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Stably extendible vector bundles over the real projective spaces and the lens spaces

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ABSTRACT. Let RP^n be the n-dimensional real projective space and let $L^n(q)$ be the $(2n+1)$-dimensional standard lens space mod q. The purpose of this paper is to prove that a complex k-dimensional vector bundle ζ over RP^n is stably equivalent to a sum of k complex line bundles if ζ is stably extendible to RP^m for every $m > n$, to prove that a real k-dimensional vector bundle ζ over $L^n(3)$ is stably equivalent to a sum of $[k/2]$ real 2-plane bundles if ζ is stably extendible to $L^m(3)$ for every $m > n$ and to study non stable extendibility of complex vector bundles over $L^n(4)$.

1. Introduction

Let F denote the real field R, the complex field C or the quaternion field H. Let X be a CW-complex and A be a subcomplex. A k-dimensional F-vector bundle ζ over A is called extendible (respectively stably extendible) to X, if there exists a k-dimensional F-vector bundle α over X whose restriction to A is equivalent (respectively stably equivalent) to ζ as F-vector bundles, that is, if the restriction $\alpha|A$ of α to A is isomorphic to ζ (respectively $(\alpha|A) \oplus \varepsilon^n$ is isomorphic to $\zeta \oplus \varepsilon^n$ for some trivial F-vector bundle ε^n over A), where $\oplus$ denotes the Whitney sum (cf. Schwarzenberger [14] and Imaoka-Kuwana [4]).

In the following we say simply a vector bundle instead of an R-vector bundle.

Concerning stably extendible F-vector bundles for $F = C$ and R, the following results are known.

THEOREM (Schwarzenberger (cf. [3], [14], [2], [13])). *Let $F = C$ or R. If a k-dimensional F-vector bundle ζ over FP^n is stably extendible to FP^m for each $m > n$, then ζ is stably equivalent to a sum of k F-line bundles.*

In the original results of Schwarzenberger, the F-vector bundles are assumed to be extendible, but the results are also valid for the stably extendible F-vector bundles.

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Recently, concerning stably extendible H-vector bundles over the quaternion projective space HP^n, M. Imaoka and K. Kuwana have proved in [4] that if a k-dimensional H-vector bundle ζ over HP^n is stably extendible to HP^m for each $m > n$ and its top non-zero Pontrjagin class is not zero mod 2, then ζ is stably equivalent to a sum of k H-line bundles provided $k \leqq n$.

We prove

THEOREM A. *If a k-dimensional C-vector bundle ζ over RP^n is stably extendible to RP^m for every $m > n$, then ζ is stably equivalent to a sum of k C-line bundles.*

THEOREM B (cf. Maki [11]). *If a k-dimensional vector bundle ζ over $L^n(3)$ is stably extendible to $L^m(3)$ for every $m > n$, then ζ is stably equivalent to a sum of $[k/2]$ 2-plane bundles.*

As for the extendibility of the tangent bundles, R. L. E. Schwarzenberger has shown in [3, p. 166] that the complex tangent bundle of the n-dimensional complex projective space CP^n is extendible to CP^{n+1} if and only if $n = 1$ (cf. [9, Remark 5.3]), and we have obtained in [8, Theorems 6.6 and 1.2] that the tangent bundle of the n-dimensional real projective space RP^n is extendible to RP^{n+1} if and only if $n = 1, 3$ or 7 and that for any integer $q > 1$ the tangent bundle of the $(2n+1)$-dimensional standard lens space mod q, $L^n(q)$, is extendible to $L^{n+1}(q)$ if and only if $n = 0, 1$ or 3. We shall show that the stable extendibility for the tangent bundle of RP^n is equivalent to the extendibility for it, and also give results about some non stable extendibility of the tangent bundle of $L^n(q)$.

This note is arranged as follows. We prove Theorem A in §2 and Theorem B in §3. In §4 we study stable extendibility of the tangent bundles of RP^n and $L^n(p)$. In §5 and §6 we study non stable extendibility of complex vector bundles over $L^n(4)$.

Related results concerning extendible F-vector bundles for $F = R$ or C over the lens spaces and the projective spaces are found in [6], [9] and [12].

2. Proof of Theorem A

Let ξ_n be the canonical line bundle over RP^n and let c denote the complexification.

The following theorem is the "stably extendible version" of Theorem 4.2 for $d = 1$ in [9].

THEOREM 2.1. *Let ζ be a k-dimensional C-vector bundle over RP^n. Assume that there is a positive integer ℓ such that*

(i) *ζ is stably equivalent to $(k+\ell)c\xi_n$, and*
(ii) $k+\ell < 2^{[n/2]}$.
Then $[n/2] < k+\ell$ and ζ is not stably extendible to RP^m for each m with $k+\ell \leqq [m/2]$.

We omit the proof since it is parallel to the "extendible case".
Theorem A is a consequence of the following

THEOREM 2.2. *Let ζ be a k-dimensional C-vector bundle over RP^n which is stably extendible to RP^m for $m = 2^{[n/2]+1}$. Then ζ is stably equivalent to a sum of k C-line bundles.*

PROOF. According to J. F. Adams [1], there is an integer ℓ such that

$$\zeta - k = (k+\ell)(c\xi_n - 1) \in \tilde{K}(RP^n) \cong Z_{2^{[n/2]}}.$$

Since $c\xi_n - 1$ is of order $2^{[n/2]}$, we may take ℓ such that $0 \leqq k+\ell \leqq 2^{[n/2]}-1$. For $m = 2^{[n/2]+1}$, $n < m$ and $k+\ell < [m/2]$. Hence ζ is not stably extendible to RP^m by Theorem 2.1 if $\ell > 0$. This contradicts our assumption. Hence $\ell \leqq 0$, as desired. q.e.d.

REMARK. If $n \neq 3$, we may replace the assumption $m = 2^{[n/2]+1}$ in Theorem 2.2 by $m = 2^{[n/2]+1} - 1$.

3. Proof of Theorem B

The following theorem is the "stably extendible version" of Theorem 1.1 in [8].

THEOREM 3.1. *Let p be an odd prime and ζ be a k-dimensional vector bundle over $L^n(p)$. Assume that there is a positive integer ℓ such that*
(i) *ζ is stably equivalent to a sum of $[k/2]+\ell$ non-trivial 2-plane bundles, and*
(ii) $[k/2]+\ell < p^{[n/(p-1)]}$.
Then $n < 2[k/2]+2\ell$ and ζ is not stably extendible to $L^m(p)$ for each m with $2[k/2]+2\ell \leqq m$.

We omit the proof since it is parallel to the "extendible case".
Theorem B is a consequence of the following

THEOREM 3.2. *Let n and s be integers with $0 \leqq n \leqq 2s$ and $s \equiv 1 \bmod 2$, and let ζ be a k-dimensional vector bundle over $L^n(3)$ which is stably extendible to $L^m(3)$ for $m = 2(3^s - 1)$. Then ζ is stably equivalent to a sum of $[k/2]$ 2-plane bundles.*

PROOF. Set $N = 2s$. Then clearly $n \leqq N < m$. Let β be a stable extension of ζ to $L^m(3)$. Then the restriction α of β to $L^N(3)$ is a stable extension of ζ to $L^N(3)$. According to [5] (cf. [10]), there is an integer ℓ such that

$$\alpha - k = ([k/2] + \ell)r(\eta_N - 1) \in \widetilde{KO}(L^N(3)) \cong Z_{3^s},$$

where η_N is the canonical complex line bundle over $L^N(3)$ and r is the forgetful map. Since $r(\eta_N - 1)$ is of order 3^s, we may take ℓ such that $0 \leqq [k/2] + \ell \leqq 3^s - 1$. Then $n < m$ and $2[k/2] + 2\ell \leqq m$. Hence α is not stably extendible to $L^m(3)$ by Theorem 3.1 if $\ell > 0$. This is a contradiction, because β is a stable extension of α. So we have $\ell \leqq 0$. Hence α is stably equivalent to a sum of $[k/2]$ 2-plane bundles. Thus ζ is also stably equivalent to a sum of $[k/2]$ 2-plane bundles. q.e.d.

4. Stable extendibility of the tangent bundles of RP^n and $L^n(p)$

The following theorem is the "stably extendible version" of Theorem 6.2 in [8].

THEOREM 4.1. *Let ζ be a k-dimensional vector bundle over RP^n. Assume that there is a positive integer ℓ such that*

(i) *ζ is stably equivalent to $(k+\ell)\xi_n$, and*

(ii) *$k + \ell < 2^{\phi(n)}$, where $\phi(n)$ is the number of integers s such that $0 < s \leqq n$ and $s \equiv 0, 1, 2$ or $4 \bmod 8$.*

Then $n < k + \ell$ and ζ is not stably extendible to $RP^{k+\ell}$.

We omit the proof since it is parallel to the "extendible case".

The following result corresponds to Theorem 6.6 in [8].

THEOREM 4.2. *The tangent bundle $\tau(RP^n)$ of RP^n is stably extendible to RP^{n+1} if and only if $n = 1, 3$ or 7.*

PROOF. If $n = 1, 3$ or 7, $\tau(RP^n)$ is trivial. Hence $\tau(RP^n)$ is extendible (and so stably extendible) to RP^m for each $m > n$.

Putting $\zeta = \tau(RP^n)$ and $\ell = 1$ in Theorem 4.1, we obtain the "only if" part. q.e.d.

Combining Theorem 6.6 in [8] with Theorem 4.2 above, we see that the stable extendibility is equivalent to the extendibility for the tangent bundles of the real projective spaces.

On the stable extendibility of the tangent bundles of lens spaces, we have

THEOREM 4.3. *Let p be an odd prime. If $n \geqq 2p - 2$, the tangent bundle $\tau(L^n(p))$ of $L^n(p)$ is not stably extendible to $L^{2n+2}(p)$.*

PROOF. Putting $\zeta = \tau(L^n(p))$ and $\ell = 1$ in Theorem 3.1, we obtain the desired result. q.e.d.

5. Some lemmas

Let η_n be the canonical complex line bundle over $L^n(4)$.

By using the structure of $\tilde{K}(L^n(4))$ (cf. [7]), we have the following

LEMMA 5.1. *If $\sum_{i=1}^{3} b_i\eta_n^i$ and $\sum_{i=1}^{3} b_i'\eta_n^i$ over $L^n(4)$ are stably equivalent, then $b_i \equiv b_i' \bmod 2^{[(n-1)/2]}$ for $1 \leqq i \leqq 3$.*

PROOF. Put $d_i = b_i - b_i'$ and $\sigma_n = \eta_n - 1 \in \tilde{K}(L^n(4))$. Then we have

$$d_1\sigma_n + d_2(\sigma_n^2 + 2\sigma_n) + d_3(\sigma_n^3 + 3\sigma_n^2 + 3\sigma_n) = 0.$$

If n is even, then σ_n, $\sigma_n^2 + 2\sigma_n$ and $\sigma_n^3 + 2\sigma_n^2 + 2^{n/2+1}\sigma_n$ are generators of

$$\tilde{K}(L^n(4)) \cong Z_{2^{n+1}} + Z_{2^{[n/2]}} + Z_{2^{[(n-1)/2]}}$$

(cf. [7, Theorem A]). Then we have $d_3 \equiv 0 \bmod 2^{[(n-1)/2]}$, $d_2 + d_3 \equiv 0 \bmod 2^{n/2}$ and $d_1 - (2^{n/2+1} - 1)d_3 \equiv 0 \bmod 2^{n+1}$. Hence, $d_i \equiv 0 \bmod 2^{[(n-1)/2]}$ for $1 \leqq i \leqq 3$.

If n is odd, then σ_n, $\sigma_n^2 + 2\sigma_n + 2^{[n/2]+1}\sigma_n$ and $\sigma_n^3 + 2\sigma_n^2$ are generators of $K(L^n(4))$, and so we have the required congruence similarly. q.e.d.

LEMMA 5.2. *Let a, r, j and k be non-negative integers with $r < a$ and $j < 2^{a-r}$. Then $\binom{n+k2^a}{j} \equiv \binom{n}{j} \bmod 2^{r+1}$ for any integer n.*

PROOF. Considering the identity $(x+1)^{n+k2^a} = (x+1)^n(x+1)^{k2^a}$, we have $\binom{n+k2^a}{j} = \sum_{i=0}^{j}\binom{n}{j-i}\binom{k2^a}{i}$. Hence

$$\binom{n+k2^a}{j} - \binom{n}{j} = \sum_{i=1}^{j}\binom{n}{j-i}\binom{k2^a}{i} \equiv 0 \bmod 2^{r+1},$$

because $\binom{k2^a}{i} \equiv 0 \bmod 2^{r+1}$ for $1 \leqq i \leqq j < 2^{a-r}$. q.e.d.

6. Non stable extendibility of complex vector bundles over $L^n(4)$

Let a, b and m be non-negative integers. Define an integer $A(a,b;m)$ as follows:

$$A(a,b;m) = \sum_{i+j=m,\, i,j \geqq 0} \binom{a}{i}\binom{b}{j}(-1)^j.$$

We have the following

THEOREM 6.1. *Assume that a k-dimensional C-vector bundle ζ over $L^n(4)$ is stably equivalent to $\sum_{i=1}^{3} b_i \eta_n^i$ for some non-negative integers b_i where $k \geqq 1$. If there exists an integer m satisfying*

(i) $k < m < 2^{[(n-3)/2]}$, *and*

(ii) $A(b_1, b_3; m) + 2b_2 A(b_1, b_3; m-1) \not\equiv 0 \bmod 4$,

then $n < m$ and ζ is not stably extendible to $L^m(4)$.

PROOF. Under the assumption of the theorem, by the fundamental properties of the K-theory Chern class, that is, γ-operation, in $\tilde{K}(L^n(4))$, we see that

$$\gamma_t(\zeta - k) = \gamma_t\left(\sum_{i=1}^{3} b_i(\eta_n^i - 1)\right) = \prod_{i=1}^{3}\{1 + (\eta_n^i - 1)t\}^{b_i}$$

$$= \prod_{i=1}^{3}\{1 + ((1+\sigma_n)^i - 1)t\}^{b_i} = \sum_{j \geqq 0}\left\{\sum_{\ell \geqq 0} B_\ell(j)\sigma_n^{\ell+j}\right\}t^j$$

for some coefficients $B_\ell(j)$. Thus we have

$$\gamma^m(\zeta - k) = \sum_{\ell \geqq 0} B_\ell(m)\sigma_n^{\ell+m} \tag{1}$$

in $\tilde{K}(L^n(4))$. We note that

$$B_0(j) = \sum_{j_1+j_2+j_3=j,\, j_1,j_2,j_3 \geqq 0} \prod_{s=1}^{3} \binom{b_s}{j_s} s^{j_s}, \quad \text{and}$$

$$B_0(m) \equiv A(b_1, b_3; m) + 2b_2 A(b_1, b_3; m-1) \bmod 4.$$

Since $k < m$ by (i), we have $\gamma^m(\zeta - k) = 0$ in $\tilde{K}(L^n(4))$. Now, suppose that $m \leqq n$. Multiplying each side of (1) by σ_n^{n-m} and taking account of the fact that $\sigma_n^{n+1} = 0$, we obtain $B_0(m)\sigma_n^n = \gamma^m(\zeta - k)\sigma_n^{n-m} = 0$. Therefore, we see that $B_0(m) \equiv 0 \bmod 4$, since the order of σ_n^n is 4. Then we have that (ii) does not hold. This contradiction shows that $n < m$.

Next, under the assumption of the theorem we suppose that ζ is stably extendible to $L^m(4)$. Let α be a stable extension of ζ to $L^m(4)$. Then we have

$$\alpha - k = \sum_{i=1}^{3} s_i(\eta_m^i - 1)$$

in $\tilde{K}(L^m(4))$ for some integer $s_i \geqq 0$. Since α is a k-dimensional C-vector bundle over $L^m(4)$ and $k < m$ by (i), we obtain $\gamma^m(\alpha - k) = 0$. On the other hand,

$$0 = \gamma^m(\alpha - k) = \sum_{\ell \geqq 0} C_\ell(m)\sigma_m^{m+\ell} \tag{2}$$

in $\tilde{K}(L^m(4))$ for some coefficients $C_\ell(m)$ $(\ell \geqq 0)$. We note that

$$\begin{aligned} C_0(m) &= \sum_{j_1+j_2+j_3=m, j_1, j_2, j_3 \geqq 0} \prod_{i=1}^{3} \binom{s_i}{j_i} i^{j_i} \\ &\equiv A(s_1, s_3; m) + 2s_2 A(s_1, s_3; m-1) \bmod 4. \end{aligned}$$

Since $\sigma_m^{m+1} = 0$, we have $C_0(m)\sigma_m^m = 0$ by (2). Since the order of σ_m^m is 4, we obtain $A(s_1, s_3; m) + 2s_2 A(s_1, s_3; m-1) \equiv C_0(m) \equiv 0 \bmod 4$.

Now, ζ is stably equivalent to $\sum_{i=1}^{3} s_i \eta_n^i$ and also to $\sum_{i=1}^{3} b_i \eta_n^i$ by the assumption. By Lemma 5.1, we have $s_i \equiv b_i \bmod 2^{[(n-1)/2]}$ for $1 \leqq i \leqq 3$. So $A(s_1, s_3; m) + 2s_2 A(s_1, s_3; m-1) \equiv A(b_1, b_3; m) + 2b_2 A(b_1, b_3; m-1) \bmod 4$ by Lemma 5.2 for $r = 1$, and consequently $A(b_1, b_3; m) + 2b_2 A(b_1, b_3; m-1) \equiv 0 \bmod 4$. This contradicts (ii). q.e.d.

Corollary 6.2. *If $n > 12$, the complexification $c\tau(L^n(4))$ of the tangent bundle $\tau(L^n(4))$ of $L^n(4)$ is not stably extendible to $L^{2n+2}(4)$.*

Proof. Since $\tau(L^n(4)) \oplus \varepsilon = (n+1)r\eta_n$ and $\eta_n^{-1} = \eta_n^3$, $c\tau(L^n(4))$ is stably equivalent to $(n+1)cr\eta_n = (n+1)(\eta_n \oplus \eta_n^3)$. Hence we have the desired result from Theorem 6.1 by putting $\zeta = c\tau(L^n(4))$, $k = 2n+1$, $b_1 = b_3 = n+1$, $b_2 = 0$ and $m = 2n+2$. q.e.d.

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