1 \leqq i \leqq p$. Without loss of generality we may assume that $\max(n(1), \dots, n(p)) = n(p)$. In order to complete the proof, we have only to show that the element $x_{n(p)+1}^*$ in A has the following property (P_1).

(P_1) For every i with $1 \leqq i \leqq p$, there exist two elements u_i, v_i in D such that $z_i = (u_i + v_i)/2$ and $(x_{n(p)+1}^*, u_i - v_i) \geqq 2\delta$.

Indeed, it easily follows from (P_1) that the following property (P_2) holds, and so the set D is non-A-midpoint-Bocce dentable.

(P_2) For every i with $1 \leqq i \leqq p$, there exist two elements u_i, v_i in D such that $z_i = (u_i + v_i)/2$ and $|(x_{n(p)+1}^*, z_i - u_i)| \equiv |(x_{n(p)+1}^*, z_i - v_i)| \geqq \delta$.

Now let us show the property (P_1). Since $z_i \in D_{n(i)}$, there exists a set $E_i \subset \{0, 1, \dots, 2^{n(i)} - 1\}$ such that

$$z_i = \left(\sum_{j\in E_i} x(n(i),j)\right) \Big/ \|E_i\|.$$

Let $q(i) = n(p) + 1 - n(i)$. Then we have by the tree property that for every $j \in E_i$,

$$\begin{aligned} x(n(i),j) &= \{x(n(i)+1,2j) + x(n(i)+1,2j+1)\}/2 \\ &= \cdots\cdots\cdots \\ &= \{x(n(i)+q(i),2^{q(i)}\cdot j) + x(n(i)+q(i),2^{q(i)}\cdot j+1) + \cdots\cdots\cdots \\ &\quad + x(n(i)+q(i),2^{q(i)}\cdot(j+1)-2) + x(n(i)+q(i),2^{q(i)}\cdot(j+1)-1)\}/2^{q(i)} \\ &= (1/2^{q(i)})\cdot \sum\{x(n(p)+1,k) : 2^{q(i)}\cdot j \leqq k \leqq 2^{q(i)}\cdot(j+1)-1\}. \end{aligned}$$

Hence we have that

$$z_i = (1/2^{q(i)}\cdot\|E_i\|)\sum_{j\in E_i}\left(\sum\{x(n(p)+1,k) : 2^{q(i)}\cdot j \leqq k \leqq 2^{q(i)}\cdot(j+1)-1\}\right).$$

So, putting

$$\begin{aligned} u_i = (1/2^{q(i)-1}\cdot\|E_i\|)\sum_{j\in E_i}\Big(\sum\{x(n(p)+1,2k) :& \\ & 2^{q(i)-1}\cdot j \leqq k \leqq 2^{q(i)-1}\cdot(j+1)-1\}\Big) \end{aligned}$$

and

$$\begin{aligned} v_i = (1/2^{q(i)-1}\cdot\|E_i\|)\sum_{j\in E_i}\Big(\sum\{x(n(p)+1,2k+1) :& \\ & 2^{q(i)-1}\cdot j \leqq k \leqq 2^{q(i)-1}\cdot(j+1)-1\}\Big), \end{aligned}$$

we then have by a simple calculation that $z_i = (u_i + v_i)/2$, and further we have that u_i and v_i are in $D_{n(p)+1} (\subset D)$. Indeed, let

$$A_i = \bigcup_{j\in E_i}\{2k : 2^{q(i)-1}\cdot j \leqq k \leqq 2^{q(i)-1}\cdot(j+1)-1\}$$

and

$$B_i = \bigcup_{j\in E_i}\{2k+1 : 2^{q(i)-1}\cdot j \leqq k \leqq 2^{q(i)-1}\cdot(j+1)-1\}.$$

Then it holds that A_i, $B_i \subset \{0,1,\ldots,2^{n(p)+1}-1\}$ and $\|A_i\| = \|B_i\| = 2^{q(i)-1}\cdot\|E_i\|$. Hence we have that

$$u_i = \left(\sum_{k\in A_i} x(n(p)+1,k)\right) \Big/ \|A_i\| \quad \text{and} \quad v_i = \left(\sum_{k\in B_i} x(n(p)+1,k)\right) \Big/ \|B_i\|,$$

and so they are in $D_{n(p)+1}$. Finally, we have that for every i with $1 \leqq i \leqq p$

$$\begin{aligned}
&(x^*_{n(p)+1}, u_i - v_i) \\
&= (1/2^{q(i)-1}\cdot\|E_i\|)\sum_{j\in E_i}\Big(\sum\{(x^*_{n(p)+1}, x(n(p)+1,2k) - x(n(p)+1,2k+1)) : \\
&\qquad\qquad 2^{q(i)-1}\cdot j \leqq k \leqq 2^{q(i)-1}\cdot(j+1)-1\}\Big) \\
&> (1/2^{q(i)-1}\cdot\|E_i\|)\sum_{j\in E_i}\Big(\sum\{2\delta : 2^{q(i)-1}\cdot j \leqq k \leqq 2^{q(i)-1}\cdot(j+1)-1\}\Big) \\
&= 2\delta,
\end{aligned}$$

as desired. Hence the proof is completed.

This result combined with Lemma 3 and Proposition 3 yields characterizations of K-weakly precompact sets A in terms of A-midpoint-Bocce-dentability.

COROLLARY 1. *Let A be a bounded subset of X and K a weak*-compact subset of X^*. Then the following statements about A and K are equivalent.*

(1) *The set A is K-weakly precompact.*

(2) *For every weak*-measurable function $f : I \to K$, the set $D(f) = \bigcup_{n\geq 0} D_n(f)$ is A-midpoint-Bocce-dentable.*

(Here, for every $n \geqq 0$, $D_n(f) = \{T^*_f(\chi_B/\lambda(B)) : B \in \varLambda_{n+1}, \lambda(B) > 0\}$).

(3) *For every weak*-measurable function $f : I \to K$, the set $T^*_f(\varDelta(I))$ is A-midpoint-Bocce-dentable.*

PROOF. (1) $\Rightarrow$ (3). This follows from Propositions 3 and 4.

(3) $\Rightarrow$ (2). Since $T^*_f(\varDelta(I)) \supset D(f)$, this also follows from Propositions 3 and 4.

(2) $\Rightarrow$ (1). This part is crucial. Suppose that (1) fails, and let $h : I \to K$ be the weak*-measurable function mentioned in Lemma 3. Then let us show that the set $D(h)$ is non-A-midpoint-Bocce-dentable. To this end, let $\{x^*(n,i) : n = 0,1,\dots; i = 0,\dots,2^n-1\}$ be the A-separated η-tree obtained in Lemma 3 and take any set $B \in \varLambda_{n+1}$ with $\lambda(B) > 0$. Then there exists a nonempty set $E \subset \{0,1,\dots,2^n-1\}$ such that $B = \bigcup_{i\in E} I(n,i)$. Hence we have that

$$\begin{aligned}
T^*_h(\chi_B/\lambda(B)) &= \sum_{i\in E}(\lambda(I(n,i))/\lambda(B))\cdot T^*_h(\chi_{I(n,i)}/\lambda(I(n,i))) \\
&= \left(\sum_{i\in E} x^*(n,i)\right)\Big/\|E\|.
\end{aligned}$$

So, $D_n(h) = \left\{ \left(\sum_{i \in E} x^*(n,i) \right) \Big/ \|E\| : E \subset \{0, 1, \ldots, 2^n - 1\}, E \neq \phi \right\}$. Thus, in virtue of Proposition 4, the set $D(h)$ is non-A-midpoint-Bocce-dentable, whence (2) fails. This completes the proof.

Note that if $\{x_n\}_{n \geq 1}$ is a $B(X^*)$-separated δ-tree separated by $\{x_n^*\}_{n \geq 1}$, then it is an $S(X^*)$-separated δ-tree separated by $\{x_n^*/\|x_n^*\|\}_{n \geq 1}$. Consequently, setting $A = S(X^*)$ in Proposition 4, we have:

Corollary 2. *Non-midpoint-Bocce-dentable bounded set contains separated δ-tree for an appropriate positive number δ. Conversely, every bounded separated δ-tree can be contained in a non-midpoint-Bocce-dentable bounded set.*

5. A-strong regularity and A-weak*-strong regularity

In this section we give characterizations of K-weakly precompact sets in terms of certain kinds of strong regularity of bounded convex sets. For that, let us give the following Definition 6, which are simple generalizations of strong regularity and weak*-strong regularity of bounded convex subsets of X^* introduced in [4]. Before giving Definition 6, we need the

Definition 5. Let D be a bounded subset of X^*. An *open* (resp. *weak**-*open*) *slice* of D is a set of the form:

$$S(x^{**}, \alpha, D) = \left\{ x^* \in D : (x^{**}, x^*) > \sup_{y^* \in D} (x^{**}, y^*) - \alpha \right\}$$

$$\left(\text{resp.}\ S(x, \alpha, D) = \left\{ x^* \in D : (x, x^*) > \sup_{y^* \in D} (x, y^*) - \alpha \right\} \right)$$

where $x^{**} \in X^{**}$ (resp. $x \in X$) and $\alpha > 0$.

Definition 6. Let H be a bounded convex subsets of X^* and A a bounded subset of X^{**}. Then the set H is said to be *A-strongly regular* (resp. *A-weak*-strongly regular*) if for every nonempty convex subset D of H and any positive number ε, there exist positive numbers $\alpha_1, \ldots, \alpha_n$ whose sum is one and open (resp. weak*-open) slices $S_1, \ldots, S_n$ of D such that

$$\sup_{x^{**} \in A} O\left(x^{**} \,\middle|\, \sum_{i=1}^{n} \alpha_i S_i \right) \left(= \sup_{x^{**} \in A} \sum_{i=1}^{n} \alpha_i O(x^{**} | S_i) \right) < \varepsilon.$$

If $A = B(X)$, then this set is simply called *strongly regular* (resp. *weak*-strongly regular*).

In order to characterize K-weakly precompact sets A in terms of A-strong regularity and A-weak*-strong regularity, we need the following notion, which acts as intermediary in our consideration.

DEFINITION 7 ([4]). A subset M of L_∞ is said to be a *set of small oscillation with respect to* λ if for every positive number ε there exists a positive measurable partition $(=\{E_1, \dots, E_n\})$ of I such that

$$\sum_{i=1}^{n} \lambda(E_i)\, \text{ess-}O\,(f|E_i) < \varepsilon$$

for every $f \in M$.

Now we are able to prove an extension of some parts of Theorem VI.16 in [4]. This is the main result of our paper.

PROPOSITION 5. *Let A be a bounded subset of X and K a weak*-compact subset of X^*. Then the following statements about A and K are equivalent.*

(1) *The set $\overline{\text{co}}^*(K)$ is A-strongly regular.*

(2) *For every weak*-measurable function $f : I \to \overline{\text{co}}^*(K)$, the set $\{x \circ f : x \in A\}$ is a set of small oscillation with respect to λ.*

(3) *For every weak*-measurable function $f : I \to K$, the set $\{x \circ f : x \in A\}$ is a set of small oscillation with respect to λ.*

(4) *The set A is K-weakly precompact.*

(5) *For every weak*-measurable function $f : I \to K$, it holds that*

$$\inf_{n \geqq 1} \left\{ \sup_{x \in A} |(x, T_f^*(r_n))| \right\} = 0.$$

(6) *For every weak*-measurable function $f : I \to \overline{\text{co}}^*(K)$, it holds that*

$$\inf_{n \geqq 1} \left\{ \sup_{x \in A} |(x, T_f^*(r_n))| \right\} = 0.$$

(7) *The set $\overline{\text{co}}^*(K)$ is A-weak*-strongly regular.*

PROOF. We note first that the equivalence among the statements (4), (5) and (6) follows in virtue of Proposition 3, since the set A is K-weakly precompact if and only if A is $\overline{\text{co}}^*(K)$-weakly precompact. So, in order to complete the proof, we are going to show that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ and $(6) \Rightarrow (7) \Rightarrow (1)$.

(i) $(1) \Rightarrow (2)$. In order to prove this, we follow the ideas of Girardi and Uhl [6]. That is, we first have the following fact: Let K be a weak*-compact subset of X^* and $f : I \to \overline{\text{co}}^*(K)$ be a weak*-measurable function. Suppose that $\overline{\mathrm{co}}^*(K)$ is A-strongly regular. Then, for every positive number ε and $B \in \Lambda^+$, there exist positive numbers $\alpha_1, \dots, \alpha_n$ whose sum is one and subsets $B_1, \dots, B_n$ of B, all of positive λ-measure, such that

$$\sup_{x \in A} O\left(x \,\middle|\, \sum_{i=1}^{n} \alpha_i T_f^*(\Delta(B_i))\right) < \varepsilon.$$

This fact immediately follows from the A-strong regularity of $\overline{\mathrm{co}}^*(K)$ and Lemma 1 in §1 of [6]. Now let us show that (1) $\Rightarrow$ (2). Assume (1). Take a positive number ε. Then, in virtue of this fact, we can use the same argument as in [6] to get a positive measurable partition $\{E_1, \dots, E_n\}$ of I such that

$$\sup_{x \in A} O\left(x \,\middle|\, \sum_{i=1}^{n} \lambda(E_i) T_f^*(\Delta(E_i))\right) < \varepsilon.$$

Since $O(x \mid T_f^*(\Delta(E))) = \text{ess-}O(x \circ f \mid E)$ for every $E \in \Lambda^+$, this inequality means that the set $\{x \circ f : x \in A\}$ is a set of small oscillation with respect to λ. Hence (2) holds.

(ii) (2) $\Rightarrow$ (3). This is trivial.

(iii) (3) $\Rightarrow$ (4). Suppose that (4) fails. Then we have a sequence $\{x_n\}_{n \geq 1}$ in A and real numbers r and δ with $\delta > 0$ such that putting $A_n = \{x^* \in K : (x_n, x^*) \leqq r\}$ and $B_n = \{x^* \in K : (x_n, x^*) \geqq r + \delta\}$, then $(A_n, B_n)_{n \geq 1}$ is an independent sequence of closed subsets of K. So, by Proposition 1, there exists a weak*-measurable function $h : I \to K$ such that

$$\int_E (x, h(t))\, d\lambda\,(t) = \int_{\phi^{-1}(\tau^{-1}(E))} (x, x^*)\, d\gamma\,(x^*) \cdots (***)$$

for every $E \in \Lambda$ and every $x \in X$.

Now we are going to show that the set $\{x \circ h : x \in A\}$ is not a set of small oscillation with respect to λ. To this end, take a positive measurable partition $\{E_1, \dots, E_m\}$ of I. Then, in virtue of Lemma 2 in [8], there exist a natural number p and a finite collection $\{i_1, \dots, i_m\}$ of non-negative integers such that

(α) $0 \leqq 2i_1, \dots, 2i_m < 2^p - 1$

and

(β) $E_j \cap I(p, 2i_j)$, $E_j \cap I(p, 2i_j + 1) \in \Lambda^+$ for every j with $1 \leqq j \leqq m$. Hence, setting $C_j = E_j \cap I(p, 2i_j)$ and $D_j = E_j \cap I(p, 2i_j + 1)$ for every j with $1 \leqq j \leqq m$, we have that

$$\text{ess-}O\ (x_p \circ h|E_j) \left(= \sup\left\{\int_I (g_1 - g_2)x_p \circ h\, d\lambda : g_1, g_2 \in \Delta(E_j)\right\}\right)$$

$$\geqq \int_{C_j} (x_p, h(t))\, d\lambda\,(t)/\lambda(C_j) - \int_{D_j} (x_p, h(t))\, d\lambda\,(t)/\lambda(D_j)$$

$$= \left\{\int_{\phi^{-1}(\tau^{-1}(C_j))} (x_p, x^*)\, d\gamma\,(x^*)\right\} \Big/ \lambda(C_j)$$

$$- \left\{\int_{\phi^{-1}(\tau^{-1}(D_j))} (x_p, x^*)\, d\gamma\,(x^*)\right\} \Big/ \lambda(D_j)$$

(by the equality $(***)$ above)

$$\geqq (r + \delta) - r = \delta,$$

for every j with $1 \leqq j \leqq m$, in virtue of the remark preceding Proposition 1. Thus we get that

$$\sup_{x \in A} \sum_{j=1}^{m} \lambda(E_j) \cdot \text{ess-}O\ (x \circ h|E_j) \geqq \sum_{j=1}^{m} \lambda(E_j) \cdot \text{ess-}O\ (x_p \circ h|E_j)$$

$$\geqq \delta \cdot \sum_{j=1}^{m} \lambda(E_j) = \delta,$$

which is the desired conclusion, and so it is contradictory to (3). Hence the proof of this part is completed.

(iv) (6) $\Rightarrow$ (7). Our proof of this part is influenced by the argument of Ghoussoub, Godfroy, Maurey, and Schachermayer ([4], Proof of Theorem VI.16). Suppose that (6) holds. In order to get the statement (7), we have only to show that for any convex subset D of $\overline{\text{co}}^*(K)$ and any positive number ε, there exist nonempty weak*-open sets $U_1, \ldots, U_n$ in D such that

$$\sup_{x \in A} O\left(x \,\middle|\, \sum_{i=1}^{n} (1/n) U_i\right) < \varepsilon.$$

Indeed, in virtue of Lemma II.1 in [4], this implies that there exist positive numbers $\alpha_1, \ldots, \alpha_n$ whose sum is one and weak*-open slices $S_1, \ldots, S_n$ of D such that

$$\sup_{x \in A} O\left(x \,\middle|\, \sum_{i=1}^{n} \alpha_i S_i\right) < \varepsilon.$$

That is, we know that (7) holds. So let us suppose that there exist a convex subset D of $\overline{\text{co}}^*(K)$ and a positive number δ such that for any nonempty weak*-open sets $U_1, \dots, U_n$ in D,

$$\sup_{x \in A} O\left(x \,\middle|\, \sum_{i=1}^{n} (1/n) U_i\right) > \delta.$$

Then we can construct by induction a sequence $\{x_n\}_{n \geq 1}$ in A and a system $\{U(n,i) : n = 0, 1, \dots; i = 0, \dots, 2^n - 1\}$ of nonempty convex weak*-open sets in D such that

(a) $U(n+1, 2i) \cup U(n+1, 2i+1) \subset U(n,i)$ and $V(n+1, 2i) \cap V(n+1, 2i+1)$ is empty for every $n \geqq 0$ and every i with $0 \leqq i \leqq 2^n - 1$, and

(b) $(1/2^n) \sum_{i=0}^{2^n-1} [\inf\{(x_{n+1}, x^*) : x^* \in V(n+1, 2i)\} - \sup\{(x_{n+1}, x^*) : x^* \in V(n+1, 2i+1)\}] > \delta$

for every $n \geqq 0$. (Here $V(n,i)$ denotes the weak*-closure of $U(n.i)$).

Indeed, assume that $\{U(n,i) : n = 0, \dots, k; i = 0, \dots, 2^k - 1\}$ and $\{x_n\}_{1 \leq n \leq k}$ have already been constructed. Take $\{U(k,i) : i = 0, \dots, 2^k - 1\}$ as a finite collection of nonempty convex weak*-open sets in D. Then, since

$$\sup_{x \in A} O\left(x \,\middle|\, \sum_{i=0}^{2^k-1} (1/2^k) U(k,i)\right) > \delta,$$

there exists an element x_{k+1} of A such that

$$O\left(x_{k+1} \,\middle|\, \sum_{i=0}^{2^k-1} (1/2^k) U(k,i)\right) > \delta,$$

that is, it holds that

$$\begin{aligned} &(1/2^k) \sum_{i=0}^{2^k-1} O(x_{k+1} \mid U(k,i)) \\ &(= (1/2^k) \sum_{i=0}^{2^k-1} [\sup\{(x_{k+1}, x^*) : x^* \in U(k,i)\} \\ &\qquad - \inf\{(x_{k+1}, x^*) : x^* \in U(k,i)\}]) > \delta. \end{aligned}$$

So, use an argument in [4] with a slight modification to get nonempty convex weak*-open sets $U(k+1, 2i)$ and $U(k+1, 2i+1)$ both contained in $U(k,i)$ for every i with $0 \leqq i \leqq 2^k - 1$ such that $V(k+1, 2i)$ and $V(k+1, 2i+1)$ $(0 \leqq i \leqq 2^k - 1)$ are disjoint and

$$\begin{aligned} &(1/2^k) \sum_{i=0}^{2^k-1} [\inf\{(x_{k+1}, x^*) : x^* \in V(k+1, 2i)\} \\ &\qquad - \sup\{(x_{k+1}, x^*) : x^* \in V(k+1, 2i+1)\}] > \delta. \end{aligned}$$

Now, for every $n \geqq 1$, let $A_n = \bigcup\{V(n, 2i+1) : 0 \leqq i \leqq 2^{n-1} - 1\}$ and $B_n = \bigcup\{V(n, 2i) : 0 \leqq i \leqq 2^{n-1} - 1\}$. Then it easily follows from (a) that $(A_n, B_n)_{n \geq 1}$ is an independent sequence of pairs of closed subsets of $\overline{\mathrm{co}}^*(K)$. Hence $\Gamma = \bigcap_{n \geq 1}(A_n \cup B_n)$ is a nonempty compact subset of $\overline{\mathrm{co}}^*(K)$. Let $g : I \to \Gamma$ ($\subset \overline{\mathrm{co}}^*(K)$) be a weak*-measurable function assured by Proposition 2 and take the sequence $\{x_n\}_{n \geq 1}$ in A obtained above. Then, since for every $n \geqq 1$

$$r_n(t) = \sum_{i=0}^{2^n-1} (-1)^i \chi_{I(n,i)}(t),$$

we have by the remark after Proposition 2 that

$$\begin{aligned}
&(x_n, T_g^*(r_n)) \ (= (T_g(x_n), r_n)) \\
&\quad = \int_I (x_n, g(t)) \cdot r_n(t)\, d\lambda(t) \\
&\quad = \sum_{i=0}^{2^n-1} (-1)^i \int_{I(n,i)} (x_n, g(t))\, d\lambda(t) \\
&\quad = \sum_{i=0}^{2^{n-1}-1} \left\{ \int_{I(n,2i)} (x_n, g(t))\, d\lambda(t) - \int_{I(n,2i+1)} (x_n, g(t))\, d\lambda(t) \right\} \\
&\quad = \sum_{i=0}^{2^{n-1}-1} \left\{ \int_{\phi^{-1}(\tau^{-1}(I(n,2i)))} (x_n, x^*)\, d\gamma(x^*) - \int_{\phi^{-1}(\tau^{-1}(I(n,2i+1)))} (x_n, x^*)\, d\gamma(x^*) \right\} \\
&\quad = \sum_{i=0}^{2^{n-1}-1} \left\{ \int_{\Gamma \cap V(n,2i)} (x_n, x^*)\, d\gamma(x^*) - \int_{\Gamma \cap V(n,2i+1)} (x_n, x^*)\, d\gamma(x^*) \right\} \\
&\quad \geqq \sum_{i=0}^{2^{n-1}-1} (1/2^{n-1})[\inf\{(x_n, x^*) : x^* \in \Gamma \cap V(n, 2i)\} \\
&\qquad\qquad - \sup\{(x_n, x^*) : x^* \in \Gamma \cap V(n, 2i+1)\}] \\
&\quad \geqq \sum_{i=0}^{2^{n-1}-1} (1/2^{n-1})[\inf\{(x_n, x^*) : x^* \in V(n, 2i)\} \\
&\qquad\qquad - \sup\{(x_n, x^*) : x^* \in V(n, 2i+1)\}] > \delta,
\end{aligned}$$

(by (b)) for every $n \geqq 1$. Hence we have that

$$\inf_{n \geqq 1} \left\{ \sup_{x \in A} |(x, T_g^*(r_n))| \right\} \geqq \delta,$$

which is contradictory to (6). Thus the proof of this part is completed.

(v) (7) $\Rightarrow$ (1). This immediately follows from definitions of A-weak*-strong regularity and A-strong regularity, since every weak*-open slice is an open slice.

Consequently, all proofs of Proposition 5 are completed.

We conclude our paper with the following summary about K-weakly precompact sets, which is obtained by combining the results established in [9], [10] and this paper.

THEOREM. *Let A be a bounded subset of X and K a weak*-compact subset of X^*. Then the following statements about A and K are equivalent.*

(1) *The set A is K-weakly precompact.*

(2) *The set $\overline{\text{co}}^*(K)$ has the $\bar{A}^*$-PCP.*

(3) *For every weak*-measurable function $f : I \to K$ and every $B \in \Lambda^+$, the set $\overline{\text{co}}^*(T_f^*(\Delta(B)))$ is weak*-$\bar{A}^*$-dentable.*

(4) *Every weak*-measurable function $f : I \to K$ is $\bar{A}^*$-Pettis decomposable.*

(5) *Every weak*-measurable function $f : I \to K$ is weak*-equivalent to a $\bar{A}^*$-measurable function $g : I \to \overline{\text{co}}^*(K)$.*

(6) *For every weak*-measurable function $f : I \to K$, the set $\{x \circ f_n : x \in A, n \geqq 1\}$ has the Bourgain property.*

(7) *For every weak*-measurable function $f : I \to K$, the set $\{x \circ \theta(f) : x \in A\}$ has the Bourgain property.*

(8) *For every weak*-measurable function $f : I \to K$, the set $T_f(A)$ is relatively norm compact.*

(9) *For every weak*-measurable function $f : I \to K$, it holds that*

$$\inf_{n\geqq 1}\left\{\sup_{x\in A}\|x\circ f_n - x\circ f_{n+1}\|_1\right\} = 0.$$

(10) *For every weak*-measurable function $f : I \to K$, it holds that*

$$\inf_{n\geqq 1}\left\{\sup_{x\in A}|(x, T_f^*(r_n))|\right\} = 0.$$

(11) *For every weak*-measurable function $f : I \to K$, the tree associated with f is not an A-δ-Rademacher tree.*

(12) *For every weak*-measurable function $f : I \to K$, the tree associated with f is not an A-separated δ-tree.*

(13) *For every weak*-measurable function $f : I \to K$, the set $T_f^*(\Delta(I))$ contains no A-δ-Rademacher tree.*

(14) *For every weak*-measurable function $f : I \to K$, the set $T_f^*(\Delta(I))$ contains no A-separated δ-tree.*

(15) *For every weak*-measurable function $f : I \to K$, the set $T_f^*(\Delta(I))$ is A-midpoint-Bocce-dentable.*

(16) *The set* $\overline{\mathrm{co}}^*(K)$ *is A-strongly regular.*

(17) *For every weak*-measurable function* $f : I \to K$, *the set* $\{x \circ f : x \in A\}$ *is a set of small oscillation with respect to* λ.

(18) *The set* $\overline{\mathrm{co}}^*(K)$ *is A-weak*-strongly regular.*

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