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Classification of proper solutions of some Emden-Fowler equations

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ABSTRACT. We classify positive proper solutions of a class of Emden-Fowler equations in terms of their asymptotic behaviors. We obtain further the uniqueness of positive decaying solutions in some cases.

1. Introduction

In this paper we consider the second order Emden-Fowler equation

$$u'' \pm (1+\varepsilon(t))e^{bt}|u|^{\lambda} \operatorname{sgn} u = 0, \qquad t \geq t_0 > 0, \tag{1.1}$$

where b and λ are real numbers satisfying $b \neq 0, \lambda \neq 0, 1$, and the function $\varepsilon(t)$ is sufficiently small in some sense. A nontrivial solution u of (1.1) is said to be proper if it is defined in some neighborhood of $+\infty$, and is nontrivial in any neighborhood of $+\infty$. We shall confine ourselves to the study of proper solutions throughout the paper.

The study of the Emden-Fowler equation

$$\frac{d}{dx}\left(x^{\rho}\frac{dy}{dx}\right) \pm x^{\mu}|y|^{\lambda} \operatorname{sgn} y = 0, \qquad x \geq x_0 > 0, \tag{1.2}$$

where ρ, μ, λ are real constants, has been one of the main objects in the field of nonlinear analysis in recent years since the appearance of the excellent monograph due to R. Bellman [1]. When $\rho \neq 1$, it is well known that a suitable change of variables $(x, y) \mapsto (t, u)$ transforms (1.2) into the equation of the form

$$\frac{d^2u}{dt^2} \pm t^{\eta}|u|^{\lambda} \operatorname{sgn} u = 0$$

with η a constant. For the case where $\lambda > 1$, the classification of all proper solutions was established by Bellman [1] in terms of their asymptotic behaviors near $+\infty$. The analogous results for the case where $\lambda \in (0, 1)$ and for the case

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where $\lambda < 0$ were established by Chanturiya [2] and Usami [7, 8], respectively. Many results for the generalized Emden-Fowler equations containing the equation (1.2) were collected by Wong [9]. On the other hand, if $\rho = 1$, the change of variables $x = e^t$, $u(t) = y(e^t)$ transforms (1.2) into the equation

$$\frac{d^2u}{dt^2} \pm e^{\xi t}|u|^\lambda \operatorname{sgn} u = 0,$$

with a constant ξ, which has the form (1.1). It seems that systematic studies for such a type of equations have not yet been carried out. Motivated by this fact, we here intend to develop an asymptotic theory for the equation (1.1), and make an attempt to find analogous results to those obtained by Bellman, Chanturiya, and others.

It is convenient to classify the equation (1.1) further into four types according by the signs of the coefficient function and the exponent b. That is, we will discuss the following four equations separately:

$$(E_{++}) \qquad u'' + (1 + \varepsilon(t))e^{\alpha t}|u|^\lambda \operatorname{sgn} u = 0,$$

$$(E_{+-}) \qquad u'' + (1 + \varepsilon(t))e^{-\alpha t}|u|^\lambda \operatorname{sgn} u = 0,$$

$$(E_{-+}) \qquad u'' - (1 + \varepsilon(t))e^{\alpha t}|u|^\lambda \operatorname{sgn} u = 0,$$

$$(E_{--}) \qquad u'' - (1 + \varepsilon(t))e^{-\alpha t}|u|^\lambda \operatorname{sgn} u = 0,$$

where $\alpha > 0$ and $\lambda \in \mathrm{R}\backslash\{0, 1\}$ are constants. When $\lambda < 0$, we often put $\lambda = -\sigma, \sigma > 0$, and hence $(E_{\pm\pm})$ can be rewritten in the form

$$u'' \pm (1 + \varepsilon(t))e^{\pm\alpha t}|u|^{-\sigma} \operatorname{sgn} u = 0.$$

The function $\varepsilon(t)$ is assumed to be of class C^1 near $+\infty$ throughout the paper.

When $\varepsilon \equiv 0$, it should be noted that the equations $(E_{\pm\pm})$ may have exact solutions of the form $u_0(t) = ce^{\beta t}$ with $c > 0$, β being real constants. In fact, we find that the functions

$$(1.3) \quad u_0(t) \equiv ce^{\beta t}, \qquad \text{where } c = |\beta|^{2/(\lambda-1)}, \qquad \beta = \frac{-\alpha}{\lambda - 1} \left[\text{resp. } \beta = \frac{\alpha}{\lambda - 1}\right]$$

solve the equations (E_{-+}) [resp. (E_{--})] with $\varepsilon \equiv 0$. However, the equations $(E_{+\pm})$ with $\varepsilon \equiv 0$ do not possess exact solutions of such a form. This exact solution u_0 will play important roles in our asymptotic theory.

The organization of the paper is as follows: In §2 we prepare auxiliary lemmas which will be employed later in various places. In §§3, 4 and 5 the asymptotic forms of positive proper solutions of (E_{+-}), (E_{-+}), and (E_{--}) are determined, respectively. In §6, we give uniqueness theorems for positive proper solutions u of (E_{--}) satisfying $\lim_{t\to\infty} u(t) = 0$ on the basis of the result in §5. Since the equation (E_{++}) has no positive proper solutions [9], this type is actually precluded from our consideration.

Finally we mention several works treating the equations $(E_{\pm\pm})$. For the equations $(E_{-\pm})$ with $\lambda > 1$, Taliaferro [6] obtained related results to ours. Kiguradze and Chanturia [4] have obtained asymptotic forms for positive solutions of $(E_{\pm -})$ and (E_{-+}) with $\lambda > 1$ and $0 < \lambda < 1$ as special cases of their asymptotic theory for more general Emden-Fowler type equations. Actually, Thorems 4.1 and 5.1 and some parts of Theorems 4.2 and 5.2 follow from [4].

2. Auxiliary lemmas

In this section we collect auxiliary lemmas which will be used later.

Lemma 2.1. *Let $f \in C^1[T, \infty)$. Assume that f' is bounded, and $\int^\infty |f(t)|^p dt < \infty$ for some $p > 1$. Then $\lim_{t\to\infty} f(t) = 0$.*

Proof. To see this, it suffices to notice the identity

$$|f(t)|^p f(t) = |f(T)|^p f(T) + (p+1) \int_T^t |f(s)|^p f'(s) ds, \qquad t \geq T.$$

Lemma 2.2. ([3, 5]) *Consider the equations*

$$v'' - mv' + lv = 0, \tag{2.1}$$

$$w'' - mw' + (l + L(t))w = 0, \tag{2.2}$$

where $m > 0, l$ are constants, L is a continuous function, and suppose that these equations are nonoscillatory. Let v_1, v_2 be linearly independent solutions of (2.1). *If*

$$\int^\infty e^{-ms} |L(s) v_1(s) v_2(s)| ds < \infty,$$

then every solution w of (2.2) *has the asymptotic behavior*

$$w(t) = c_1(1 + o(1)) v_1(t) + c_2(1 + o(1)) v_2(t) \qquad as\ t \to \infty,$$

where c_1, c_2 are constants.

Lemma 2.3. ([1]) *Consider the system*

$$w' = (A + B(t))w, \tag{2.3}$$

where A is a constant matrix with simple characteristic roots and $B(t) \in C[T, \infty)$, $\|B(t)\| \to 0$ as $t \to \infty$. Then, corresponding to any characteristic root κ of A, *there is a solution* w *of* (2.3) *satisfying the inequalities*

$$c_1 \exp\left[(\mathrm{Re}\,\kappa)t - d_1 \int_T^t \|B(s)\| ds\right] \le \|w(t)\|$$

$$\le c_2 \exp\left[(\mathrm{Re}\,\kappa)t + d_2 \int_T^t \|B(s)\| ds\right],$$

for some constants $c_1, c_2, d_1, d_2 > 0$.

The following is a variant of l'Hospital's rule:

LEMMA 2.4. *Let* $f(t)$ *and* $g(t)$ *be continuously differentiable functions defined near* ∞ *and* $g'(t) \neq 0$. *Then, we have*

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \le \liminf_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}$$

if either $\lim_{t\to\infty} g(t) = \infty$ *or* $\lim_{t\to\infty} g(t) = \lim_{t\to\infty} f(t) = 0$ *holds.*

Finally, we present a simple lemma. We use this lemma when we show that some positive proper solutions of $(E_{-\pm})$ are asymptotic to u_0 given by (1.3).

LEMMA 2.5. *Let* u *be a positive proper solution of* $(E_{-\pm})$ *and* u_0 *be the positive proper solution of* $(E_{-\pm})$ *with* $\varepsilon(t) \equiv 0$ *given by* (1.3)*:*

$$u_0 = ce^{\beta t}.$$

Put $v = u/u_0$. *Then* v *satisfies the equation*

$$v'' + 2\beta v' + \beta^2(v - (1 + \varepsilon(t))v^\lambda) = 0. \tag{2.4}$$

3. The equation (E_{+-})

The next result is simple. But for future reference, we prove the next result.

THEOREM 3.1. *If* $\lambda \in \mathrm{R}\backslash\{0,1\}$, $\alpha > 0$ *and* $\lim_{t\to\infty} \varepsilon(t) = 0$, *then all positive proper solutions* u *of*

$$u'' + (1 + \varepsilon(t))e^{-\alpha t}u^\lambda = 0 \tag{E_{+-}}$$

possess one of the following asymptotic forms

$$u = a_1 t + a_2 - \frac{(a_1 t)^\lambda}{\alpha^2} e^{-\alpha t}(1 + o(1)); \tag{3.1}$$

$$u = a - \frac{a^\lambda}{\alpha^2} e^{-\alpha t}(1 + o(1)), \tag{3.2}$$

where $a > 0, a_1 > 0, a_2$ *are constants.*

PROOF. *Case 1. The case where* $\lambda > 0$. Since $u'' < 0$, u' is monotone decreasing, and hence there are two possibilities for u' as $t \to \infty$:
(A) $u' \searrow 0$;
(B) $u' \searrow a_1$, where a_1 is a positive constant.

When case (A) occurs, u is monotone increasing, since $u' > 0$. Furthermore there are two possibilities for u as $t \to \infty$:
(A-1) $u \nearrow +\infty$;
(A-2) $u \nearrow a$, where a is a positive constant.

Let us first show that the case (A-1) is impossible. Since $u' \searrow 0$, we get an estimate for u

(3.3) $$u \leq t \qquad \text{for large } t.$$

Since $u'(\infty) = 0$, integrating (E_{+-}) we obtain

$$u(t) = c_1 + \int_{t_0}^{t} \int_{s}^{\infty} (1 + \varepsilon(r))e^{-\alpha r}[u(r)]^{\lambda} drds,$$

where c_1 is a constant. Hence by considering the estimate (3.3), we have $u(\infty) < \infty$, which leads to a contradiction.

Let case (A-2) occur. Since $u'(\infty) = 0$ and $u(\infty) = a$, integrating (E_{+-}) from t to ∞ twice, we get

$$u(t) = a - \int_{t}^{\infty} \int_{s}^{\infty} (1 + \varepsilon(r))e^{-\alpha r}[u(r)]^{\lambda} drds.$$

From this identity we can easily get the asymptotic formula (3.2).

When case (B) occurs, the asymptotic behavior of u is $u \sim a_1 t$. We easily find that $u - a_1 t$ has a finite limit a_2 as $t \to \infty$ by the assumption for $\varepsilon(t)$. By integrating (E_{+-}) from t to ∞ twice, we get

$$u = a_1 t + a_2 - \int_{t}^{\infty} \int_{s}^{\infty} (1 + \varepsilon(r))e^{-\alpha r}[u(r)]^{\lambda} drds,$$

which implies the asymptotic form (3.1).

Case 2. The case where $\lambda < 0$. As in the proof for Case 1, we obtain the possibilities (A) and (B). When case (A-1) occurs, we obtain a contradiction from the estimate

$$u \geq c_2 \qquad \text{for large } t,$$

where c_2 is a positive constant. The remainder of the proof is carried out as in Case 1. The proof of Theorem 3.1 is complete.

4. The equation (E_{-+})

The following result follows directly from [4, Theorem 20.27].

THEOREM 4.1. *If* $\lambda > 1$ *and* $\varepsilon(t) \to 0$ *as* $t \to \infty$, *then all positive proper solutions* u *of*

$$(E_{-+}) \qquad u'' - (1+\varepsilon(t))e^{\alpha t}u^{\lambda} = 0$$

possess the asymptotic form

$$(4.1) \qquad u \sim u_0 \equiv ce^{\beta t}, \qquad \beta = -\frac{\alpha}{\lambda - 1}, \qquad c = |\beta|^{2/(\lambda-1)}.$$

The next result can be shown by [4, Theorem 20.23] when the condition (4.2) below is assumed.

THEOREM 4.2. *Let* $0 < \lambda < 1$ *and either*

$$(4.2) \qquad \lim_{t\to\infty} \varepsilon(t) = 0, \qquad \int^{\infty} |\varepsilon'(t)|dt < \infty$$

or

$$(4.3) \qquad \lim_{t\to\infty} \varepsilon(t) = 0, \qquad \int^{\infty} [\varepsilon(t)]^2 dt < \infty$$

hold. Then all positive proper solutions u *of* (E_{-+}) *possess the asymptotic form* (4.1).

PROOF. It remains to prove the results under the condition (4.3).
As the first step we will show that

$$(4.4) \qquad \limsup_{t\to\infty} \frac{u(t)}{e^{(\alpha/(1-\lambda))t}} < \infty.$$

From the equation (E_{-+}), u' is of constant sign near $+\infty$. When $u' \le 0$ near $+\infty$, clearly (4.4) holds. Hence we shall consider the case where $u' > 0$ near $+\infty$. It is easily seen from the equation (E_{-+}) that $u'(\infty) = \infty$, and hence $\lim_{t\to\infty} u(t)/t = \infty$. We may assume that $u' > 0$ on $[T, \infty)$. Integrating (E_{-+}) twice, we have

$$\begin{aligned} u(t) &= c_1 + c_2 t + \int_T^t \int_T^s (1+\varepsilon(r))e^{\alpha r}[u(r)]^{\lambda} dr ds \\ &\le |c_1| + |c_2|t + c_3 e^{\alpha t}[u(t)]^{\lambda}, \qquad t \ge T, \end{aligned}$$

for some constants c_1, c_2, and $c_3 > 0$. In view of the relation $\lim_{t\to\infty} u(t)/t = \infty$, it is easy to see that (4.4) holds.

Put $v = u/u_0$. We know by Lemma 2.5 that v solves

$$v'' + 2\beta v' + \beta^2(v - (1 + \varepsilon(t))v^\lambda) = 0, \tag{4.5}$$

and by (4.4) that

$$\limsup_{t\to\infty} v(t) < \infty. \tag{4.6}$$

On the basis of (4.6) we next show that $v'(t)$ is bounded on $[T, \infty)$. Clearly (4.6) implies that $\liminf_{t\to\infty}|v'(t)| = 0$. Suppose the contrary that $\limsup_{t\to\infty} \cdot |v'(t)| = \infty$. We can find a sequence $\{t_n\}$ such that

$$\lim_{t\to\infty} t_n = \infty, \qquad \lim_{t\to\infty} |v'(t_n)| = \infty, \qquad v''(t_n) = 0, \qquad n \in \mathrm{N}$$

by Rolle's theorem. Putting $t = t_n$ in equation (4.5), we obtain

$$2\beta v'(t_n) + \beta^2(v(t_n) - (1 + \varepsilon(t_n))[v(t_n)]^\lambda) = 0, \qquad n \in \mathrm{N},$$

implying that

$$|v'(t_n)| \le \frac{\beta}{2}|v(t_n) - (1 + \varepsilon(t_n))[v(t_n)]^\lambda|, \qquad n \in \mathrm{N}.$$

Since v is bounded, this inequality contradicts the hypothesis $|v'(t_n)| \to \infty$, $n \to \infty$. Accordingly, $v'(t)$ is bounded on $[T, \infty)$. Furthermore we know from (4.5) that $v''(t)$ is also bounded.

Multiplying (4.5) by v' and integrating the resulting equation on $[T, t]$, we obtain

$$\frac{(v')^2}{2} + 2\beta \int_T^t (v')^2 ds + \frac{\beta^2 v^2}{2} - \frac{\beta^2 v^{\lambda+1}}{\lambda + 1} = c_4 + \beta^2 \int_T^t \varepsilon(s) v^\lambda v' ds, \qquad t \ge T, \tag{4.7}$$

where c_4 is a constant. Invoking Schwarz's inequality, we have

$$\int_T^t |\varepsilon(s) v^\lambda v'| ds \le c_5 \left(\int_T^\infty [\varepsilon(s)]^2 ds\right)^{1/2} \left(\int_T^\infty [v'(s)]^2 ds\right)^{1/2},$$

where $c_5 > 0$ is a constant. Since $v, v' = O(1)$ and $\beta < 0$, (4.7) implies that

$$\int^\infty [v'(t)]^2 dt < \infty. \tag{4.8}$$

From the above consideration we know that $\lim_{t\to\infty} v'(t) = 0$ via Lemma 2.1. Accodingly it follows from (4.7) that the limit of $(v^2/2 - v^{\lambda+1}/(\lambda + 1))$ must exist as a finite value, that is, v has a finite limit l. Letting $t \to \infty$ in (4.5), we have $\lim_{t\to\infty} v'' = \beta^2(l^\lambda - l)$. If $l^\lambda - l \ne 0$, then the boundedness of v' is violated. Hence $l^\lambda - l = 0$, i.e., $l = 0$ or 1. Below we show that $l = 1$.

Suppose that $l=0$. Noting (4.5) and the fact that $0<\lambda<1$, we easily see that there exist a large t_1 such that $v'<0$ for $t\ge t_1$. From (4.5) we obtain

$$v''>\beta^2((1+\varepsilon(t))v^\lambda - v)>\left(\frac{1}{2}+\varepsilon(t)\right)\beta^2 v^\lambda > \frac{\beta^2}{4}v^\lambda,$$

for large t, since $0<\lambda<1$ and $v\searrow 0$. Multiplying the above inequality by v' and integrating from t to ∞, we find

$$-v'(t)\ge \frac{\beta}{\sqrt{2(\lambda+1)}}v^{(\lambda+1)/2}, \qquad t\ge t_1 \qquad \text{for large } t_1.$$

Integrating again from t_1 to t, we conclude that

$$v(t_1)^{(-\lambda+1)/2}\ge \frac{\beta(1-\lambda)}{2\sqrt{2(\lambda+1)}}(t-t_1), \qquad t\ge t_1,$$

and this is a contradiction. Hence, $l=1$, i.e., $u\sim ce^{\beta t}$. The proof of Theorem 4.2 is complete.

THEOREM 4.3. *Let* $\lambda=-\sigma<0$.

(i) *If* $0<\sigma<1$ *and* $\lim_{t\to\infty}\varepsilon(t)=0$, *then all positive proper solutions u of*

$$(E_{-+}) \qquad u''-(1+\varepsilon(t))e^{\alpha t}u^{-\sigma}=0$$

have the asymptotic form (4.1).

(ii) *If* $\sigma\ge 1$, *and either* (4.2) *or* (4.3) *holds, then all positive proper solutions of* (E_{-+}) *have the asymptotic form* (4.1).

PROOF. We first show that

$$0<\liminf_{t\to\infty}\frac{u(t)}{e^{\beta t}}\le \limsup_{t\to\infty}\frac{u(t)}{e^{\beta t}}<\infty. \tag{4.9}$$

From (E_{-+}), we have $u''u^\sigma=(1+\varepsilon(t))e^{\alpha t}$ and integrating from t_0 to t, we obtain

$$u^\sigma u'-\sigma\int_{t_0}^t u^{\sigma-1}(u')^2 ds=\frac{1}{\alpha}e^{\alpha t}+\int_{t_0}^t \varepsilon(s)e^{\alpha s}ds+c_1.$$

This yields

$$u^\sigma u'\ge c_2 e^{\alpha t},$$

where $c_1, c_2>0$ are constants. Integrating the above inequality from t_0 to t, we have

$$u\ge c_3 e^{\beta t}, \qquad \text{for large } t, \tag{4.10}$$

where c_3 is a positive constant.

On the other hand, by integrating (E_{-+}) from t_0 to t twice, and from (4.10), we get

$$u \leq c_4 e^{\beta t}, \qquad \text{for large } t,$$

where c_4 is a positive constant. Hence we conclude that (4.9) holds.

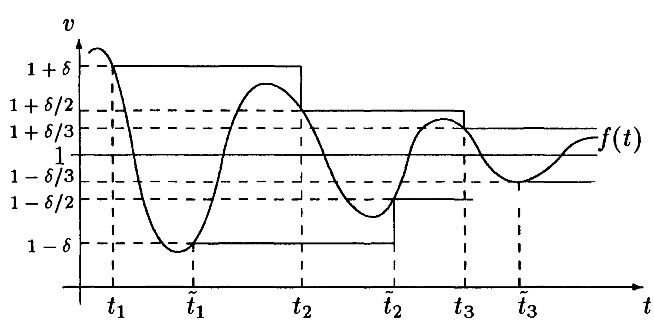
Let $v = u/u_0$. We then have (4.5). By (4.9) we know that

$$0 < k = \liminf_{t \to \infty} v(t) \leq \limsup_{t \to \infty} v(t) = l < \infty.$$

We first consider the case (i). Define the new function f by $f(t) = (1 + \varepsilon(t))^{1/(1+\sigma)}$ for sufficiently large t. Clearly $\lim_{t \to \infty} f(t) = 1$. If $v' = 0$ and $v > f(t)$, then $v'' > 0$ there, by (4.5). This means that only minima can occur in the region $v > f(t)$. Similarly, if $v' = 0$ and $0 < v < f(t)$, then $v'' < 0$ there. This means only maxima can occur in the region $0 < v < f(t)$. This simple observation works essentially below.

We may assume that $f(t)$ oscillates around 1 as $t \to \infty$ since the other case is even simpler. For sufficiently small $\delta > 0$, we can find two strictly increasing sequences $\{t_n\}$ and $\{\tilde{t}_n\}$ satisfying

$$\begin{cases} \lim_{n \to \infty} t_n = \lim_{n \to \infty} \tilde{t}_n = \infty, \\ f(t_n) = 1 + \dfrac{\delta}{n}, \qquad f(t) < 1 + \dfrac{\delta}{n}, \qquad t > t_n, \\ f(\tilde{t}_n) = 1 - \dfrac{\delta}{n}, \qquad f(t) > 1 - \dfrac{\delta}{n}, \qquad t > \tilde{t}_n, \qquad n \in \mathrm{N}. \end{cases}$$



Obviously, if $v(t)$ attains extrema at points (t, v) satisfying

$$t_n \leq t \leq t_{n+1}, \qquad v(t) > 1 + \frac{\delta}{n-1},$$

then only minima can occur. Similarly, if $v(t)$ attains extrema at points (t, v)

satisfying

$$\tilde{t}_n \le t \le \tilde{t}_{n+1}, \qquad 0 < v(t) < 1 - \frac{\delta}{n-1},$$

then only maxima can occur. This observation implies that $k \le 1 \le l$. Invoking Lemma 2.4, we obtain

$$\liminf_{t\to\infty} \frac{u''}{u_0''} \le k \le 1 \le l \le \limsup_{t\to\infty} \frac{u''}{u_0''}, \tag{4.11}$$

and hence

$$\liminf_{t\to\infty} \frac{(1+\varepsilon(t))e^{\alpha t}u^{-\sigma}}{e^{\alpha t}u_0^{-\sigma}} \le k \le 1 \le l \le \limsup_{t\to\infty} \frac{(1+\varepsilon(t))e^{\alpha t}u^{-\sigma}}{e^{\alpha t}u_0^{-\sigma}}. \tag{4.12}$$

This means that $l^{-\sigma} \le k \le 1 \le l \le k^{-\sigma}$, i.e., $1 \le kl^{\sigma}$ and $k^{\sigma}l \le 1$. Since $0 < \sigma < 1$, the inequalities (4.11) and (4.12) can hold only if $k = l = 1$. This implies $\lim_{t\to\infty} v(t) = 1$, i.e., $u \sim ce^{\beta t}$.

Next we consider the case (ii). As in the sublinear case, we can establish the boundedness of v' and v'', and $\int^{\infty}(v')^2 ds < \infty$. We find therefore that $\lim_{t\to\infty} v'(t) = 0$. Proceeding exactly as before we can get the desired conclusion $\lim_{t\to\infty} v(t) = 1$, i.e., $u \sim ce^{\beta t}$. The proof of Theorem 4.3 is complete.

5. The equation (E_{--})

For the equation (E_{--}), the next result also follows from [4, Theorem 20.26].

THEOREM 5.1. *If $\lambda > 1$, $\alpha > 0$, and $\varepsilon(t) \to 0$ as $t \to \infty$, then all positive proper solutions of*

$$u'' - (1+\varepsilon(t))e^{-\alpha t}u^{\lambda} = 0 \tag{E_{--}}$$

possess one of the following asymptotic forms

$$u = a_1 t + a_2 + \frac{(a_1 t)^{\lambda}}{\alpha^2} e^{-\alpha t}(1 + o(1)); \tag{5.1}$$

$$u = a + \frac{a^{\lambda}}{\alpha^2} e^{-\alpha t}(1 + o(1)); \tag{5.2}$$

$$u \sim u_0 \equiv ce^{\beta t}, \qquad \beta = \frac{\alpha}{\lambda - 1}, \qquad c = |\beta|^{2/(\lambda-1)}, \tag{5.3}$$

where $a > 0, a_1 > 0, a_2$ are constants.

The next result can be shown by [4, Theorem 20.21] when (5.4) below is assumed.

THEOREM 5.2. *Let* $0 < \lambda < 1$ *and either*

$$\lim_{t\to\infty} \varepsilon(t) = 0, \qquad \int^{\infty} |\varepsilon'(t)|dt < \infty \tag{5.4}$$

or

$$\lim_{t\to\infty} \varepsilon(t) = 0, \qquad \int^{\infty} [\varepsilon(t)]^2 dt < \infty \tag{5.5}$$

hold. Then all positive proper solutions of (E_{--}) *possess one of the asymptotic forms* (5.1), (5.2), *and* (5.3).

PROOF. Note that in this case $\beta < 0$. It remains to prove the assertion under the condition (5.5). The asymptotic forms (5.1) and (5.2) are obtained by the same method as in the proof of Theorem 3.1. Put $v = u/u_0$. We know by Lemma 2.5 that v solves

$$v'' + 2\beta v' + \beta^2(v - (1+\varepsilon(t))v^{\lambda}) = 0. \tag{5.6}$$

We shall show that $v \to 1$ or $v \to \infty$ as $t \to \infty$.

Step 1. We first show that v has a finite or infinite limit.

Let v be unbounded. To see $v(\infty) = \infty$, suppose the contrary that v has no limits. Then for sufficiently large N there exists an increasing sequence $\{t_k\}$ with $\lim_{t\to\infty} t_k = \infty$ such that $v(t_k) = N, v(t) \le N$ for $t_{2k-1} \le t \le t_{2k}, v(t) \ge N$ for $t_{2k} \le t \le t_{2k+1}$. Let $\{\xi_{2k}\}$ and $\{\xi_{2k-1}\}$ be the sequences such that $v(\xi_{2k}) = \max\{v(t) : t_{2k} \le t \le t_{2k+1}\}$ and $v(\xi_{2k-1}) = \min\{v(t) : t_{2k-1} \le t \le t_{2k+1}\}$, and let $v_l = v(\xi_l)$. We may assume that $\lim_{k\to\infty} v_{2k} = \infty$. Multiplying (5.6) by v', and integrating the resulting equation on $[\xi_{2k}, \xi_{2k+1}]$, we obtain

$$2\beta\int_{\xi_{2k}}^{\xi_{2k+1}} (v')^2 ds + \frac{\beta^2}{2}(v_{2k+1}^2 - v_{2k}^2) - \frac{\beta^2}{\lambda+1}(v_{2k+1}^{\lambda+1} - v_{2k}^{\lambda+1}) \tag{5.7}$$
$$= \beta^2\int_{\xi_{2k}}^{\xi_{2k+1}} \varepsilon(s)v^{\lambda}v' ds, \qquad k \in \mathrm{N}.$$

Since $\{v_{2k+1}\}$ is a bounded sequence, and $\beta < 0$, we find that

$$\frac{\beta^2}{2}v_{2k}^2 \le c_1 + \frac{\beta^2}{\lambda+1}v_{2k}^{\lambda+1} + \beta^2\left|\int_{\xi_{2k}}^{\xi_{2k+1}} \varepsilon(s)v^{\lambda}v' ds\right|, \qquad k \in \mathrm{N},$$

where $c_1 > 0$ is a constant. Schwarz's inequality implies that

$$(5.8)\qquad \int_{\xi_{2k}}^{\xi_{2k+1}} |\varepsilon(s)v^{\lambda}v'|ds \le v_{2k}^{\lambda}\left(\int_{\xi_{2k}}^{\xi_{2k+1}} [\varepsilon(s)]^2 ds\right)^{1/2}\left(\int_{\xi_{2k}}^{\xi_{2k+1}} (v')^2 ds\right)^{1/2}.$$

Let us put for simplicity $I_k = \left(\int_{\xi_{2k}}^{\xi_{2k+1}} (v')^2 ds\right)^{1/2}$, and $\varepsilon_k = \left(\int_{\xi_{2k}}^{\xi_{2k+1}} [\varepsilon(s)]^2 ds\right)^{1/2}$. From (5.7) and (5.8), we have

$$-\frac{\beta^2}{2}v_{2k}^2 + \frac{\beta^2}{\lambda+1}v_{2k}^{\lambda+1} + c_2 + c_3\varepsilon_k v_{2k}^{\lambda} I_k + 2\beta I_k^2 \ge 0, \qquad k \in \mathrm{N},$$

where $c_2, c_3 > 0$ are constants. This ineqality can be rewritten as

$$\left(I_k + \frac{c_3}{4\beta}\varepsilon_k v_{2k}^{\lambda}\right)^2 - \frac{c_3^2}{16\beta^2}\varepsilon_k^2 v_{2k}^{2\lambda} - \frac{\beta}{4}v_{2k}^2 + \frac{\beta}{2(\lambda+1)}v_{2k}^{\lambda+1} + \frac{c_2}{2\beta} \le 0,$$

from which we have

$$\frac{-\beta}{4}v_{2k}^2 - \frac{c_3^2}{16\beta^2}\varepsilon_k^2 v_{2k}^{2\lambda} + \frac{\beta}{2(\lambda+1)}v_{2k}^{\lambda+1} + \frac{c_2}{2\beta} \le 0.$$

Since $\varepsilon_k \to 0$ as $k \to \infty$, this contradicts the assumption that $\lim_{t\to\infty} v_{2k} = \infty$. Therefore $\lim_{t\to\infty} v(t) = \infty$.

Next let v be bounded. We show that a finite $\lim_{t\to\infty} v(t)$ exists. Multiplying (5.6) by v', and integrating the resulting equation on $[t_0, t]$, we obtain

$$(5.9)\qquad \frac{(v')^2}{2} - (-2\beta)\int_{t_0}^{t} (v')^2 ds + \frac{\beta^2 v^2}{2} - \frac{\beta^2 v^{\lambda+1}}{\lambda+1} - c_4 = \beta^2 \int_{t_0}^{t} \varepsilon(s)v^{\lambda}v' ds, \qquad t \ge t_0,$$

where c_4 is a constant. As in the proof of Theorem 4.2, invoking Schwarz's inequality in the right hand side of this formula, we have

$$\frac{(v')^2}{2} \ge c_5 + (-2\beta)\int_{t_0}^{t} (v')^2 ds - c_6\left(\int_{t_0}^{t} (v')^2 ds\right)^{1/2}, \qquad t \ge t_0,$$

where $c_5, c_6 > 0$ are constants. We can easily see that $\int^{\infty} (v')^2 ds < \infty$, and Lemma 2.1 shows that $\lim_{t\to\infty} v'(t) = 0$. That is, returning to (5.9), we find that $\lim_{t\to\infty} v(t)$ exists as a finite value.

From the above consideration it follows that

$$\lim_{t\to\infty} v(t) = \infty, \qquad \lim_{t\to\infty} v(t) = 0 \qquad \text{or} \qquad \lim_{t\to\infty} v(t) = 1.$$

Step 2. We show that actually $\lim_{t\to\infty} v(t) = 0$ does not occur.

Suppose the contrary that $\lim_{t\to\infty} v(t) = 0$. Recall, then, that $\lim_{t\to\infty} v'(t) = 0$ and $\int^{\infty} (v')^2 ds < \infty$. We easily see that there exists a large t_1 such that $v' < 0$ for $t \geq t_1$. From (5.6) we obtain

$$v'' \leq \beta^2(1+\varepsilon(t))v^{\lambda} \qquad \text{for large } t.$$

Multiplying the above inequality by v' and integrating from t to ∞, we have

$$\frac{(v')^2}{2} \leq \frac{\beta^2}{\lambda+1} v^{\lambda+1} - \beta^2 \int_t^{\infty} \varepsilon(s) v^{\lambda} v' ds.$$

It should be noted that the condition (5.5) ensure the convergence of the integral on the right hand side. l'Hospital's rule therefore implies that

$$\begin{aligned} \frac{(v')^2}{2} &\leq \frac{\beta^2}{\lambda+1} v^{\lambda+1}(1+o(1)) \\ &\leq \frac{2\beta^2 v^{\lambda+1}}{\lambda+1}(1+\delta_1) \qquad \text{for large } t, \end{aligned} \tag{5.10}$$

where $\delta_1 > 0$ is a sufficiently small constant. Since $0 < \lambda < 1$, for such a δ_1 we can choose δ such that $2\lambda(1+\delta_1)/(\lambda+1) < \delta < 1$. Furthermore for this δ, there exists t_2 such that

$$(1-\delta)v^{\lambda} - v > 0 \qquad \text{for } t \geq t_2.$$

We apply this estimate to (5.6). Then

$$v'' + 2\beta v' \geq \delta\beta^2 v^{\lambda}.$$

Multiply the above inequality by $v^{-\lambda}$ and integrate from t_1 to t and apply the estimate (5.10). Then,

$$\left(\delta\beta^2 - \frac{2\beta^2\lambda(1+\delta_1)}{\lambda+1}\right)(t-t_1) \leq c_1,$$

where c_1 is a constant, this contradicts the assumption for δ. Thus the limit of v is 1 or ∞.

To show that the solution v with infinite limit corresponds to the solution of (E_{--}) satisfying (5.1) or (5.2), we use Lemma 2.2.

A linearized equation of the equation (5.6) is

$$w'' + 2\beta w' + \beta^2 w = 0.$$

This equation has linearly independent solutions $w_1 = e^{-\beta t}, w_2 = te^{-\beta t}$. To use this lemma, we must show

$$\int^{\infty} |1+\varepsilon(s)| sv(s)^{\lambda-1} ds < \infty. \tag{5.11}$$

Since $v \to \infty$, for arbitrary constant $\gamma \in (0, 1/2)$ there exists t_2 such that

$$v'' + 2\beta v' + \beta^2 v > 0,$$

$$v'' + 2\beta v' + \beta^2(1-\gamma)v > 0, \qquad \text{for } t \le t_2.$$

By performing change of variables $v(t) = e^{-\beta t}x(t)$ in the above two inequalities, we get

$$x'' > 0, \tag{5.12}$$

$$x'' < \gamma\beta^2 x, \tag{5.13}$$

respectively. According to (5.12) we find that x' is eventually of definite sign.

When $x' > 0$ for $t \ge t_2$, integrating (5.12) from t_2 to t twice, we have

$$x(t) > x'(t_2)t + c_2,$$

where c_2 is a constant. This means that there exists a constant $c_3 > 0$ such that $v(t) \ge c_3 te^{-\beta t}$ for large t, and hence (5.11) is satisfied.

When $x' < 0$ for $t \ge t_3$, x has a finite limit. If $\lim_{t\to\infty} x(t) > 0$, then the condition of Lemma 2.2 is satisfied. If $\lim_{t\to\infty} x(t) = 0$, then multiplying (5.13) by x' and integrating from t to ∞, we get

$$\frac{x'}{x} > \beta\sqrt{\gamma}.$$

Integrating further from t_2 to t, we obtain

$$\log x(t) > \beta\sqrt{\gamma}(t - t_2) + c_4,$$

where c_4 is a constant. Consequently, it follows that

$$x(t) > c_5 e^{\beta\sqrt{\gamma}t},$$

where $c_5 > 0$ is a constant. This means that (5.11) is satisfied. Hence $v(t) = O(te^{-\beta t})$, and so $u(t) = O(t)$. The proof of Theorem 5.2 is complete.

THEOREM 5.3. *Let* $\lambda = -\sigma < 0$.

(i) *If* $0 < \sigma < 1$ *and* $\lim_{t\to\infty} \varepsilon(t) = 0$, *then all positive proper solutions of*

$$u'' - (1 + \varepsilon(t))e^{-\alpha t}u^{-\sigma} = 0 \tag{E_{--}}$$

have one of the asymptotic forms (5.1), (5.2) *or* (5.3).

(ii) *If* $\sigma \ge 1$, *and either* (5.4) *or* (5.5) *holds, then all positive proper solutions of* (E_{--}) *have one of the asymptotic forms* (5.1), (5.2) *or* (5.3).

PROOF. In this case $\beta < 0$. Positive solutions u satisfying $u' > 0$ near ∞ have the asymptotic form (5.1). Positive solutions satisfying $u' < 0$ near ∞ and

$u(\infty) \in (0, \infty)$ have the form (5.2). We show that positive solutions satisfying $u' < 0$ near ∞ and $u(\infty) = 0$ have the form (5.3).

As in the first step of the proof of Theorem 4.3, rewriting (E_{--}) as $u''u^\sigma = (1 + \varepsilon(t))e^{-\alpha t}$ and integrating on $[t, \infty)$, we obtain

$$-u'u^\sigma \geq \int_t^\infty (1 + \varepsilon(s))e^{-\alpha s}ds.$$

One more integration yields

$$u^{\sigma+1} \geq c_1 e^{-\alpha t}$$

for large t with some $c_1 > 0$. Thus we find that

$$0 < \liminf_{t\to\infty} \frac{u(t)}{e^{\beta t}} \leq \limsup_{t\to\infty} \frac{u(t)}{e^{\beta t}} < \infty. \tag{5.14}$$

Put $v = u/u_0$. Hence (5.14) implies

$$0 < \liminf_{t\to\infty} v(t) \leq \limsup_{t\to\infty} v(t) < \infty.$$

The remainder of the proof will be carried out by the same method as in the proofs of Theorems 4.2 and 4.3. We leave the detail to the reader.

6. Uniqueness of positive decaying proper solutions to the equation (E_{--})

By the results in §5 we find that the equation (E_{--}) may have positive proper solutions u satisfying $\lim_{t\to\infty} u(t) = 0$. Actually, when $0 < \lambda < 1$, Chanturiya [2] showed the existence of such solutions. When $\lambda < 0$, Usami [7] constructed such solutions. Therefore, it is natural to ask whether positive proper solutions u of (E_{--}) satisfying $\lim_{t\to\infty} u(t) = 0$ are unique or not. In the final section we answer this question affirmatively in the case where $\lambda \in (-\infty, 1)\backslash\{0\}$.

THEOREM 6.1. *Let $0 < \lambda < 1$. Suppose that the assumptions of Theorem 5.2 hold. Then, (E_{--}) has at most one positive proper solution u satisfying $\lim_{t\to\infty} u(t) = 0$.*

PROOF. Let $x(t)$ and $y(t)$ be positive proper solutions of (E_{--}) satisfying $x(\infty) = y(\infty) = 0$. By Theorem 5.2 we know that

$$x(t), y(t) \sim ce^{\beta t} \qquad \text{as } t \to \infty, \tag{6.1}$$

where c and β are given as in Theorem 5.2. Furthermore, it is easy to see that

$$x'(t), y'(t) \sim \beta ce^{\beta t} \qquad \text{as } t \to \infty. \tag{6.2}$$

Put $z(t) = x(t) - y(t)$. We show that $z \equiv 0$. By using the mean value theorem, we know that there exists a continuous function $\xi(t)$ such that

$$z'' = \lambda(1 + \varepsilon(t))e^{-\alpha t}[\xi(t)]^{\lambda-1}z, \tag{6.3}$$

and $\xi(t)$ lies between $x(t)$ and $y(t)$. By (6.1), $\xi(t) \sim ce^{\beta t}$, and this implies

$$\lim_{t\to\infty} \lambda(1 + \varepsilon(t))e^{-\alpha t}[\xi(t)]^{\lambda-1} = \lambda\beta^2.$$

Hence we can rewrite (6.3) as

$$z'' = (\lambda\beta^2 + \tilde{\varepsilon}(t))z, \tag{6.4}$$

where $\tilde{\varepsilon}(t) \to 0$ as $t \to \infty$. Now we reduce (6.4) to a first-order system by introducing the new variables $w_1 = z, w_2 = z'$:

$$\frac{dw}{dt} = (A + B(t))w, \tag{6.5}$$

where

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ \lambda\beta^2 & 0 \end{pmatrix}, \qquad B(t) = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\varepsilon}(t) \end{pmatrix}.$$

The eigenvalues of A are $\pm\sqrt{\lambda}\beta$. By Lemma 2.3, there exist solutions $(\bar{w}_1, \bar{w}_2)$ and $(\underline{w}_1, \underline{w}_2)$ of (6.5) satisfying

$$\begin{aligned} &c_1 \exp[(-\sqrt{\lambda}\beta + o(1))t] \le |\bar{w}_1| + |\bar{w}_2| \le c_2 \exp[(-\sqrt{\lambda}\beta + o(1))t], \\ &c_3 \exp[(\sqrt{\lambda}\beta + o(1))t] \le |\underline{w}_1| + |\underline{w}_2| \le c_4 \exp[(\sqrt{\lambda}\beta + o(1))t], \end{aligned} \tag{6.6}$$

where $c_1, c_2, c_3, c_4 > 0$ are constants. It is easily seen from (6.6) that $\{(\bar{w}_1, \bar{w}_2), (\underline{w}_1, \underline{w}_2)\}$ is a basis of the solution set of (6.5). Hence z is represented by

$$z = c_5\bar{w}_1 + c_6\underline{w}_1, \tag{6.7}$$

$$z' = c_5\bar{w}_2 + c_6\underline{w}_2 \tag{6.8}$$

where c_5 and c_6 are constants. We will show that $c_5 = c_6 = 0$. If $c_5 \ne 0$, then we find from (6.7) and (6.8) that

$$|\bar{w}_1| + |\bar{w}_2| \le \frac{|z| + |z'|}{|c_5|} + \left|\frac{c_6}{c_5}\right|(|\underline{w}_1| + |\underline{w}_2|). \tag{6.9}$$

By (6.6) we know that the left hand side of (6.9) tends to ∞ as $t \to \infty$, while the right hand side remains bounded. Hence we must have $c_5 = 0$, that is,

$$z = c_6\underline{w}_1, \qquad z' = c_6\underline{w}_2.$$

It follows therefore that

$$\frac{|z|+|z'|}{e^{\beta t}} = \frac{|c_6|(|\underline{w}_1|+|\underline{w}_2|)}{e^{\beta t}}. \tag{6.10}$$

By (6.1) and (6.2), the left hand side of (6.10) is bounded, while the right hand side of (6.10) is estimated as

$$\frac{|c_6|(|\underline{w}_1|+|\underline{w}_2|)}{e^{\beta t}} \geq c_3|c_6| \exp[(\beta(\sqrt{\lambda}-1)+o(1))t].$$

Since $0<\lambda<1$, if $c_6 \neq 0$, then the right hand side of (6.10) diverges as $t \to \infty$ and this is a contradiction. Hence $c_6 = 0$, i.e., $z \equiv 0$. The proof of Theorem 6.1 is complete.

THEOREM 6.2. *Let $\lambda = -\sigma < 0$. Suppose that assumptions of Theorem 5.3 hold. Suppose moreover that*

$$\limsup_{t\to\infty} \varepsilon'(t) < \frac{4\alpha}{\sigma+1}.$$

Then (E_{--}) has at most one positive proper solution u satisfying $\lim_{t\to\infty} u(t) = 0$.

PROOF. Let $x(t)$ and $y(t)$ be positive proper solutions of (E_{--}) satisfying $x(\infty) = y(\infty) = 0$. We know that

$$x(t), y(t) \sim ce^{\beta t} \quad \text{and} \quad x'(t), y'(t) \sim \beta ce^{\beta t} \quad \text{as } t \to \infty, \tag{6.11}$$

where c and β are given in Theorem 5.3.

Firstly we consider the case where $x(t) - y(t)$ is of definite sign for all sufficiently large t. Without loss of generality we may assume that $x(t) \geq y(t)$. Integrating (E_{--}) (with $u = x, y$) from t to ∞ twice, we have

$$\begin{aligned} x(t) &= \int_t^\infty \int_s^\infty (1+\varepsilon(r))e^{-\alpha r}(x(r))^{-\sigma} drds \\ &\leq \int_t^\infty \int_s^\infty (1+\varepsilon(r))e^{-\alpha r}(y(r))^{-\sigma} drds = y(t). \end{aligned}$$

Hence $x \equiv y$ near $+\infty$. Moreover we can easily show that $x \equiv y$ on the whole interval.

Secondly we consider the case where $x(t) - y(t)$ changes the sign infinitely many times near ∞. Put $z(t) = x(t)/y(t)$, and $p(t) = (1+\varepsilon(t))e^{-\alpha t}$ for simplicity. A computation gives

$$\frac{d^2z}{dt^2} + \frac{2x'(t)}{x(t)}\frac{dz}{dt} + \frac{p(t)}{[x(t)]^{1+\sigma}}(z - z^{-\sigma}) = 0, \qquad t \geq t_0.$$

The change of variable $\tau = \int_{t_1}^{t}[x(r)]^{-2}dr$ transforms this equation into

$$\frac{d^2z}{d\tau^2} + p(t(\tau))[x(t(\tau))]^{3-\sigma}(z - z^{-\sigma}) = 0, \qquad \tau \geq 0.$$

Moreover the change of variable $s = \int_0^{\tau} p(t(\xi))^{1/2}[x(t(\xi))]^{(3-\sigma)/2}d\xi$ transforms this equation into

$$\ddot{z} - f(s)\dot{z} + z - z^{-\sigma} = 0, \qquad s \geq 0 \tag{6.12}$$

where $\cdot = d/ds$, and

$$f(s) = -\tfrac{1}{2}[x(t)]^{(\sigma-1)/2}[p(t)]^{-3/2}\{p'(t)x(t) + (3-\sigma)p(t)x'(t)\}, \qquad s \geq 0.$$

Our assumption (6.11) implies that $f(s) > 0$ near $+\infty$, and $z(\infty) = 1$. Multiplying (6.12) by $\dot{z}$, and integrating on $[s_0, s]$, we have

$$\frac{[\dot{z}(s)]^2}{2} - \int_{s_0}^{s} f(r)(\dot{z}(r))^2dr + \frac{(z(s))^2}{2} - \frac{(z(s))^{1-\sigma}}{1-\sigma} = c_1, \qquad s \geq s_0 \tag{6.13}$$

where c_1 is a constant. We show that $\int^{\infty} f(s)[\dot{z}(s)]^2ds < \infty$. In fact, assuming the contrary that $\int^{\infty} f(s)[\dot{z}(s)]^2ds = \infty$, we obtain $\lim_{s\to\infty}(\dot{z}(s))^2 = \infty$ since $z(\infty) = 1$. But obviously this is a contradiction. Hence $\int^{\infty} f(s)(\dot{z}(s))^2ds < \infty$ as stated above. Returning to (6.13) we find that $\lim_{s\to\infty}(\dot{z}(s))^2$ exists as a finite value. The boundedness of z asserts therefore that $\lim_{s\to\infty}\dot{z}(s) = 0$.

On the other hand, there is an increasing sequence $\{s_n\}$ such that $\lim_{n\to\infty}s_n = \infty$, and $z(s_n) = 1, n \in \mathrm{N}$. This is an immediate consequence of the fact that $z(s)$ oscillates around 1. Multiplying (6.12) by $\dot{z}(s)$, and integrating on $[s_n, s_{n+1}]$, we have

$$\frac{[\dot{z}(s_{n+1})]^2}{2} - \frac{[\dot{z}(s_n)]^2}{2} = \int_{s_n}^{s_{n+1}} f(s)[\dot{z}(s)]^2ds.$$

This is an obvious contradiction unless $\dot{z} \equiv 0$ near ∞. Hence $z \equiv 1$ near ∞, that is, $x \equiv y$ on the whole interval. The proof of Theorem 6.2 is complete.

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