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On hypercoverings

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ABSTRACT. In the paper [3], a new notion of hypercovering was introduced to compute the higher cohomology groups of hypercoverings. This hypercovering has no degeneracy maps, and hence it is not a simplicial scheme.

In this paper, we study the basic properties of these hypercoverings. It has the notion of cosquelton like usual hypercoverings, and one can construct the hypercovering inductively using cosqueltons.

We construct intermediate objects corresponding to each simplicial ordered set. For example, the n-th cosquelton corresponds to $n-1$ sphere, and the n-th hypercovering corresponds to n-simplex. To each simplicial map corresponds a morphism contravariantly. Our main theorem says that this morphism is always a covering map.

1. Ordered system

DEFINITION 1.1. Let $\mathcal{N}$ be the category of *finite strictly ordered sets*. Namely, its objects are finite sets $\{\phi, \{1\}, \{1,2\}, \{1,2,3\}, \dots\}$, and morphisms are strictly increasing, say $f(i) < f(j)$ when $i < j$ (hence always injective). We denote the finite ordered set $\{1,2,\dots,n\}$ simply by $\mathbf{n}$ when confusion is unlikely. With this notation, the objects of $\mathcal{N}$ are $\{\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3},\dots\}$.

Define the morphism $\partial_i : \mathbf{n} \to \mathbf{n+1}$ for $i = 1,2,\dots,n+1$ by

$$\partial_i(j) = \begin{cases} j & (i > j) \\ j+1 & (i \le j) \end{cases}.$$

The truncated ordered set $\mathcal{N}_{[n]}$ is the full subcategory of $\mathcal{N}$ with objects $\{\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3},\dots,\mathbf{n}\}$.

REMARK 1.1.1. It is easy to check the following identity.

$$\partial_i \circ \partial_j = \begin{cases} \partial_j \circ \partial_{i-1} & (i > j) \\ \partial_{j+1} \circ \partial_i & (i \le j) \end{cases} \tag{1}$$

Any morphism $f : \mathbf{n} \to \mathbf{m}$ can be written as a composition of ∂_i's. In fact, when the set $\{i_1, \dots, i_{m-n}\}$ is the complement of the set $\{f(1), f(2), \dots, f(n)\}$

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in $\{1, 2, \ldots, m\}$ with $i_1 > i_2 > \cdots > i_{m-n}$, then we have

$$f = \partial_{i_1} \circ \partial_{i_2} \circ \cdots \circ \partial_{i_{m-n}} \tag{2}$$

In particular, the representation of f in this form is unique. On the other hand, for any composition of ∂_i's, by using relation (1), one can reduce the composition into the form as (2) with $i_1 > i_2 > \cdots > i_{m-n}$. Therefore, the morphisms in $\mathcal{N}$ are generated by ∂_i's, subject to the relation (1).

DEFINITION 1.2. *A (Scheme valued) ordered system* is a contravariant functor F from $\mathcal{N}$ to the category of schemes.

A partial orderd system (of level n) is a contravariant functor from $\mathcal{N}_{[n]}$.

REMARK 1.2.1. By Remark 1.1.1, to determine a (partial) ordered system F, it is enough to determine schemes $F(\mathbf{n})$ and morphisms $F(\partial_i) : F(\mathbf{n}) \to F(\mathbf{n-1})$ such that

$$F(\partial_j) \circ F(\partial_i) = \begin{cases} F(\partial_{i-1}) \circ F(\partial_j) & (i > j) \\ F(\partial_i) \circ F(\partial_{j+1}) & (i \leq j) \end{cases}.$$

DEFINITION 1.3. Let $A \subset \{1, 2, 3, \ldots\}$ be a finite subset of natural numbers with number of elements d. When F is a (partial) ordered system, then we define $F(A)$ to be $F(d) = F(\sharp A)$. When F is a partial ordered set of level n, then we assume that $d \leq n$.

REMARK 1.3.1. When $A \subset B \subset \{1, 2, 3, \ldots\}$ are finite subsets of natural numbers, then there is a canonical morphism from $F(B) \to F(A)$. For example, when $A = \{2, 4, 6\}$ and $B = \{2, 4, 5, 6\}$, then the natural map is $F(\partial_3) : F(B) \to F(A)$, because A skips the third element in B.

DEFINITION 1.4. Let A and B be finite subsets of natural numbers. By Remark 1.3.1, there are canonical morphisms $F(A) \to F(A \cap B)$ and $F(B) \to F(A \cap B)$. Define the *incidental subscheme* Z of $F(A) \times F(B)$ by the following fibre diagram:

$$\begin{array}{ccc} Z & \longrightarrow & F(A) \times F(B) \\ \downarrow & \square & \downarrow \\ F(A \cap B) & \xrightarrow{\Delta} & F(A \cap B) \times F(A \cap B) \end{array}$$

In other words, the incidental subscheme Z is the fibre product of $F(A)$ and $F(B)$ over $F(A \cap B)$.

DEFINITION 1.5. Let F be a (partially) ordered system, and $A_1, \ldots, A_r \subset \{1, 2, 3, \ldots\}$ be finite subsets of natural numbers. We define *the marked* *product* $F[A_1, A_2, \dots, A_r]$ be the closed subscheme of $F(A_1) \times \cdots \times F(A_r)$, defined in the following fiber diagram:

$$\begin{array}{ccc} F[A_1,\dots,A_r] & \longrightarrow & F(A_1) \times \cdots \times F(A_r) \\ \downarrow & \square & \downarrow \\ \prod_{i<j} Z_{ij} & \longrightarrow & \prod_{i<j} F(A_i) \times F(A_j) \end{array}$$

Here Z_{ij} is the incidental subscheme of $F(A_i) \times F(A_j)$.

REMARK 1.5.1. Set theoretically, a geometric point of $F[A_1, \dots, A_r]$ is a point $(x_1, \dots, x_r) \in F(A_1) \times \cdots \times F(A_r)$ such that any two components x_i and x_j are compatible in $F(A_i \cap A_j)$, the largest possible common image from $F(A_i)$ and $F(A_j)$.

REMARK 1.5.2. There is a natural projection map $F[A_1, \dots, A_r] \to F(A_i)$, which is the composition of $F[A_1, \dots, A_r] \to F(A_1) \times \cdots \times F(A_r)$ and the projection $F(A_1) \times \cdots \times F(A_r) \to F(A_i)$.

LEMMA 1.5.3. *Let B and $A_1, \dots, A_r$ be finite subsets of natural numbers. Assume that $A_i \supset B$ for some i. Then one can compose the projection $F[A_1, \dots, A_r] \to F(A_i)$ defined in Remark* 1.5.2 *and the morphism $F(A_i) \to F(B)$ defined in Remark* 1.3.1. *Then the composition $F[A_1, \dots, A_r] \to F(B)$ is independent of the choice of A_i which contains B.*

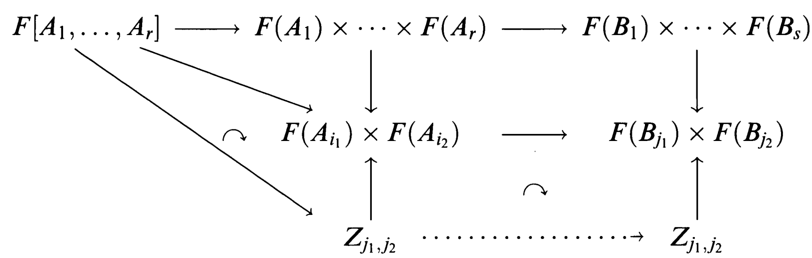
PROOF: Assume that A_j is another set which contains B. Consider the composition of canonical morphisms $F[A_1, \dots, A_r] \to F(A_i) \times F(A_j) \to F(B) \times F(B)$. It is enough to show that this morphism factors through $F(B) \xrightarrow{\Delta} F(B) \times F(B)$. Noticing that $F(A_i) \times F(A_j) \to F(B) \times F(B)$ factors through $F(A_i \cap A_j) \times F(A_i \cap A_j)$, we consider the diagram below:

$$\begin{array}{ccccc} F[A_1,\dots,A_r] & \longrightarrow & F(A_i \cap A_j) \times F(A_i \cap A_j) & \longrightarrow & F(B) \times F(B) \\ & \searrow^{\cdots} & \uparrow \Delta & \curvearrowright & \uparrow \Delta \\ & & F(A_i \cap A_j) & \longrightarrow & F(B) \end{array}$$

It is enough to show that there exists a morphism along the dotted arrow which makes the diagram commtative. It is obvious from the definition of the marked product. □

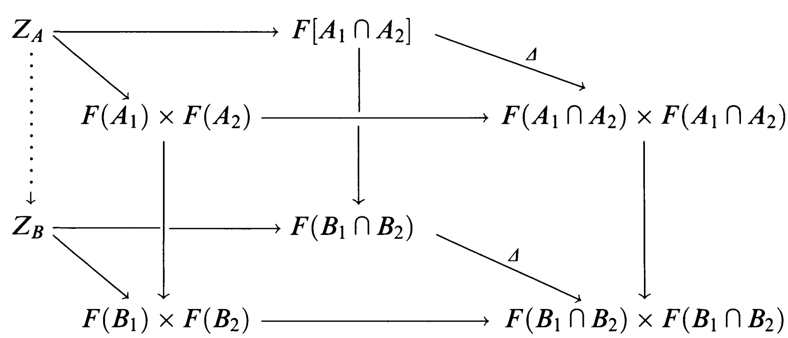
PROPOSITION 1.6. *Let F be an ordered system, and $A_1, \dots, A_r$ and $B_1, \dots, B_s$ be finite subsets of natural numbers. Assume that for each B_j, there exists some A_i such that $A_i \supset B_j$. Then there is a natural morphism $F[A_1, \dots, A_r] \to F[B_1, \dots, B_s]$ such that the composition with the projections $F[B_1, \dots, B_s] \to F(B_j)$ are the morphisms defined in Lemma* 1.5.3.

PROOF: By Lemma 1.5.3, there is a natural morphism $F[A_1, \dots, A_r] \to F(B_1) \times \cdots \times F(B_s)$. We need to show that this morphism factors through the marked product $F[B_1, \dots, B_s]$. We need only to show that the composition $F[A_1, \dots, A_r] \to F(B_1) \times \cdots \times F(B_s) \to F(B_{j_1}) \times F(B_{j_2})$ factors through the incidental subscheme $Z_{j_1, j_2} \subset F(B_{j_1}) \times F(B_{j_2})$ for all pair B_{j_1} and B_{j_2}, by construction of the marked product $F[B_1, \dots, B_s]$. Assume that $A_{i_1} \supset B_{j_1}$ and $A_{i_2} \supset B_{j_2}$. Consider the diagram below, where Z_{i_1, i_2} and Z_{j_1, j_2} are suitable incidental schemes:



It is enough to show that there is a mophism $Z_{i_1, i_2} \to Z_{j_1, j_2}$ as dotted arrow in the diagram, which makes the square commute.

Now denote Z_{i_1, i_2} as Z_A, and Z_{j_1, j_2} as Z_B. Also we rename A_1 and A_2 for A_{i_1} and A_{i_2}, and B_1 and B_2 for B_{j_1} and B_{j_2}. Let us consider the diagram:



We need to fill in the dotted arrow. The top and the bottom squares are fiber diagrams, and the front and the right behined vertical squares are commutative. Hence an easy diagram chasing shows that the composition

$$Z_A \to F(A_1) \times F(A_2) \to F(B_1) \times F(B_2) \to F(B_1 \cap B_2) \times F(B_1 \cap B_2)$$

is same as the composition

$$Z_A \to F(A_1 \cap A_2) \to F(B_1 \cap B_2) \to F(B_1) \times F(B_2) \to F(B_1 \cap B_2) \times F(B_1 \cap B_2)$$

which implies that there is a unique morphism $Z_A \to Z_B$ which makes all the squares commute. □

Corollary 1.6.1. *Let F be an ordered system and $A_1, \dots, A_r$ be finite subsets of natural numbers. When $A_i \subset A_j$ for some $i \neq j$, then omitting A_i does not change the marked product, namely $F[A_1, \dots, A_r] \simeq F[A_1, \dots, \hat{A_i}, \dots, A_r]$.*

Proof: By Proposition 1.6, there are natural morphisms

$$f : F[A_1, \dots, A_r] \to F[A_1, \dots, \hat{A_i}, \dots, A_r]$$

and

$$g : F[A_1, \dots, \hat{A_i}, \dots, A_r] \to F[A_1, \dots, A_r].$$

Then $f \circ g$ and $g \circ f$ are again natural morphisms, hence the identity morphisms. Therefore f and g are inverse to each other, and $F[A_1, \dots, A_r]$ and $F[A_1, \dots, \hat{A_i}, \dots, A_r]$ are isomorphic to each other. □

Definition 1.7. When $A_i \not\subset A_j$ for all $i \neq j$, we call the marked product $F[A_1, \dots, A_r]$ to have the reduced representation.

Proposition 1.8. *Let F be a (partial) ordered system, and $A_1, \dots, A_r$ and $B_1, \dots, B_s$ finte subsets of natural numbers. Then the marked product $F[A_1, \dots, A_r, B_1, \dots, B_s]$ is the fibre product of $F[A_1, \dots, A_r]$ and $F[B_1, \dots, B_s]$ over $F[A_1 \cap B_1, A_1 \cap B_2, \dots, A_i \cap B_j, \dots, A_r \cap B_s]$.*

Proof: By Proposition 1.6, there are canonical morphisms

$$F[A_1, \dots, A_r, B_1, \dots, B_s] \to F[A_1, \dots, A_r]$$

and

$$F[A_1, \dots, A_r, B_1, \dots, B_s] \to F[B_1, \dots, B_r]$$

whose compositions to $F[A_1 \cap B_1, \dots, A_r \cap B_s]$ agree. Therefore there is a canonical morphism from $F[A_1, \dots, A_r, B_1, \dots, B_s]$ to the fiber product of $F[A_1, \dots, A_r]$ and $F[B_1, \dots, B_s]$ over $F[A_1 \cap B_1, A_1 \cap B_2, \dots, A_i \cap B_j, \dots, A_r \cap B_s]$. These two schemes are closed subschemes of $F(A_1) \times \cdots \times F(A_r) \times F(B_1) \times \cdots \times F(B_s)$. Therefore, it is enough to show that the morphism from the fibre product of $F[A_1, \dots, A_r]$ and $F[B_1, \dots, B_s]$ over $F[A_1 \cap B_1, A_1 \cap B_2, \dots, A_i \cap B_j, \dots, A_r \cap B_s]$ to $F(A_i) \times F(A_j)$, $F(A_i) \times F(B_j)$ and $F(B_i) \times F(B_j)$ factor through the incidental subchemes. We need only to check that the morphism to $F(A_i) \times F(B_j)$ factors through the incidental subscheme, but that is exatly the condition enforced by taking the fiber product over $F[A_1 \cap B_1, \dots, A_r \cap B_s]$. □

2. Hypercovering

Definition 2.1. A collection of morphisms $\mathscr{C} = \{f : X \to Y\}$ determines a Grothendieck topology when it satisfies the following axioms.

1. For each $f \in \mathscr{C}$, f is epi.
2. $\mathscr{C}$ is stable under base extensions, namely for any morphism $\tilde{Y} \to Y$, the base extension $\tilde{f} : X \times_Y \tilde{Y} \to \tilde{Y}$ of f is in $\mathscr{C}$.
3. $\mathscr{C}$ is stable under compositions, namely when $f : X \to Y$ and $g : Y \to Z$ are in $\mathscr{C}$, then $g \circ f$ is also contained in $\mathscr{C}$.

When $f \in \mathscr{C}$, we call f as a *covering map.*

EXAMPLE 2.2. The collection of surjective étale morphisms determines a Grothendieck topology. Also the collection of proper surjective morphisms determines a Grothendieck topology.

In this section, we fix a Grothendieck topology $\mathscr{C}$, and discuss covering maps in terms of $\mathscr{C}$.

DEFINITION 2.3. Let F be a (partial) ordered system. For each $i = 1, 2, \dots, n$, let $A_i = \{1, 2, \dots, \hat{i}, \dots, n\}$ be the subset of $\{1, 2, \dots, n\}$ with $n-1$-elements. We call $F[A_1, \dots, A_n]$ as *the cosquelton* of F, and denote it as $\mathrm{cosk}_n(F)$.

REMARK 2.3.1. By Proposition 1.6, there is a canonical morphism $F(\mathbf{n}) \to F[A_1, \dots, A_n] = \mathrm{cosk}_n(F)$. This morphism plays an important role in the theory of hypercovering.

REMARK 2.3.2. One can construct ordered system inductively using cosqueltons. Let F be a partial ordered system of level $n-1$, and let $\mathrm{cosk}_n(F)$ the cosquelton. Choose any scheme, name it $F(\mathbf{n})$, and take any morphism $F(\mathbf{n}) \to \mathrm{cosk}_n(F)$, then defining $F(\partial_i)$ to be the compositon $F(\mathbf{n}) \to \mathrm{cosk}_n(F) \to F(A_i) = F(\mathbf{n-1})$, the condition of Remark 1.2.1 is automatically satisfied, hence F is extended to partial ordered system of level n.

DEFINITION 2.4. Let F be an ordered system. We call F to be a *hypercovering* if for any n, the canonial morphism $F(\mathbf{n}) \to \mathrm{cosk}_n(F)$ is a covering map.

LEMMA 2.4.1. *Let F be a (partial) ordered system, and $A_1, \dots, A_r$ be finite subsets of natural numbers. Let $n := \sharp A_1$, and define $B_1, \dots, B_n$ be the subsets of A_1 with exactly $n-1$ elements. Consider the morphism $\varphi(A_1)$: $F[A_1, A_2, \dots, A_r] \to F[B_1, \dots, B_n, A_2, \dots, A_r]$ as in Proposition* 1.6. *When A_1 is contained in some other A_i, then $\varphi(A_1)$ is an isomorphism, and otherwise, it is a base extension of $F(\mathbf{n}) \to \mathrm{cosk}_n(F)$.*

PROOF: The case $A_1 \subset A_i$ is obvious from Corollary 1.6.1. Assume that A_1 is not contained in any other A_i. By Proposition 1.8, the marked product $F[A_1, \dots, A_r]$ is the fiber produt of $F[B_1, \dots, B_n, A_2, \dots, A_r]$ and $F[A_1]$ over $F[A_1 \cap B_1, \dots, A_1 \cap B_n, A_1 \cap A_2, \dots, A_1 \cap A_r]$. Because A_1 is not contained in any A_i with $i \neq 1$, $A_1 \cap A_i$ is contained in some B_j. Also we have $A_1 \cap B_j = B_j$. So by Corollary 1.6.1, $F[A_1 \cap B_1, \ldots, A_1 \cap B_n, A_1 \cap A_2, \ldots, A_1 \cap A_r]$ is isomorphic to $F[B_1, \ldots, B_n]$, and hence the morphism $F[A_1, \ldots, A_n] \to F[B_1, \ldots, B_n, A_2, \ldots, A_r]$ is the base extension of $F[A_1] \to F[B_1, \ldots, B_n]$, which can be identified with $F[\mathbf{n}] \to \operatorname{cosk}_n(F)$. □

Definition 2.5. Let A be a finite subset of natural numbers with n elemetns. Define $d(A) := \{B_1, \ldots, B_n\}$ be the subsets of A with exactly $n-1$ elements. When $B_1, \ldots, B_n$ are elements of $d(A)$ and $A_1, \ldots, A_r$ any finite subsets of natural numbers, we will denote the marked product $F[B_1, , \ldots, B_n, A_1, \ldots, A_r]$ simply by $F[d(A), A_1, \ldots, A_r]$.

Theorem 2.6. *Let F be a hypercovering, $A_1, \ldots, A_r, B_1, \ldots, B_s$ finite subsets of natural numbers such that any B_j is contained in some A_i. Then the canonical morphism $F[A_1, \ldots, A_r] \to F[B_1, \ldots, B_s]$ is a coverint map.*

Proof: Consider the factorization of the morphism $F[A_1, \ldots, A_r] \to F[B_1, \ldots, B_s]$ into

$$\begin{aligned} F[A_1, A_2, \ldots, A_r] &= F[A_1, \ldots, A_r, B_1, \ldots, B_s] \\ &\to F[d(A_1), A_2, \ldots, A_r, B_1, \ldots, B_s] \\ &\to F[d(A_1), d(A_2), A_3, \ldots, B_1, \ldots, B_s] \\ &\to \cdots \\ &\to F[d(A_1), \ldots, d(A_r), B_1, \ldots, B_s] \end{aligned}$$

Then by Lemma 2.4.1, each step is either isomorphism or a base extension of the morphism $F(\mathbf{n}) \to \operatorname{cosk}_n(F)$ for some n, hence a covering map. Therefore, the composition

$$F[A_1, \ldots, A_r] \to F[d(A_1), \ldots, d(A_r), B_1, \ldots, B_s]$$

is also a covering map. Now using the induction on $\operatorname{Max}(\sharp A_i - \sharp B_j)$, where the maximum is taken over all $A_i \supset B_j$, we have

$$F[d(A_1), \ldots, d(A_r), B_1, \ldots, B_s] \to F[B_1, \ldots, B_s]$$

is a covering map. So the composition $F[A_1, \ldots, A_r] \to F[B_1, \ldots, B_s]$ is also a covering map. □

3. Conjectures

This research is conducted in the hope that it might be of help to prove Conjecture 3.1. In this section, we explain what is needed for that goal.

Conjecture 3.1. *Let F be a hypercover with all $F(n)$ Alexander (e.g., smooth). Then for any bivariant sheaf $\mathscr{F}$ on $F(0)$, the Čech cohomology $\check{H}^i(\mathscr{F}, F) = 0$ for all $i > 0$.*

Remark 3.1.1. The cohomology group of a bivariant sheaf $\mathscr{F}$ on X coincides with the inductive limit $\varinjlim \check{H}^{\cdot}(\mathscr{F}, F)$, where F runs over the hypercover with $F(0) = X$. Conjecture 3.1 implies that once we construct a hypercover F with $F(0) = X$ and $F(n)$ Alexander for $n > 0$, then $H^i(X, \mathscr{F}) \simeq \check{H}^i(\mathscr{F}, F)$.

Conjecture 3.2. *Let F be a hypercover such that $F(n)$ is Alexander for all n. Then there exist cycles $c_n \in A_* F(n)$ for all n which satisfy the following conditions* (1) *and* (2).

(1) $\partial_{0*} c_n = c_{n-1}$

(2) *Consider the diagram below:*

$$\begin{array}{ccc} F(\{0,1,\dots,\hat{\imath},\dots,n-1\},\{1,2,\dots,n-1\}) & \longrightarrow & F(\{0,1,\dots,\hat{\imath},\dots,n-1\}) \\ \downarrow & \square & \downarrow \partial_0 \\ F(\{1,2,\dots,n-1\}) & \longrightarrow & F(\{1,2,\dots,\hat{\imath},\dots,n-1\}) \end{array}$$

Then the cycle $c_{n+1} \in A_* F(n+1)$ is pushed-forward to $\partial_0^! c_n \in A_* F(\{0,1,\dots,\hat{\imath},\dots,n-1\},\{1,2,\dots,n-1\})$ for $i = 1,2,\dots,n-1$.

Proposition 3.3. *Conjecture* 3.2 *implies Conjecture* 3.1.

Proof: Consider $c_n \in A_* F(n) \simeq A(F(n) \to F(n-1))$ as a bivariant intersection class, where the morphism $F(n) \to F(n-1)$ is ∂_0. Then c_n induces a correspondence $h_n : F(n-1) \vdash F(n)$, and the conditions imply that $\{h_n\}$ determine a homotopy for the identity map $F(\bullet) \to F(\bullet)$, therefore, for any bivariant sheaf $\mathscr{F}$ on $F(0)$, the complex $\mathscr{F}(F(\bullet))$ is exact. □

References

[1] M. Artin and B. Mazur, Etale Homotopy, Lecture Notes in Mathematics 100, Springer-Verlag, 1969.

[2] Pierre Deligne, Théorie de hodge III, Inst. Hautes Études Sci. Publ. Math. **44** (1974), 5–77.

[3] Shun-ichi Kimura, Is alexander property étale local? preprint, 1998.

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