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Chemotactic collapse in a parabolic system of mathematical biology

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ABSTRACT. In 1970, Keller and Segel proposed a parabolic system describing the chemotactic feature of cellular slime molds and recently, several mathematical works have been devoted to it. In the present paper, we study its blowup mechamism and prove the following. First, chemotactic collapse occurs at each isolated blowup point. Next, any blowup point is isolated, provided that the Lyapunov function is bounded from below. Finally, only the origin can be a blowup point of radially symmetric solutions.

1. Introduction

A system of parabolic partial differential equations of mathematical biology is attracting interest. It was proposed by Nanjundiah [22] in 1973, as a simplified model of the Keller and Segel system [16] describing a chemotactic feature, the aggregation of some organisms (cellular slime molds) sensitive to gradient of a chemical substance. Precisely, with $u(x,t)$ and $v(x,t)$ standing for the density of the organism and the concentration of the chemical substance at the position $x \in \Omega$ and the time $t \in (0,T)$, respectively, it is given as

$$\text{(KS)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\nabla u - \chi u \nabla v) & \text{in } \Omega \times (0,T) \\ \tau \dfrac{\partial v}{\partial t} = \Delta v - \gamma v + \alpha u & \text{in } \Omega \times (0,T) \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T) \\ u(\cdot,0) = u_0, \qquad v(\cdot,0) = v_0 & \text{on } \Omega, \end{cases}$$

where
(A1) τ, α, γ and χ are positive constants
(A2) Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$
(A3) n denotes the unit outer normal vector

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(A4) u_0 and v_0 are smooth, nonnegative, and nontrivial initial values on $\bar{\Omega}$.
The first equation describes the conservation of mass. Flux of u is given by

$$\mathscr{F} = -\nabla u + \chi u \nabla v$$

so that the effect of diffusion $\nabla \cdot \nabla u$ and that of chemotaxis $\nabla \cdot \chi u \nabla v$ are competing for u to vary. The second equation is linear, and v is produced proportionarily to u, diffuses, and is destroyed by a certain rate.

The phenomenon of the blowup in a finite time of the solution is important from both mathematical and biological points of view. There are conjectures by Nanjundiah [22], Childress [5], and Childress and Percus [6]; $c_* = c^* = c = 8\pi/(\alpha\chi)$ is the threshold number in the following sense: if $\|u_0\|_{L^1(\Omega)} < c_*$ then the solution exists globally in time and if $\|u_0\|_{L^1(\Omega)} > c^*$ then $u(x,t)$ can form a delta function singularity in a finite time. The latter case is referred to as the *chemotactic collapse*. The arguments were heuristic, making use of numerical computations for the stationary problem, while recent studies are supporting their validity rigorously ([12], [14], [19], [20] and [21]).

First, the existence of such numbers c_* and c^* was proven by Jäger and Luckhaus [14] for a simplified system. Later, Nagai [19] treated another system, (KS) with $\tau = 0$, referred to as *N model* in the present paper; as [6] conjectured, $8\pi/(\alpha\chi)$ is actually the threshold number in the above sense for radially symmetric solutions. Then, several works were devoted to the full system, (KS) with $\tau > 0$. Particularly, Herrero and Velázquez [12] constructed a radially symmetric solution with u collapsing at the origin in a finite time, having the concentrated mass equal to $8\pi/(\alpha\chi)$. Its counter part was shown by Nagai, Senba, and Yoshida [21]; radial solutions exist globally in time with uniformly bounded, provided that $\|u_0\|_{L^1(\Omega)} < 8\pi/(\alpha\chi)$. In this way, conjecture [5] has been almost settled down in the affirmative for radially symmetric solutions.

As for the general case, contrarily to the conjecture, [21] gave only

$$\|u_0\|_{L^1(\Omega)} < 4\pi/(\alpha\chi)$$

as a criterion for the existence of global solutions. (The same result is obtained by Biler [3] and Gajewski and Zacharias [7] independently.) But this number $4\pi/(\alpha\chi)$ is also realized to be best possible and the reason for the discrepancy between radial and non-radial cases has been clarified by Nagai, Senba and Suzuki [20] and Senba and Suzuki [23]. Namely, the former studied N model and showed, among others, that if $4\pi/(\alpha\chi) \le \|u_0\|_{L^1(\Omega)} < 8\pi/(\alpha\chi)$ and the solution blows up in a finite time then the concentration toward $\partial\Omega$ occurs to u. (This phenomenon is proven also for the full system recently by Senba and Suzuki [25] and Harada, Nagai, Senba, and Suzuki [10].) On the other hand the latter studied the stationary problem in details;

the underlying variational structure and its effects to the dynamics. In particular, it asserts that many non-radial stationary solutions, missed by [6], exist and take roles in non-stationary problems even in the case that Ω is a disc.

Through those studies we are led to the following conjecture:

Component u forms a delta function singularity at each blowup point $x_0 \in \bar{\Omega}$ *with the concentrated mass equal to* $8\pi/(\alpha\chi)$ *and* $4\pi/(\alpha\chi)$ *according to* $x_0 \in \Omega$ *and* $x_0 \in \partial\Omega$*, respectively.*

Actually Senba and Suzuki [24] studied N model and proved the above phenomenon with the mass greater than or equal to the expected values. The present paper studies the full system and proves the following; if the solution (u, v) blows-up in a finite time, then u forms a delta function singularity at each *isolated* blowup point, and any blowup point is isolated, provided that the Lyapunov function described below is bounded. Finally, only the origin can be a blowup point of radially symmetric solutions.

2. Summary

Let us put that

$$\tau = \alpha = \gamma = \chi = 1$$

for simplicity. The following facts are known.

1. ([27], [3]) *Given smooth nonnegative initial data* $u_0 \not\equiv 0$ *and* v_0*, we have a unique classical solution* $(u(\cdot, t), v(\cdot, t))$ *to (KS) defined on the maximal time interval* $[0, T_{\max})$*. The solution is smooth and positive on* $\bar{\Omega} \times (0, T_{\max})$*. If* $T_{\max} < +\infty$*, then it holds that*

$$\limsup_{t \uparrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = +\infty.$$

2. ([21], [3], [7]) *Putting*

$$W(t) = \int_\Omega \left\{ u \log u - uv + \frac{1}{2}(|\nabla v|^2 + v^2) \right\} dx,$$

we have

$$\frac{d}{dt} W(t) + \int_\Omega v_t^2 \, dx + \int_\Omega u |\nabla \cdot (\log u - v)|^2 dx = 0.$$

In particular, $W(t)$ *is a Lyapunov function. It is monotone decreasing, so that either*

$$\inf_{0 \le t < T_{\max}} W(t) > -\infty \tag{1}$$

or

$$\lim_{t\uparrow T_{\max}} W(t) = -\infty$$

holds.

We prepare several notations and definitions.

Notation

(i) $B(x_0, R) = \{x \in \mathbf{R}^2 \mid |x - x_0| < R\}$, where $x_0 \in \mathbf{R}^2$ and $R > 0$

(ii) $A(x_0, r, R) = B(x_0, R) \backslash B(x_0, r)$

(iii) $\mathscr{M}(\mathscr{S}) = \{\text{Radon measures on } \mathscr{S}\}$, where $\mathscr{S}$ denotes a compact Hausdorff space

(iv) $w^* - \lim =$ weak star limit in $\mathscr{M}(\mathscr{S})$

(v) $\delta(\cdot) =$ Dirac's delta function concentrated at $x = 0$ in $\mathbf{R}^2$ and $\delta_{x_0}(\cdot) = \delta(\cdot - x_0)$ for $x_0 \in \mathbf{R}^2$

(vi) $|\Omega| =$ the Lebegue measure of $\Omega \subset \mathbf{R}^2$

DEFINITION

(i) In the case of $T_{\max} < +\infty$, we say that $x_0 \in \bar{\Omega}$ is a blowup point of u if there exist $\{t_k\}_{k=1}^{\infty} \subset [0, T_{\max})$ and $\{x_k\}_{k=1}^{\infty} \subset \bar{\Omega}$ satisfying $u(x_k, t_k) \to +\infty$, $t_k \to T_{\max}$, and $x_k \to x_0$ as $k \to \infty$. The set of blowup points of u is denoted by $\mathscr{B}$.

(ii) We say that $x_0 \in \mathscr{B}$ is isolated if there exists $R > 0$ such that

$$\sup_{0 \le t < T_{\max}} \|u(\cdot, t)\|_{L^\infty(A(x_0, r, R) \cap \Omega)} < +\infty$$

for any $r \in (0, R)$. The set of isolated blowup points of u is denoted by $\mathscr{B}_I$.

(iii) System (KS) is called radially symmetric if $\Omega = \{x \in \mathbf{R}^2 \mid |x| < 1\}$ and $u_0 = u_0(|x|)$, $v_0 = v_0(|x|)$.

Our results are stated as follows.

THEOREM 1. *Given $x_0 \in \mathscr{B}_I$, we have $0 < R \ll 1$, $m \ge m_*$, and*

$$f \in L^1(B(x_0, R) \cap \Omega) \cap C(\overline{B(x_0, R) \cap \Omega} \backslash \{x_0\}),$$

satisfying $f \ge 0$ and

$$w^* - \lim_{t\uparrow T_{\max}} u(\cdot, t)dx = m\delta_{x_0}(dx) + f\,dx \tag{2}$$

in $\mathscr{M}(\overline{B(x_0, R) \cap \Omega})$, where

$$m_* = \begin{cases} 8\pi & (x_0 \in \Omega) \\ 4\pi & (x_0 \in \partial\Omega) \end{cases}$$

THEOREM 2. *If* (1) *occurs, then* $\mathscr{B} = \mathscr{B}_I$.

THEOREM 3. *If (KS) is radially symmetric and* $T_{\max} < +\infty$, *then* $\mathscr{B} = \{0\}$.

In our notation, the delta function $\delta_{x_0}(dx) \in \mathscr{M}(\bar{\Omega})$ acts as

$$\langle \eta, \delta_{x_0}(dx) \rangle = \eta(x_0)$$

for $x_0 \in \bar{\Omega}$ and $\eta \in C(\bar{\Omega})$. It is easy to see that L^1 norm of $u(\cdot, t)$ is preserved (see section 3). Therefore, Theorem 1 implies that the number of isolated blowup points is finite. More precisely,

$$2 \times \sharp(\mathscr{B}_I \cap \Omega) + \sharp(\mathscr{B}_I \cap \partial\Omega) \leq \|u_0\|_{L^1(\Omega)}/4\pi.$$

Condition (1) actually holds for the blowup solution constructed by [12]. However, except for this example any criteria for (1) have not been known. In this connection, it may be worth mentioning about the semilinear heat equation

$$u_t = \Delta u + |u|^{p-1}u \qquad \text{in } \Omega \times (0, T) \qquad \text{with } u|_{\partial\Omega} = 0 \tag{3}$$

on a bounded domain $\Omega \subset \mathbf{R}^n$. For the subcritical case $1 < p < \dfrac{n+2}{n-2}$, it is known that blowup occurs if and only if $\lim_{t \uparrow T_{\max}} J(u(t)) = -\infty$, where

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}$$

stands for the Lyapunov function ([8], [13], e.g.). To our knowledge, it has not been clarified whether $\lim_{t \uparrow T_{\max}} J(u(t)) > -\infty$ and $T_{\max} < +\infty$ can occur for the other cases of (3). But those relations between Lyapunov functions and blowup mechanisms may suggest that (KS) with two space dimension obeys some features of (3); in the former case the boundedness of the Lyapunov function implies the finiteness of blowup points.

Our theorems are proven through localized energy estimates, particularly the localized Lyapunov function. Concluding the section, we describe it in short.

The localized Lyapunov function is defined by

$$W_\varphi(t) = \int_\Omega \left\{ u \log u - uv + \frac{1}{2}(|\nabla v|^2 + v^2) \right\} \varphi \, dx,$$

where φ is a nonnegative C^∞ function. If $\varphi \equiv 1$, $W_\varphi(t)$ is equal to $W(t)$, but usually φ is a cut-off function satisfying

$$0 \leq \varphi \leq 1 \qquad \text{in } \mathbf{R}^2, \qquad \frac{\partial \varphi}{\partial n} = 0 \qquad \text{on } \partial\Omega. \tag{4}$$

Actually it is taken in the following way.

Given $x_0 \in \Omega$, we have $0 < R' < R$ with $B(x_0, 2R) \subset \Omega$. Then we take φ satisfying

$$\varphi(x) = \begin{cases} 1 & (x \in B(x_0, R')) \\ 0 & (x \in \mathbf{R}^2 \backslash B(x_0, R)). \end{cases} \tag{5}$$

Given $x_0 \in \partial\Omega$, we first prepare $\zeta \in C_0^\infty(\mathbf{R}^2)$ satisfying $\zeta = \zeta(|y|)$, $0 \le \zeta \le 1$ in $\mathbf{R}^2$, and

$$\zeta(y) = \begin{cases} 1 & (y \in B(0, 1/2)) \\ 0 & (y \in \mathscr{R}^2 \backslash B(0, 1)). \end{cases}$$

Next, we take a smooth conformal mapping $X : B(x_0, 2R) \to \mathcal{O} \subset \mathbf{R}^2$ satisfying $x_0 \mapsto 0$ and

$$X(B(x_0, R) \cap \Omega) \subset \{(x_1, x_2) | x_2 > 0\}$$

$$X(B(x_0, R) \cap \partial\Omega) \subset \{(x_1, x_2) | x_2 = 0\}$$

$$X(B(x_0, R')) \subset B(0, 1/2)$$

$$X(\mathbf{R}^2 \backslash B(x_0, R)) \subset \mathbf{R}^2 \backslash B(0, 1)$$

for $0 < R' < R \ll 1$. Then we set $\varphi(x) = \zeta(X(x))$. It holds that

$$\frac{\partial}{\partial n} \zeta \circ X = \frac{\partial X}{\partial n} \cdot (\nabla \zeta \circ X) = 0 \qquad \text{on } \partial\Omega$$

because $(\partial X)/(\partial n)$ is proportional to $(0, -1)$ on $\partial\Omega$, and such φ satisfies (4) and (5).

We have the following.

LEMMA 2.1. *It holds that*

$$\frac{d}{dt} W_\varphi(t) + \int_\Omega v_t^2 \varphi \, dx + \int_\Omega u |\nabla(\log u - v)|^2 \varphi \, dx = \frac{d}{dt} \int_\Omega u\varphi \, dx + R_1(u, v, \varphi), \tag{6}$$

where

$$R_1(u, v, \varphi) = \int_\Omega [(1 - v)\nabla u - (u \log u - uv + v_t)\nabla v] \cdot \nabla\varphi \, dx + \int_\Omega (u \log u) \triangle \varphi \, dx.$$

PROOF. Multiplying $(\log u - v)\varphi$ by the first equation of (KS) and using Green's formula, we have

$$\begin{aligned} &\int_\Omega u_t (\log u - v)\varphi \, dx \\ &= \int_\Omega \nabla \cdot (\nabla u - u\nabla v)(\log u - v)\varphi \, dx \\ &= -\int_\Omega u |\nabla(\log u - v)|^2 \varphi \, dx - \int_\Omega (\log u - v)(\nabla u - u\nabla v) \cdot \nabla\varphi \, dx. \end{aligned} \tag{7}$$

Here, it holds that

$$\int_\Omega u_t(\log u - v)\varphi\,dx = \frac{d}{dt}\int_\Omega (u\log u - uv)\varphi\,dx - \frac{d}{dt}\int_\Omega u\varphi\,dx + \int_\Omega uv_t\varphi\,dx \tag{8}$$

and

$$\begin{aligned}\int_\Omega (\log u)\nabla u\cdot\nabla\varphi\,dx &= -\int_\Omega u\nabla\cdot(\log u\nabla\varphi)dx + \int_{\partial\Omega}(u\log u)\frac{\partial\varphi}{\partial n}dx \\ &= -\int_\Omega \{(u\log u)\triangle\varphi + \nabla u\cdot\nabla\varphi\}dx. \end{aligned}\tag{9}$$

In use of the second equation of (KS), we have

$$\begin{aligned}\int_\Omega uv_t\varphi\,dx &= \int_\Omega (v_t - \triangle v + v)v_t\varphi\,dx \\ &= \int_\Omega\left\{v_t^2 + \frac{1}{2}\frac{\partial}{\partial t}(|\nabla v|^2 + v^2)\right\}\varphi\,dx + \int_\Omega v_t\nabla v\cdot\nabla\varphi\,dx,\end{aligned}$$

which, together with (7), (8) and (9), leads to

$$\begin{aligned}&\frac{d}{dt}W_\varphi + \int_\Omega v_t^2\varphi\,dx + \int_\Omega u|\nabla(\log u - v)|^2\varphi\,dx \\ &\quad = \frac{d}{dt}\int_\Omega u\varphi\,dx + \int_\Omega (u\log u)\triangle\varphi\,dx \\ &\qquad + \int_\Omega [(1-v)\nabla u - (u\log u - uv + v_t)\nabla v]\cdot\nabla\varphi\,dx.\end{aligned}$$

The proof is complete. □

We sometimes write $\varphi = \varphi_{x_0,R',R}$.

Now we describe the way of proof and some technical difficulties. Theorem 1 is proven by the method of [20], localizing estimates of [21]. The crucial point for the proof of Theorem 2 is showing finiteness of blowup points. As is described in [24], it follows if local L^1 norms of u have bounded variations in time, and this actually holds if the Lyapunov function is bounded. (In N model, it can be shown that the local L^1 norms have always bounded variation in time thanks to remarkable properties of the Green's function. See [24].) Finally, Theorem 3 is a consequence of those arguments.

3. Preliminaries

Regard $-\triangle + 1$ as a closed operator in $L^p(\Omega)$ $(1 < p < \infty)$, denoted by A_p, by

$$D(A_p) = \left\{ v \in W^{2,p}(\Omega) \,\middle|\, \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

It is sectorial so that $-A_p$ generates an analytic semigroup denoted by $\{T_p(t)\}$ (see [15]). The spectrum $\sigma(A_p)$ is independent of p and satisfies $\sigma(A_p) \subset \{z \in \mathbf{C} \mid Re(z) \geq 1\}$. We have the following because $\Omega \subset \mathbf{R}^2$ is bounded and $\partial\Omega$ is smooth (see [26]):

$T_p(t)$ is an operator of integration with the symmetric kernel $G(x,y,t)$ independent of p, satisfying

$$|D_x^\alpha D_y^\beta G(x,y,t)| \leq \frac{C}{t^{(2+|\alpha|+|\beta|)/2}} \exp\left(-\frac{|x-y|^2}{Ct}\right) e^{-\delta t} \tag{10}$$

for $|\alpha| \leq 2$, $|\beta| \leq 2$, and $(x,y,t) \in \Omega \times \Omega \times (0,+\infty)$, where $C > 0$ is a constant and $0 < \delta < 1$.

An immediate consequence is

$$\|T_p(t)\|_{\mathscr{L}(L^p(\Omega),L^p(\Omega))} \leq C_p,$$

where $C_p > 0$ is a constant determined by $p \in (1,\infty)$.

For $\beta \in [0,1]$ the fractional powers A_p^β of A_p is defined, and the domain $X_p^\beta = D(A_p^\beta)$ is a Banach space under the norm $\|u\|_{X_p^\beta} = \|A_p^\beta u\|_{L^p(\Omega)}$. We have the following ([11]):

$X_p^\beta \subset W^{k,q}(\Omega)$ and $X_p^\beta \subset C^\mu(\bar{\Omega})$, provided that $k - \frac{2}{q} < 2\beta - \frac{2}{p}$, $q \geq p$ and $0 \leq \mu < 2\beta - \frac{2}{p}$, respectively.

Making use of those estimates instead of the elliptic estimate for the second equation, we get the following similarly to N model (see [20]). We have $v(t_0) \in D(A_p)$ for $0 < t_0 < T_{\max}$ and henceforth suppose that $v_0 \in D(A_p)$.

PROPOSITION 3.1. *The following relations hold for the solution (u,v) to (KS), where $C_{q,\varepsilon} > 0$ is a constant determined by $q \in (1,2)$ and $\varepsilon \in (0,1/2)$:*

$$\|u(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \tag{11}$$

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)} \leq C_{q,\varepsilon}(\|A_q^{1/2+\varepsilon} v_0\|_{L^q(\Omega)} + \|u_0\|_{L^1(\Omega)}) \tag{12}$$

PROOF. Integrating the equations of (KS) over Ω, we have

$$\frac{d}{dt}\int_\Omega u(x,t)dt = 0 \tag{13}$$

and

$$\frac{d}{dt}\int_{\Omega} v(x,t)dt = -\int_{\Omega} v(x,t)dt + \int_{\Omega} u(x,t)dt. \tag{14}$$

Equality (13) implies (11) because $u > 0$. Then,

$$\|v(\cdot,t)\|_{L^1(\Omega)} = e^{-t}\|v_0\|_{L^1(\Omega)} + (1-e^{-t})\|u_0\|_{L^1(\Omega)}$$

follows from (14) and $v > 0$.

Poincaré-Wirtinger's inequality assures the equivalence

$$\|v\|_{W^{1,q}(\Omega)} \approx \|\nabla v\|_{L^q(\Omega)} + \|v\|_{L^1(\Omega)},$$

so that (12) is reduced to

$$\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le C_{q,\varepsilon}(\|A_q^{1/2+\varepsilon}v_0\|_{L^q(\Omega)} + \|u_0\|_{L^1(\Omega)}). \tag{15}$$

Rewrite the second equation of (KS) as

$$v(\cdot,t) = T_q(t)v_0 + \int_0^t T_q(t-s)u(\cdot,s)ds.$$

Inequality (15) will follow from

$$\left\|\nabla\int_0^t T_q(t-s)u(\cdot,t)ds\right\|_{W^{1,q}(\Omega)} \le C_q\|u_0\|_{L^1(\Omega)}$$

and

$$\|T_q(t)v_0\|_{W^{1,q}(\Omega)} \le C_{q,\varepsilon}\|A_q^{1/2+\varepsilon}v_0\|_{W^{1,q}(\Omega)}.$$

In fact, we have

$$\begin{aligned}\left\|\nabla\int_0^t T_q(t-s)u(\cdot,t)ds\right\|_{L^q(\Omega)}^q &= \int_\Omega\left|\int_0^t\int_\Omega \nabla_x G(x,y,t-s)u(y,s)dyds\right|^q dx\\ &\le \int_\Omega I(t)^{q-1}II(x,t)dx\end{aligned}$$

with

$$I = \int_0^t\int_\Omega (t-s)^{(6-5q)/(4q-4)}u(y,s)e^{\delta q(s-t)/(2q-2)}\,dyds$$

and

$$II = \int_0^t\int_\Omega (t-s)^{(5q-6)/4}|\nabla_x G(x,y,t-s)|^q u(y,s)e^{\delta q(t-s)/2}\,dyds.$$

If $q \in (1,2)$ then $(6-5q)/(4q-4) > -1$, so that we have

$$I(t) \leq C_q \|u_0\|_{L^1(\Omega)}.$$

On the other hand, inequality (10) gives that

$$\int_\Omega II(x,t)dx \leq C \int_\Omega \int_0^t \int_{\mathbf{R}^2} (t-s)^{-(q+6)/4} \exp\left(-\frac{q|x-y|^2}{C(t-s)}\right) u(y,s) e^{\delta q(s-t)/2}\, dxdsdy$$

$$\leq C \int_0^\infty t^{-(q+2)/4} e^{-\delta q t/2}\, dt \|u_0\|_{L^1(\Omega)} = C_q \|u_0\|_{L^1(\Omega)}.$$

Therefore, we get

$$\left\| \nabla \int_0^t T_q(t-s) u(\cdot,s) ds \right\|_{L^q(\Omega)}^q \leq C_q \|u_0\|_{L^1(\Omega)}^q.$$

Finally,

$$\begin{aligned} \|T_q(t)v_0\|_{W^{1,q}(\Omega)} &\leq C_{q,\varepsilon} \|A_q^{1/2+\varepsilon} T_q(t) v_0\|_{L^q(\Omega)} \\ &= C_{q,\varepsilon} \|T_q(t) A_q^{1/2+\varepsilon} v_0\|_{L^q(\Omega)} \leq C'_{q,\varepsilon} \|A_q^{1/2+\varepsilon} v_0\|_{L^q(\Omega)} \end{aligned}$$

and (15) follows. The proof is complete. □

We note that inequality (122) with $1 \leq q < 2$ implies

$$\|v(\cdot,t)\|_p \leq C_p \tag{16}$$

for $1 < p < \infty$.

Lemma 5.10 of Adams [1] reads;

$$\|w\|_{L^2(\Omega)}^2 \leq K^2 (\|w\|_{L^1(\Omega)}^2 + \|\nabla w\|_{L^1(\Omega)}^2) \tag{17}$$

for $w \in W^{1,1}(\Omega)$, where $K > 0$ is a constant determined by Ω. Inequality (17) implies some estimates on u.

Recall the cut-off function $\varphi_{x_0,R',R}$ intoduced at the end of section 2. Then, $\psi = (\varphi_{x_0,R',R})^6$ satisfies

$$\psi(x) = \begin{cases} 1 & (x \in B(x_0,R')) \\ 0 & (x \in \mathbf{R}^2 \backslash B(x_0,R)) \end{cases}$$

$$0 \leq \psi \leq 1 \quad \text{in } \mathbf{R}^2, \qquad \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \partial\Omega$$

$$|\nabla \psi| \leq A\psi^{5/6}, \qquad |\triangle \psi| \leq B\psi^{2/3} \quad \text{in } \mathbf{R}^2,$$

where $A > 0$ and $B > 0$ are constants determined by $0 < R' < R \ll 1$.

Lemma 3.2 *The following inequalities hold for any $s > 1$, where $C > 0$ is a constant:*

$$\int_\Omega u^2\psi\,dx \le 2K^2\int_{B(x_0,R)\cap\Omega} u\,dx\cdot\int_\Omega u^{-1}|\nabla u|^2\psi\,dx + K^2\left(\frac{A^2}{2}+1\right)\|u\|^2_{L^1(\Omega)} \tag{18}$$

$$\int_\Omega u^2\psi\,dx \le \frac{4K^2}{\log s}\int_{B(x_0,R)\cap\Omega}(u\log u + e^{-1})dx\cdot\int_\Omega u^{-1}|\nabla u|^2\psi\,dx$$

$$+ C\|u\|^2_{L^1(\Omega)} + 3s^2|\Omega| \tag{19}$$

$$\int_\Omega u^3\psi\,dx \le \frac{72K^2}{\log s}\int_{B(x_0,R)\cap\Omega}(u\log u + e^{-1})dx\cdot\int_\Omega |\nabla u|^2\psi\,dx$$

$$+ C\|u\|^3_{L^1(B(x_0,R)\cap\Omega)} + 10|\Omega|s^3 \tag{20}$$

Proof. Putting $w = u\psi^{1/2}$, we have

$$\begin{aligned}\left\{\int_\Omega|\nabla w|dx\right\}^2 &\le 2\left\{\int_\Omega|\nabla u|\psi^{1/2}\,dx\right\}^2 + 2\left\{\int_\Omega u|\nabla\psi^{1/2}|dx\right\}^2\\ &\le 2\left\{\int_\Omega|\nabla u|\psi^{1/2}\,dx\right\}^2 + \frac{A^2}{2}\|u\|^2_{L^1(\Omega)}\\ &\le 2\int_{B(x_0,R)\cap\Omega}u\,dx\cdot\int_\Omega u^{-1}|\nabla u|^2\psi\,dx + \frac{A^2}{2}\|u\|^2_{L^1(\Omega)}.\end{aligned}$$

Hence (18) follows from (17) and $\|w\|_{L^1(\Omega)} \le \|u\|_{L^1(\Omega)}$.

We turn to (19). Take $w = (u-s)_+\psi^{1/2}$ with $a_+ = \max\{a, 0\}$. We have

$$\begin{aligned}\|w\|^2_{L^2(\Omega)} &= \int_{\{u>s\}}(u-s)^2\psi\,dx\\ &\ge \int_{\{u>s\}}\left(\frac{1}{2}u^2 - s^2\right)\psi\,dx\\ &= \int_\Omega\frac{1}{2}u^2\psi\,dx - \int_{\{u\le s\}}\frac{1}{2}u^2\psi\,dx - \int_\Omega s^2\psi\,dx\\ &\ge \frac{1}{2}\int_\Omega u^2\psi\,dx - \frac{3}{2}s^2|\Omega|.\end{aligned}$$

On the other hand we have $\|w\|^2_{L^1(\Omega)} \le \|u\|^2_{L^1(\Omega)}$ and

$$\|\nabla w\|_{L^1(\Omega)}^2 \le \left\{\int_{\{u>s\}} (|\nabla u|\psi^{1/2} + (u-s)_+|\nabla\psi^{1/2}|)dx\right\}^2$$

$$\le 2\left\{\int_{\{u>s\}} |\nabla u|\psi^{1/2}\,dx\right\}^2 + 2\left\{\int_\Omega u|\nabla\psi^{1/2}|dx\right\}^2$$

$$\le 2\left\{\int_{\{u>s\}} |\nabla u|\psi^{1/2}\,dx\right\}^2 + \frac{A^2}{2}\|u\|_{L^1(\Omega)}^2.$$

Here,

$$\left\{\int_{\{u>s\}} |\nabla u|\psi^{1/2}\,dx\right\}^2 \le \int_{B(x_0,R)\cap\{u>s\}} u\,dx \cdot \int_{\{u>s\}} u^{-1}|\nabla u|^2\psi\,dx$$

$$\le \frac{1}{\log s}\int_{B(x_0,R)\cap\Omega} (u\log u + e^{-1})dx \cdot \int_\Omega u^{-1}|\nabla u|^2\psi\,dx$$

because $s\log s \ge -e^{-1}$ for any $s > 0$. This implies (19).

Finally, take $w = (u-s)_+^{3/2}\psi^{1/2}$. We have

$$\|w\|_{L^2(\Omega)}^2 = \int_{\{u>s\}} (u-s)_+^3\psi\,dx$$

$$\ge \int_{\{u>s\}} \left(\frac{1}{4}u^3 - s^3\right)\psi\,dx$$

$$\ge \frac{1}{4}\int_\Omega u^3\psi\,dx - \frac{5}{4}s^3|\Omega|.$$

Because

$$|\nabla w| \le \frac{3}{2}(u-s)_+^{1/2}|\nabla u|\psi^{1/2} + \frac{1}{2}A(u-s)_+^{3/2}\psi^{1/3}$$

we have

$$\|\nabla w\|_{L^1(\Omega)}^2 \le \frac{9}{2}\left\{\int_{\{u>s\}} (u-s)^{1/2}|\nabla u|\psi^{1/2}\,dx\right\}^2 + \frac{A^2}{2}\left\{\int_{\{u>s\}} (u-s)^{3/2}\psi^{1/3}\,dx\right\}^2.$$

Here, it holds that

$$\left\{\int_{\{u>s\}}(u-s)^{1/2}|\nabla u|\psi^{1/2}\,dx\right\}^2 \le \left\{\int_{\{u>s\}}u^{1/2}|\nabla u|\psi^{1/2}\,dx\right\}^2$$
$$\le \int_{B(x_0,R)\cap\{u>s\}}u\,dx\cdot\int_{\{u>s\}}|\nabla u|^2\psi\,dx$$
$$\le \frac{1}{\log s}\int_{B(x_0,R)\cap\Omega}(u\log u+e^{-1})dx\cdot\int_\Omega|\nabla u|^2\psi\,dx$$

and

$$\left\{\int_{\{u>s\}}(u-s)^{3/2}\psi^{1/3}\,dx\right\}^2 \le \left\{\int_{\{u>s\}}u\psi^{1/3}u^{1/2}\,dx\right\}^2$$
$$\le \left\{\int_\Omega u^3\psi\,dx\right\}^{2/3}\|u\|_{L^1(B(x_0,R)\cap\Omega)}|\Omega|^{1/3}$$
$$\le \varepsilon\int_\Omega u^3\psi\,dx + C_\varepsilon|\Omega|\|u\|^3_{L^1(B(x_0,R)\cap\Omega)}, \tag{21}$$

where $C_\varepsilon > 0$ is a constant determined by $\varepsilon > 0$. Therefore,

$$\|\nabla w\|^2_{L^1(\Omega)} \le \frac{9}{2\log s}\int_{B(x_0,R)\cap\Omega}(u\log u+e^{-1})dx\cdot\int_\Omega|\nabla u|^2\psi\,dx$$
$$+\frac{A^2}{2}\varepsilon\int_\Omega u^3\psi\,dx+\frac{A^2}{2}C_\varepsilon|\Omega|\,\|u\|^3_{L^1(B(x_0,R)\cap\Omega)}.$$

Since $\psi^{1/2}\le\psi^{1/3}$, it follows from (21) that

$$\|w\|^2_{L^1(\Omega)} \le \varepsilon\int_\Omega u^3\psi\,dx + C_\varepsilon|\Omega|\|u\|^3_{L^1(B(x_0,R)\cap\Omega)}.$$

We get

$$\left(\frac{1}{4}-K^2\left(\frac{A^2}{2}+1\right)\varepsilon\right)\int_\Omega u^3\psi\,dx$$
$$\le \frac{9K^2}{\log s}\int_{B(x_0,R)\cap\Omega}(u\log u+e^{-1})dx\cdot\int_\Omega|\nabla u|^2\psi\,dx$$
$$+K^2C_\varepsilon|\Omega|\left(\frac{A^2}{2}+1\right)\|u\|^3_{L^1(B(x_0,R)\cap\Omega)}+\frac{5}{4}s^3|\Omega|$$

by (17). Taking $\varepsilon > 0$ as

$$\frac{1}{4} - K^2\left(\frac{A^2}{2} + 1\right)\varepsilon = \frac{1}{8},$$

we obtain (20). □

4. Finiteness of blowup points

This section is devoted to the proof of Therems 2 and 3. First, a technical estimate is derived for local norms of the solution (u, v) to (KS). Henceforth, we always asume $T_{\max} < +\infty$ and a generic positive constant (possibly changing from line to line) is denoted by C.

LEMMA 4.1. *It holds that*

$$\frac{d}{dt}\int_\Omega (u\log u)\psi\,dx + \frac{1}{2}\int_\Omega u^{-1}|\nabla u|^2\psi\,dx + \int_\Omega uv\psi\,dx$$
$$+\frac{1}{2}\int_\Omega v_t^2\psi\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla v|^2 + v^2)\psi\,dx \le 2\int_\Omega u^2\psi\,dx + C. \quad (22)$$

PROOF. We show the following equality first:

$$\frac{d}{dt}\int_\Omega (u\log u)\psi\,dx + \int_\Omega u^{-1}|\nabla u|^2\psi\,dx + \int_\Omega uv\psi\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla v|^2 + v^2)\psi\,dx$$
$$= \int_\Omega u^2\psi\,dx - \int_\Omega v_t^2\psi\,dx - I - II - III - IV, \quad (23)$$

where

$$I = \int_\Omega v_t\nabla v\cdot\nabla\psi\,dx$$

$$II = \int_\Omega (1 + \log u)\nabla u\cdot\nabla\psi\,dx$$

$$III = \int_\Omega v(1 + \log u)\nabla u\cdot\nabla\psi\,dx$$

$$IV = \int_\Omega (uv\log u)\,\triangle\psi\,dx.$$

This can be derived by (6), but here we prove it directly by (KS).

In fact, multiplying $(\log u)\psi$ by the first equation of (KS), we get that

$$\frac{d}{dt}\int_\Omega (u\log u)\psi\,dx = \int_\Omega u_t(\log u)\psi\,dx + \int_\Omega u_t\psi\,dx$$
$$= \int_\Omega \{\nabla\cdot(\nabla u - u\nabla v)\}(1 + \log u)\psi\,dx$$

$$= -\int_\Omega \nabla u \cdot \nabla\{(1+\log u)\psi\} dx + \int_\Omega u\nabla v \cdot \nabla\{(1+\log u)\psi\} dx$$
$$= -V + VI.$$

Here,

$$VI = \int_\Omega u\nabla v \cdot (u^{-1}\psi\nabla u + (1+\log u)\nabla\psi) dx$$
$$= \int_\Omega (\nabla v \cdot \nabla u)\psi\, dx + \int_\Omega u(1+\log u)\nabla v \cdot \nabla\psi\, dx$$
$$= -\int_\Omega u\nabla \cdot (\psi\nabla v) dx + \int_\Omega u(1+\log u)\nabla v \cdot \nabla\psi\, dx$$
$$= -\int_\Omega u\psi \triangle v\, dx + \int_\Omega (u\log u)\nabla v \cdot \nabla\psi\, dx$$
$$= -\int_\Omega u(v_t + v - u)\psi\, dx + \int_\Omega (u\log u)\nabla v \cdot \nabla\psi\, dx$$
$$= -\int_\Omega u(v_t + v - u)\psi\, dx - \int_\Omega v(u\log u)\Delta\psi\, dx - \int_\Omega v(1+\log u)\nabla u \cdot \nabla\psi\, dx$$

by the second equation of (KS). On the other hand,

$$V = \int_\Omega \nabla u \cdot \{u^{-1}\psi\nabla u + (1+\log u)\nabla\psi\} dx$$
$$= \int_\Omega u^{-1}|\nabla u|^2\psi\, dx + \int_\Omega (1+\log u)\nabla u \cdot \nabla\psi\, dx.$$

Therefore, it holds that

$$\frac{d}{dt}\int_\Omega (u\log u)\psi\, dx + \int_\Omega u^{-1}|\nabla u|^2\psi\, dx + \int_\Omega uv\psi\, dx$$
$$= \int_\Omega (u^2 - v_t u)\psi\, dx - \int_\Omega (1+\log u)\nabla u \cdot \nabla\psi\, dx + \int_\Omega (u\log u)\nabla v \cdot \nabla\psi\, dx$$
$$= \int_\Omega (u^2 - v_t u)\psi\, dx - II - III - IV.$$

On the other hand we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla v|^2 + v^2)\psi\, dx = \int_\Omega (\nabla v_t \cdot \nabla v + v_t v)\psi\, dx$$
$$= \int_\Omega v_t(-\Delta v + v)\psi\, dx - \int_\Omega v_t\nabla v \cdot \nabla\psi\, dx$$
$$= \int_\Omega (-v_t^2 + v_t u)\psi\, dx - I.$$

Equality (23) has been proven.

Now we proceed to the proof of (22). First, in use of (18), we get that

$$|II| \le C\int_{\Omega} u^{1/6}\cdot u^{1/3}|1+\log u|\psi^{1/3}\cdot u^{-1/2}|\nabla u|\psi^{1/2}\,dx$$

$$\le C\|u\|_{L^1(\Omega)}^{1/6}\left\{\int_{\Omega} u|1+\log u|^3\psi\,dx\right\}^{1/3}\left\{\int_{\Omega} u^{-1}|\nabla u|^2\psi\,dx\right\}^{1/2}.$$

Recall the elementary inequality: Let $1\le\alpha<2$ and $\beta>0$. Then

$$u^{\alpha}(1+|\log u|)^{\beta}\le C(u^2+1)\qquad (u>0).$$

We obtain

$$|II| \le C\|u_0\|_{L^1(\Omega)}^{1/6}\left\{\int_{\Omega} u^2\psi\,dx+1\right\}^{1/3}\left\{\int_{\Omega} u^{-1}|\nabla u|^2\psi\,dx\right\}^{1/2}$$

$$\le \frac{1}{4}\int_{\Omega} u^{-1}|\nabla u|^2\psi\,dx+\frac{1}{4}\int_{\Omega} u^2\psi\,dx+C$$

by (11). Similarly, we have

$$|III| \le C\int_{\Omega} u^{-1/2}|\nabla u|\psi^{1/2}\cdot u^{1/2}|1+\log u|\psi^{1/3}\cdot v\,dx$$

$$\le C\left\{\int_{\Omega} u^{-1}|\nabla u|^2\psi\,dx\right\}^{1/2}\left\{\int_{\Omega} u^{3/2}|1+\log u|^3\psi\,dx\right\}^{1/3}\|v\|_{L^6(\Omega)}$$

$$\le \frac{1}{4}\int_{\Omega} u^{-1}|\nabla u|^2\psi\,dx+\frac{1}{4}\int_{\Omega} u^2\psi\,dx+C$$

and

$$|IV| \le \int_{\Omega} |uv\log u|\psi^{2/3}\,dx$$

$$\le \left\{\int_{\Omega} |u\log u|^{3/2}\psi\,dx\right\}^{2/3}\|v\|_{L^3(\Omega)}$$

$$\le \frac{1}{4}\int_{\Omega} u^2\psi\,dx+C.$$

by (16). Finally,

$$|I| \le A\int_{\Omega} |v_t\nabla v|\psi^{5/6}\,dx$$

$$\le \frac{1}{4}\int_{\Omega} v_t^2\psi\,dx+A^2\int_{\Omega} |\nabla v|^2\psi^{2/3}\,dx.$$

Here,

$$\begin{aligned}
\int_\Omega |\nabla v|^2 \psi^{2/3}\,dx &= -\int_\Omega v\nabla\cdot(\psi^{2/3}\nabla v)dx \\
&= \int_\Omega \left\{ v(-\Delta v)\psi^{2/3} - \frac{2}{3}\psi^{-1/3} v\nabla\psi\cdot\nabla v \right\} dx \\
&= \int_\Omega v(-v_t + u - v)\psi^{2/3}\,dx - \frac{2}{3}\int_\Omega \psi^{-1/3} v\nabla\psi\cdot\nabla v\,dx \\
&\le \int_\Omega v\psi^{1/6}\cdot u\psi^{1/2}\,dx + \left|\int_\Omega v_t\psi^{1/2}\cdot v\psi^{1/6}\,dx\right| \\
&\quad + \frac{2A}{3}\int_\Omega |\nabla v|\psi^{1/3}\cdot v\psi^{1/6}\,dx \\
&\le \frac{1}{8A^2}\int_\Omega u^2\psi\,dx + 16A^2\|v\|^2_{L^2(\Omega)} + \frac{1}{8A^2}\int_\Omega v_t^2\psi\,dx \\
&\quad + 32A^2\|v\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega |\nabla v|^2\psi^{2/3}\,dx + \frac{2A^2}{9}\|v\|^2_{L^2(\Omega)}.
\end{aligned}$$

Therefore, it holds that

$$\int_\Omega |\nabla v|^2\psi^{2/3}\,dx \le \frac{1}{4A^2}\int_\Omega v_t^2\psi\,dx + \frac{1}{4A^2}\int_\Omega u^2\psi\,dx + C,$$

which implies

$$|I| \le \frac{1}{2}\int_\Omega v_t^2\psi\,dx + \frac{1}{4}\int_\Omega u^2\psi\,dx + C.$$

Inequality (22) has been proven. □

We show a key fact for the proof of Theorems.

Proposition 4.2. *Suppose $T_{\max} < +\infty$ and let $x_0 \in \bar{\Omega}$ and $0 < R \ll 1$. Then, if a solution (u, v) to (KS) satisfies*

$$\sup_{0\le t<T_{\max}} \int_{B(x_0,R)\cap\Omega} u\log u\,dx < +\infty \tag{24}$$

it holds that

$$\sup_{0\le t<T_{\max}} \|u(\cdot,t)\|_{L^\infty(B(x_0,r)\cap\Omega)} < +\infty \tag{25}$$

for any $r \in (0, R)$.

PROOF. We divide the argument in five steps. Take $R' \in (0, R)$ and let $\psi = (\varphi_{x_0, R', R})^6$.

Step 1 We show that (24) with $T_{\max} < +\infty$ implies

$$\int_0^{T_{\max}} \int_{B(x_0, R') \cap \Omega} v_t^2 \, dxdt < +\infty. \tag{26}$$

Therefore, taking $R > 0$ smaller, we can assume that

$$\int_0^{T_{\max}} \int_{B(x_0, R) \cap \Omega} v_t^2 \, dxdt < +\infty. \tag{27}$$

In fact, inequality (19) with

$$M = \sup_{0 \le t < T_{\max}} \int_{B(x_0, R) \cap \Omega} (u \log u + e^{-1}) dx < +\infty \tag{28}$$

gives that

$$\int_\Omega u^2 \psi \, dx \le \frac{4K^2 M}{\log s} \int_\Omega u^{-1} |\nabla u|^2 \psi \, dx + C + 3s^2 |\Omega|.$$

Therefore, taking $s > 1$ as $8K^2 M / (\log s) < 1/2$, we have (26) by (22).

Step 2 Multiplying $u\psi$ by the first equation of (KS), we have

$$\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 \psi \, dx + \int_\Omega |\nabla u|^2 \psi \, dx + \int_\Omega u \nabla u \cdot \nabla \psi \, dx$$
$$= \int_\Omega u \psi \nabla v \cdot \nabla u \, dx + \int_\Omega u^2 \nabla v \cdot \nabla \psi \, dx. \tag{29}$$

From the second equation of (KS) follows that

$$\begin{aligned}
\int_\Omega u \psi \nabla v \cdot \nabla u \, dx &= \frac{1}{2} \int_\Omega \psi \nabla v \cdot \nabla u^2 dx \\
&= -\frac{1}{2} \int_\Omega u^2 \psi \triangle v \, dx - \frac{1}{2} \int_\Omega u^2 \nabla v \cdot \nabla \psi \, dx \\
&= \frac{1}{2} \int_\Omega u^3 \psi \, dx - \frac{1}{2} \int_\Omega u^2 (v_t + v) \psi \, dx - \frac{1}{2} \int_\Omega u^2 \nabla v \cdot \nabla \psi \, dx \\
&\le \frac{1}{2} \int_\Omega u^3 \psi \, dx - \frac{1}{2} \int_\Omega u^2 v_t \psi \, dx - \frac{1}{2} \int_\Omega u^2 \nabla v \cdot \nabla \psi \, dx.
\end{aligned}$$

Therefore, in use of

$$\int_\Omega u^2 \nabla v \cdot \nabla \psi \, dx = -\int_\Omega v \nabla u^2 \cdot \nabla \psi \, dx - \int_\Omega u^2 v \triangle \psi \, dx$$

and

$$\int_\Omega u\nabla u\cdot\nabla\psi\,dx=-\frac{1}{2}\int_\Omega u^2\triangle\psi\,dx,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\psi\,dx+\int_\Omega|\nabla u|^2\psi\,dx\le\frac{1}{2}\int_\Omega u^3\psi\,dx-\frac{1}{2}\int_\Omega u^2v_t\psi\,dx$$
$$+\frac{1}{2}\int_\Omega v\nabla u^2\cdot\nabla\psi\,dx+\frac{1}{2}\int_\Omega u^2(v+1)\triangle\psi\,dx. \quad (30)$$

Here, last three terms of the right-hand side are dominated as follows. First, inequality (16) gives that

$$\begin{aligned}\frac{1}{2}\left|\int_\Omega u^2(v+1)\Delta\psi\,dx\right|&\le\frac{B}{2}\int_\Omega(v+1)\cdot u^2\psi^{2/3}\,dx\\&\le\frac{B}{2}(\|v\|_{L^3(\Omega)}+|\Omega|^{1/3})\left\{\int_\Omega u^3\psi\,dx\right\}^{2/3}\\&\le\frac{1}{3}\int_\Omega u^3\psi\,dx+C.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{1}{2}\left|\int_\Omega v\nabla u^2\cdot\nabla\psi\,dx\right|&\le 3A\int_\Omega v\cdot u\psi^{1/3}\cdot|\nabla u|\psi^{1/2}\,dx\\&\le 3A\|v\|_{L^6(\Omega)}\left\{\int_\Omega u^3\psi\,dx\right\}^{1/3}\left\{\int_\Omega|\nabla u|^2\psi\right\}^{1/2}\\&\le\frac{1}{8}\int_\Omega|\nabla u|^2\psi\,dx+\frac{1}{3}\int_\Omega u^3\psi\,dx+C.\end{aligned}$$

Finally, Gagliardo-Nirenberg's inequality

$$\|w\|_{L^4(\Omega)}\le K(\|\nabla w\|^{1/2}_{L^2(\Omega)}\|w\|^{1/2}_{L^2(\Omega)}+\|w\|_{L^2(\Omega)})$$

to $w=u\psi^{1/2}$ implies that

$$\begin{aligned}\frac{1}{2}\left|\int_\Omega u^2v_t\psi\,dx\right|&\le\frac{1}{2}\left\{\int_{B(x_0,R)\cap\Omega}v_t^2\,dx\right\}^{1/2}\left\{\int_\Omega u^4\psi^2\,dx\right\}^{1/2}\\&\le C\left\{\int_{B(x_0,R)\cap\Omega}v_t^2\,dx\right\}^{1/2}\\&\quad\times(\|\nabla(u\psi^{1/2})\|_{L^2(\Omega)}\|u\psi^{1/2}\|_{L^2(\Omega)}+\|u\psi^{1/2}\|^2_{L^2(\Omega)})\end{aligned}$$

$$\leq \frac{1}{16}\int_\Omega |\nabla(u\psi^{1/2})|^2 dx + C\int_{B(x_0,R)\cap\Omega} v_t^2\,dx \cdot \|u\psi^{1/2}\|_{L^2(\Omega)}^2$$
$$+ C\left\{\int_{B(x_0,R)\cap\Omega} v_t^2\,dx\right\}^{1/2} \|u\psi^{1/2}\|_{L^2(\Omega)}^2$$
$$\leq \frac{1}{16}\int_\Omega |\nabla(u\psi^{1/2})|^2 dx + C\left(\int_{B(x_0,R)\cap\Omega} v_t^2\,dx + 1\right)\|u\psi^{1/2}\|_{L^2(\Omega)}^2$$

Here, we have

$$\int_\Omega |\nabla(u\psi^{1/2})|^2 dx \leq 2\int_\Omega |\nabla u|^2\psi\,dx + \frac{A^2}{2}\int_\Omega u^2\psi^{2/3}\,dx$$
$$\leq 2\int_\Omega |\nabla u|^2\psi\,dx + \frac{16}{3}\int_\Omega u^3\psi\,dx + C,$$

so that

$$\frac{1}{2}\left|\int_\Omega u^2 v_t\psi\,dx\right| \leq \frac{1}{8}\int_\Omega |\nabla u|^2\psi\,dx + \frac{1}{3}\int_\Omega u^3\psi\,dx$$
$$+ C\left(\int_{B(x_0,R)\cap\Omega} v_t^2\,dx + 1\right)\cdot\int_\Omega u^2\psi\,dx + C.$$

In this way, inequality (30) has been reduced to

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\psi\,dx + \frac{3}{4}\int_\Omega |\nabla u|^2\psi\,dx$$
$$\leq 2\int_\Omega u^3\psi\,dx + C\left(\int_{B(x_0,R)\cap\Omega} v_t^2\,dx + 1\right)\int_\Omega u^2\psi\,dx + C$$
$$\leq 3\int_\Omega u^3\psi\,dx + C\left(\int_{B(x_0,R)\cap\Omega} v_t^2\,dx\cdot\int_\Omega u^2\psi\,dx + 1\right). \tag{31}$$

We can make use of (20) for the first term of the right-hand side. It holds that

$$\int_\Omega u^3\psi\,dx \leq \frac{72K^2M}{\log s}\int_\Omega |\nabla u|^2\psi\,dx + C + 10|\Omega|s^3,$$

where $M > 0$ is the constant defined in (28). Making $s > 1$ large, this term is absorbed into the left-hand side of (31). We obtain

$$\frac{d}{dt}\int_\Omega u^2\psi\,dx + \int_\Omega |\nabla u|^2\psi\,dx \le C\left(\int_{B(x_0,R)\cap\Omega} v_t^2\,dx \cdot \int_\Omega u^2\psi\,dx + 1\right).$$

In particular, $g(t) = \int_\Omega u^2\psi\,dx$ solves

$$\frac{dg}{dt} \le hg + C \qquad (0 \le t < T_{\max})$$

with a continuous function $h(t) \ge 0$ satisfying $\int_0^{T_{\max}} h(t)dt < +\infty$. This implies

$$\sup_{0\le t<T_{\max}} g(t) = \sup_{0\le t<T_{\max}} \int_\Omega u^2\psi\,dx < +\infty. \tag{32}$$

Setp 3 We take $R'' \in (0, R')$ and set $\psi_1 = (\varphi_{x_0,R'',R'})^6$. Multiplying $u^2\psi_1$ by the first equation of (KS), we have

$$\frac{1}{3}\frac{d}{dt}\int_\Omega u^3\psi_1\,dx + 2\int_\Omega u|\nabla u|^2\psi_1\,dx + \int_\Omega u^2\nabla u\cdot\nabla\psi_1\,dx$$
$$= 2\int_\Omega u^2\psi_1\nabla v\cdot\nabla u\,dx + \int_\Omega u^3\nabla v\cdot\nabla\psi_1\,dx.$$

This means that

$$\frac{1}{3}\frac{d}{dt}\int_\Omega w^2\psi_1\,dx + \frac{8}{9}\int_\Omega |\nabla w|^2\psi_1\,dx + \frac{2}{3}\int_\Omega w\nabla w\cdot\nabla\psi_1\,dx$$
$$= \frac{4}{3}\int_\Omega w\psi_1\nabla v\cdot\nabla w\,dx + \int_\Omega w^2\nabla v\cdot\nabla\psi_1\,dx \tag{33}$$

for $w = u^{3/2}$. From (32) and

$$w\log w \le 3w^{4/3} = 3u^2$$

we have

$$\sup_{0\le t<T_{\max}} \int_{B(x_0,R')\cap\Omega} (w\log w)dx < +\infty.$$

Relation (33) is similar to (29). Inequality (20) holds with u replaced by w, and $\|w\|_{L^1(B(x_0,R')\cap\Omega)} \le C$ follows from (32). Finally, if we make use of the second equation of (KS), we obtain

$$\int_\Omega w\psi_1\nabla v\cdot\nabla w\,dx \le \frac{1}{2}\int_\Omega w^{8/3}\psi_1\,dx - \frac{1}{2}\int_\Omega w^2 v_t\psi_1\,dx - \frac{1}{2}\int_\Omega w^2\nabla v\cdot\nabla\psi_1\,dx$$
$$\le \int_\Omega w^3\psi_1\,dx - \frac{1}{2}\int_\Omega w^2 v_t\psi_1\,dx - \frac{1}{2}\int_\Omega w^2\nabla v\cdot\nabla\psi_1\,dx + C$$

similarly to (30). Under those circumstances we can repeat the arguments in Step 2 and get

$$\sup_{0\le t<T_{\max}}\int_{B(x_0,R'')\cap\Omega} u^3\,dx = \sup_{0\le t<T_{\max}}\int_{B(x_0,R'')\cap\Omega} w^2\,dx < +\infty. \tag{34}$$

If we repeat the arguments once more, we get

$$\sup_{0\le t<T_{\max}} \|u(\cdot,t)\|_{L^4(B(x_0,r)\cap\Omega)} < +\infty. \tag{35}$$

for any $r \in (0,R)$.

Step 4 Put $u_1 = u\chi_{B(x_0,r)}$, $u_2 = u - u_1$, and let v_1, v_2 be the solutions for

$$\begin{cases} v_t = \triangle v - v + f & \text{in } \Omega\times(0,T_{\max}),\\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega\times(0,T_{\max}),\\ v(\cdot,0) = 0 & \text{in } \Omega. \end{cases}$$

with $f = u_1$, u_2, respectively. It holds that

$$v_2 = \int_0^t \int_{\Omega\setminus B(x_0,r)} G(x,y,t-s)u(y,s)dyds,$$

so that

$$\sup_{0\le t<T_{\max}} \|v_2(\cdot,t)\|_{W^{2,\infty}(B(x_0,r')\cap\Omega)} < +\infty$$

for $r' \in (0,r)$ by (10) and (11).

To handle with $v_1(x,t)$, we recall the operator A_p in Section 3. Let $5/6 < \beta < 1$ and $p = 3$. Then we have

$$\begin{aligned} \sup_{0\le t<T_{\max}} \|v_1(\cdot,t)\|_{X_p^\beta} &= \sup_{0\le t<T_{\max}} \|A_p^\beta v_1(\cdot,t)\|_{L^p(\Omega)} \\ &\le \sup_{0\le t<T_{\max}} \int_0^t \|A_p^\beta T_p(t-s)u_1(\cdot,s)\|_{L^p(\Omega)}ds \\ &\le C \sup_{0\le t<T_{\max}} \int_0^t (t-s)^{-\beta}\|u_1(\cdot,s)\|_{L^p(\Omega)}ds < +\infty \end{aligned}$$

by (34). Inclusion $X_p^\beta \subset C^1(\bar{\Omega})$ holds and hence

$$\sup_{0\le t<T_{\max}} \|v(\cdot,t)\|_{C^1(B(x_0,r)\cap\Omega)} < +\infty \tag{36}$$

for any $r \in (0,R)$.

Step 5 Take $r' \in (0,r)$ and put $\psi_1 = (\varphi_{x_0,r',r})^6$. We multiply $u^p\psi_1^{p+1}$ by the first equation of (KS) and get

$$\frac{d}{dt}\frac{1}{p+1}\int_\Omega (u\psi_1)^{p+1}dx = -\int_\Omega \nabla(u^p\psi_1^{p+1})\cdot\nabla u\,dx + \int_\Omega u\nabla(u^p\psi_1^{p+1})\cdot\nabla v\,dx$$
$$= -I + II.$$

Here,

$$\begin{aligned}
I &= \int_\Omega (pu^{p-1}\psi_1^{p+1}\nabla u + u^p\nabla\psi_1^{p+1})\cdot\nabla u\,dx \\
&= \frac{4p}{(p+1)^2}\int_\Omega |\nabla u^{(p+1)/2}|^2\psi_1^{p+1}\,dx + \frac{1}{p+1}\int_\Omega \nabla\psi_1^{p+1}\cdot\nabla u^{p+1}\,dx \\
&= \frac{4p}{(p+1)^2}\int_\Omega |\nabla u^{(p+1)/2}|^2\psi_1^{p+1}\,dx \\
&\quad + \frac{4}{p+1}\int_\Omega \psi_1^{(p+1)/2}\nabla u^{(p+1)/2}\cdot u^{(p+1)/2}\nabla\psi_1^{(p+1)/2}\,dx \\
&= \left\{\frac{4p}{(p+1)^2} - \frac{2}{p+1}\right\}\int_\Omega |\nabla u^{(p+1)/2}|^2\psi_1^{p+1}\,dx \\
&\quad + \frac{2}{p+1}\int_\Omega |\nabla(u\psi_1)^{(p+1)/2}|^2dx - \frac{2}{p+1}\int_\Omega u^{p+1}|\nabla\psi_1^{(p+1)/2}|^2dx \\
&\geq \frac{2}{p+1}\int_\Omega |\nabla(u\psi_1)^{(p+1)/2}|^2dx - \frac{p+1}{2}\int_\Omega u^{p+1}\psi_1^{p-1}|\nabla\psi_1|^2dx \\
&\geq \frac{2}{p+1}\int_\Omega |\nabla(u\psi_1)^{(p+1)/2}|^2dx - \frac{A^2(p+1)}{2}\int_\Omega (u\psi_1)^{p+(2/3)}\cdot u^{1/3}\,dx \\
&\geq \frac{2}{p+1}\int_\Omega |\nabla(u\psi_1)^{(p+1)/2}|^2dx - \frac{A^2(p+1)}{2}\|u_0\|_{L^1(\Omega)}^{1/3}\left\{\int_\Omega (u\psi_1)^{1+(3/2)p}dx\right\}^{2/3}.
\end{aligned}$$

Furthermore, (36) implies

$$\begin{aligned}
II &\leq C\int_\Omega |u\nabla(u^p\psi_1^{p+1})|dx \\
&\leq C\left\{\frac{p}{p+1}\int_\Omega |\nabla(u\psi_1)^{p+1}|dx + (p+1)\int_\Omega u^{p+1}\psi_1^p|\nabla\psi_1|dx\right\} \\
&\leq C\frac{2p}{p+1}\int_\Omega (u\psi_1)^{(p+1)/2}|\nabla(u\psi_1)^{(p+1)/2}|dx + CA(p+1)\int_\Omega (u\psi_1)^{p+(5/6)}u^{1/6}\,dx \\
&\leq \frac{1}{p+1}\int_\Omega |\nabla(u\psi_1)^{(p+1)/2}|^2dx + C(p+1)\int_\Omega (u\psi_1)^{p+1}dx \\
&\quad + CA(p+1)\|u_0\|_{L^1(\Omega)}^{1/6}\left\{\int_\Omega (u\psi_1)^{1+(6/5)p}\right\}^{5/6}.
\end{aligned}$$

It holds that

$$\frac{d}{dt}\int_{\Omega} u_1^{p+1}\,dx \le -\int_{\Omega} |\nabla u_1^{(p+1)/2}|^2 dx + C(p+1)^2\int_{\Omega} u_1^{p+1}\,dx$$
$$+ C(p+1)^2\left(\left\{\int_{\Omega} u_1^{1+(3/2)p}\,dx\right\}^{2/3} + \left\{\int_{\Omega} u_1^{1+(6/5)p}\,dx\right\}^{5/6}\right), \quad (37)$$

where $u_1 = u\psi_1$. Here, $C > 0$ is independent of $p \ge 1$ and we can apply an iteration scheme of Moser's type (see Alikakos [2]). To this end we make use of Gagliardo-Nirenberg's inequality in the form of

$$\|w\|_{L^q(\Omega)} \le K(\|\nabla w\|_{L^2(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2)^{(1-(1/q))/2}\|w\|_{L^1(\Omega)}^{1/q}, \quad (38)$$

where $K > 0$ independent of $q \in [1, q_0]$ for given $q_0 > 1$.

First, apply (38) for $w = u_1^{(p+1)/2}$ and $q = \dfrac{3p+2}{p+1} \in \left[\dfrac{5}{2}, 3\right)$. We have

$$C(p+1)^2\left\{\int_{\Omega} u_1^{1+(3/2)p}\,dx\right\}^{2/3}$$
$$\le C(p+1)^2\left\{\int_{\Omega} |\nabla u_1^{(p+1)/2}|^2 dx + \int_{\Omega} u_1^{p+1}\,dx\right\}^{(2p+1)/(3p+3)}\left\{\int_{\Omega} u_1^{(p+1)/2}\,dx\right\}^{2/3}.$$

Because $\dfrac{2p+1}{3p+3} < \dfrac{2}{3}$, the right-hand side is dominated by

$$C(p+1)^2\left\{\int_{\Omega}(|\nabla u_1^{(p+1)/2}|^2 + u_1^{p+1})dx + 1\right\}^{2/3}\cdot\left\{\int_{\Omega} u_1^{(p+1)/2}\,dx\right\}^{2/3}$$
$$\le \frac{1}{6}\left\{\int_{\Omega}(|\nabla u_1^{(p+1)/2}|^2 + u_1^{p+1})dx + 1\right\} + C(p+1)^6\left\{\int_{\Omega} u_1^{(p+1)/2}\,dx + 1\right\}^2.$$

Second, apply (38) for $w = u_1^{(p+1)/2}$ and $q = \dfrac{12p+10}{5p+5} \in \left[\dfrac{22}{10}, \dfrac{12}{5}\right)$. We have

$$C(p+1)^2\left\{\int_{\Omega} u_1^{1+(6/5)p}\,dx\right\}^{5/6}$$
$$\le C(p+1)^2\left\{\int_{\Omega} |\nabla u_1^{(p+1)/2}|^2 dx + \int_{\Omega} u_1^{p+1}\,dx\right\}^{(7p+5)/(12p+12)}\left\{\int_{\Omega} u_1^{(p+1)/2}\,dx\right\}^{5/6}$$
$$\le C(p+1)^2\left\{\int_{\Omega}(|\nabla u_1^{(p+1)/2}|^2 + u_1^{p+1})dx + 1\right\}^{7/12}\left\{\int_{\Omega} u_1^{(p+1)/2}\,dx\right\}^{5/6}$$

$$\leq \frac{1}{6}\left\{\int_\Omega (|\nabla u_1^{(p+1)/2}|^2 + u_1^{p+1})dx + 1\right\} + C(p+1)^{24/5}\left\{\int_\Omega u_1^{(p+1)/2}\,dx\right\}^2$$

$$\leq \frac{1}{6}\left\{\int_\Omega (|\nabla u_1^{(p+1)/2}|^2 + u_1^{p+1})dx + 1\right\} + C(p+1)^6\left\{\int_\Omega u_1^{(p+1)/2}\,dx + 1\right\}^2.$$

Finally, apply (38) for $w = u_1^{(p+1)/2}$ and $q = 2$. We have

$$C(p+1)^2\int_\Omega u_1^{p+1}\,dx$$

$$\leq C(p+1)^2\left\{\int_\Omega (|\nabla u_1^{(p+1)/2}|^2 + u_1^{p+1})dx\right\}^{1/2}\left\{\int_\Omega u_1^{(p+1)/2}\,dx\right\}$$

$$\leq \frac{1}{6}\left\{\int_\Omega (|\nabla u_1^{(p+1)/2}|^2 + u_1^{p+1})dx + 1\right\} + C(p+1)^4\left\{\int_\Omega u_1^{(p+1)/2}\,dx\right\}^2.$$

Inequality (37) has been reduced to

$$\frac{d}{dt}\int_\Omega u_1^{p+1}\,dx + \frac{1}{2}\int_\Omega |\nabla u_1^{(p+1)/2}|^2 dx$$

$$\leq \frac{1}{2}\int_\Omega u_1^{p+1}\,dx + C(p+1)^6\left\{\int_\Omega u_1^{(p+1)/2}\,dx + 1\right\}^2.$$

However, again (38) for $q = 2$ implies

$$\|w\|^2_{L^2(\Omega)} \leq K^2(\|\nabla w\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)})^{1/2}\|w\|_{L^1(\Omega)}$$

$$\leq \frac{1}{2}(\|\nabla w\|^2_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)}) + C\|w\|^2_{L^1(\Omega)}$$

and hence

$$\|u_1^{(p+1)/2}\|^2_{L^2(\Omega)} \leq \tfrac{1}{2}\|\nabla u_1^{(p+1)/2}\|^2_{L^2(\Omega)} + C\|u_1^{(p+1)/2}\|^2_{L^1(\Omega)}.$$

We obtain

$$\frac{d}{dt}\int_\Omega u_1^{p+1}\,dx + \frac{1}{4}\int_\Omega u_1^{p+1}\,dx \leq C(p+1)^6\left\{\int_\Omega u_1^{(p+1)/2}\,dx + 1\right\}^2$$

and hence

$$\sup_{0\leq t<T_{\max}}\left\{\int_\Omega u_1^{p+1}\,dx + 1\right\}$$

$$\leq C\max\left\{(p+1)^6\sup_{0\leq t<T_{\max}}\left\{\int_\Omega u_1^{(p+1)/2}\,dx + 1\right\}^2, \|u_0\|^{p+1}_{L^\infty(\Omega)}|\Omega| + 1\right\}.$$

Therefore,

$$\Phi_k = \sup_{0 \le t < T_{\max}} \int_\Omega u_1^{2^k}\, dx + 1$$

satisfies

$$\begin{aligned}\Phi_{k+1} &\le C \max\{2^{6(k+1)}\Phi_k^2,\ (|\Omega|+1)(\|u_0\|_{L^\infty(\Omega)}+1)^{2^{k+1}}\} \\ &\le C2^{6(k+1)} \max\{\Phi_k^2, (\|u_0\|_{L^\infty(\Omega)}+1)^{2^{k+1}}\} \qquad (39)\end{aligned}$$

for $k = 1, 2, \dots$.

Let $d = \|u_0\|_{L^\infty(\Omega)} + 1$. Then, (39) is reduced to

$$\Phi_{k+1} \le C^{2k-3} \cdot 2^{\sum_{\ell=2}^{k} 6(\ell+1)2^{k-\ell}} \cdot \max\{\Phi_2^{2^{k-1}}, d^{2^{k+1}}\}$$

for $k = 2, 3, \dots$. We have

$$\begin{aligned}\sup_{0\le t<T_{\max}} \left\{\int_\Omega u_1^{2^{k+1}}\, dx\right\}^{1/2^{k+1}} &\le \Phi_{k+1}^{1/2^{k+1}} \\ &\le C^{(2k-3)/2^{k+1}} \cdot 2^{6\sum_{j=1}^{\infty} j2^{-j}} \cdot \max\{\Phi_2^{1/4}, d\},\end{aligned}$$

and letting $k \to +\infty$,

$$\sup_{0\le t<T_{\max}} \|u_1(\cdot,t)\|_{L^\infty(\Omega)} \le C \max\left\{\left(\sup_{0\le t<T_{\max}} \|u_1(\cdot,t)\|_{L^4(\Omega)}^4 + 1\right)^{1/4}, d\right\}$$

follows. In use of (35), we obtain

$$\sup_{0\le t<T_{\max}} \|u_1(\cdot,t)\|_{L^\infty(\Omega)} = \sup_{0\le t<T_{\max}} \|u(\cdot,t)\psi_1\|_{L^\infty(\Omega)} < +\infty.$$

Since $r' \in (0, r)$ and $r \in (0, R)$ are arbitrary, we have (25).

The proof is complete. □

Theorems 2 and 3 are immediate consequences of the following.

PROPOSITION 4.3. *Let (u, v) be a solution to (KS) and $T_{\max} < +\infty$. Then, any $x_0 \in \mathscr{B}$ and $0 < R \ll 1$ admit*

$$\limsup_{t\uparrow T_{\max}} \int_{B(x_0,R)\cap\Omega} u(x,t)dx \ge \frac{1}{16K^2}. \qquad (40)$$

PROOF. Take $r \in (0, R)$ and $\psi = (\phi_{x_0,r,R})^6$. If (40) does not hold, then (18) implies

$$\int_\Omega u^2\psi\,dx \le 2K^2\int_{B(x_0,R)\cap\Omega} u\,dx \cdot \int_\Omega u^{-1}|\nabla u|^2\psi\,dx + C\|u\|^2_{L^1(\Omega)}$$

$$\le \frac{1}{8}\int_\Omega u^{-1}|\nabla u|^2\psi\,dx + C$$

for $0 < T_{\max} - t \ll 1$. Then (22) gives

$$\limsup_{t\uparrow T_{\max}} \int_\Omega (u\log u)\psi\,dx < +\infty,$$

and hence

$$\limsup_{t\uparrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(B(x_0,r')\cap\Omega)} < +\infty$$

follows from Lemma 4.2, where $r' \in (0,r)$. We get $x_0 \notin \mathscr{B}$ and the proof is complete. □

PROOF of THEOREM 3. In this case it holds that

$$u = u(|x|,t). \tag{41}$$

If $x_0 \in \mathscr{B}\backslash\{0\}$, we have $\mathscr{S} \equiv \{x\,|\,|x| = |x_0|\} \subset \mathscr{B}$.

Given a positive integer m, we take $0 < R \ll 1$ and $x_1,\dots,x_m \in \mathscr{S}$ satisfying $B(x_i,R)\cap B(x_j,R) = \emptyset$ for $i \ne j$. Relation (40) admits a sequence $t_k \uparrow T_{\max}$ satisfying

$$\int_{B(x_j,R)\cap\Omega} u(x,t_k)dx > \frac{1}{18K^2}$$

for $j = 1$ and hence for $j = 2,\dots,m$ by (41). Therefore,

$$\|u(\cdot,t_k)\|_{L^1(\Omega)} \ge \sum_{j=1}^m \int_{B(x_j,R)\cap\Omega} u(x,t_k)dx > \frac{m}{18K^2}$$

follows, which contradicts (11) if $m \ge 18K^2\|u_0\|_{L^1(\Omega)}$. The proof is complete. □

PROOF of THEOREM 2. If the solution satisfies (1), then

$$\int_0^{T_{\max}}\int_\Omega (v_t^2 + u|\nabla(\log u - v)|^2)dxdt < +\infty$$

follows.

Take $x_0 \in \mathscr{B}$, $0 < R \ll 1$ and let $\varphi = \varphi_{x_0, R/2, R}$. We have

$$\begin{aligned} \left|\frac{d}{dt}\int_\Omega u\varphi\, dx\right| &= \left|\int_\Omega u_t\varphi\, dx\right| = \left|\int_\Omega (\nabla u - u\nabla v)\cdot \nabla\varphi\, dx\right| \\ &\le C\int_{A(x_0, R/2, R)} u|\nabla(\log u - v)|dx \\ &\le C\left\{\|u\|_{L^1(\Omega)} + \int_\Omega u|\nabla(\log u - v)|^2 dx\right\} \end{aligned} \tag{42}$$

and hence

$$\int_0^{T_{\max}} \left|\frac{d}{dt}\int_\Omega u\varphi\, dx\right| dt < +\infty.$$

This assures the existence of $\lim_{t\uparrow T_{\max}} \int_\Omega u\varphi\, dx$. In use of (40) we have

$$\begin{aligned} \liminf_{t\uparrow T_{\max}} \int_{B(x_0, R)\cap\Omega} u &\ge \lim_{t\uparrow T_{\max}} \int_\Omega u\varphi\, dx \\ &\ge \limsup_{t\uparrow T_{\max}} \int_{B(x_0, R/2)\cap\Omega} u\, dx \\ &\ge \frac{1}{16K^2}. \end{aligned}$$

Since $x_0 \in \mathscr{B}$ and $0 < R \ll 1$ is arbitrary, this implies that

$$\sharp\mathscr{B} \le 16K^2\|u_0\|_{L^1(\Omega)} < +\infty$$

by (11), and in particular, any blow-up point is isolated. □

5. Isolated blowup points

In this section we study the behavior of u around the isolated blowup points more precisely and prove Theorem 1.

We first note the following.

LEMMA 5.1. *Let (u, v) be a solution to* (KS) *and $x_0 \in \mathscr{B}_I$. Then there exist $0 < R \ll 1$ and $\theta \in (0, 1/2)$ such that*

$$\|u\|_{C^{2+2\theta, 1+\theta}((A(x_0, r, R)\cap\Omega)\times[0, T_{\max}))} + \|v\|_{C^{2+2\theta, 1+\theta}((A(x_0, r, R)\cap\Omega)\times[0, T_{\max}))} < +\infty \tag{43}$$

for any $r \in (0, R)$.

PROOF. Because $x_0 \in \mathscr{B}_I$, there exists $R_0 > 0$ such that

$$\sup_{0\le t<T_{\max}} (\|u(\cdot, t)\|_{L^\infty(A(x_0, r_0, R_0)\cap\Omega)} + \|v(\cdot, t)\|_{L^\infty(A(x_0, r_0, R_0)\cap\Omega)}) < +\infty$$

for any $r_0 \in (0, R_0)$. Then the parabolic estimate for the second equation of (KS) (see [26]) gives that

$$\sup_{0 \le t < T_{\max}} \|\nabla v(\cdot, t)\|_{L^\infty(A(x_0, r, R) \cap \Omega)} < +\infty$$

for R and r in $r_0 < r < R < R_0$ and the standard theory for the first equation (see Theorem 10.1 of Chapter IV of [17]) applies; R' and r' in $r < r' < R' < R$ admit $\theta \in (0, 1/2)$ such that

$$\|u\|_{C^{2\theta, \theta}((A(x_0, r', R') \cap \Omega) \times [0, T_{\max}))} < +\infty.$$

Now Theorem 10.1 in Section IV of [17] is available for the second and the first equation in turn, and, given R'' and r'' in $r' < r'' < R'' < R'$ we have $\theta' \in (0, 1/2)$ such that

$$\|v\|_{C^{2+2\theta', 1+\theta'}((A(x_0, r'', R'') \cap \Omega) \times [0, T_{\max}))} < +\infty$$

and

$$\|u\|_{C^{2+2\theta', 1+\theta'}((A(x_0, r'', R'') \cap \Omega) \times [0, T_{\max}))} < +\infty.$$

Since r'' is arbitrary, proof is complete. □

An immediate consequence is the following.

LEMMA 5.2. *Let $x_0 \in \mathscr{B}_I$ and $\varphi = \varphi_{x_0, R', R}$ for $0 < R' < R \ll 1$. Then we have*

$$\sup_{0 \le t < T_{\max}} W_\varphi(t) < +\infty \tag{44}$$

and

$$\limsup_{t \uparrow T_{\max}} \int_\Omega |\nabla v|^2 \varphi \, dx = +\infty. \tag{45}$$

PROOF. Recall (6) and put

$$F(t) = W_\varphi(t) - \int_0^t R_1(u, v, \varphi) ds - \int_\Omega u\varphi \, dx.$$

Relations (11) and (43) imply

$$\left| \int_\Omega u\varphi \, dx \right| \le \|u_0\|_{L^1(\Omega)} \qquad \text{and} \qquad \sup_{0 \le t < T_{\max}} |R_1(u, v, \varphi)| < +\infty,$$

respectively. By Lemma 2.1, F is monotone decreasing in $[0, T_{\max})$ and (44) follows. Then we have

$$\int_\Omega (u \log u)\varphi\, dx \le C + \int_\Omega uv\varphi\, dx,$$

and

$$\limsup_{t \uparrow T_{\max}} \int_\Omega uv\varphi\, dx = +\infty$$

follows from Proposition 4.2. In use of Young's inequality we have

$$\begin{aligned} a\int_\Omega uv\varphi\, dx &\le \int_\Omega (u\log u)\varphi\, dx + \frac{1}{e}\int_\Omega e^{av}\varphi\, dx \\ &\le W_\varphi + \int_\Omega uv\varphi\, dx + \frac{1}{e}\int_\Omega e^{av}\varphi\, dx \\ &\le C + \int_\Omega uv\varphi\, dx + \frac{1}{e}\int_\Omega e^{av}\varphi\, dx, \end{aligned}$$

and hence

$$(a-1)\int_\Omega uv\varphi\, dx \le \frac{1}{e}\int_\Omega e^{av}\varphi\, dx + C.$$

If $a > 1$, we have

$$\limsup_{t \uparrow T_{\max}} \int_\Omega e^{av}\varphi\, dx = +\infty,$$

which implies (45) by the following Lemma. □

LEMMA 5.3. *Let* $a > 0$, $x_0 \in \mathscr{B}_I$, *and* $\varphi = \varphi_{x_0, R', R}$ *for* $0 < R' < R \ll 1$. *Then, the inequality*

$$\int_\Omega e^{av}\varphi\, dx \le C \exp\left(\frac{a^2}{8\pi}\int_\Omega |\nabla v|^2 \varphi\, dx\right) \tag{46}$$

holds on $[0, T_{\max})$. *If* $x_0 \in \Omega$, *then*

$$\int_\Omega e^{av}\varphi\, dx \le C \exp\left(\frac{a^2}{16\pi}\int_\Omega |\nabla v|^2 \varphi\, dx\right). \tag{47}$$

PROOF. We recall the following inequalities due to Moser [18] and Chang and Yang [4]: *There exists a constant* K *determined by* Ω *such that*

$$\log\left(\int_\Omega e^w\, dx\right) \le \frac{1}{2\pi^*}\|\nabla w\|_{L^2(\Omega)}^2 + \frac{1}{|\Omega|}\int_\Omega w\, dx + K$$

for $w \in X$, *where*

$$\pi^* = \begin{cases} 4\pi & \text{if } X = H^1(\Omega) \\ 8\pi & \text{if } X = H_0^1(\Omega). \end{cases}$$

Because $x_0 \in \mathscr{B}_I$, we have

$$\sup_{0 \le t < T_{\max}} \|v(\cdot, t)\|_{L^\infty(A(x_0, R', R) \cap \Omega)} < +\infty.$$

Therefore, we get

$$\begin{aligned} \int_\Omega e^{av} \varphi \, dx &\le \int_{B(x_0, R') \cap \Omega} e^{av} \, dx + \int_{A(x_0, R', R) \cap \Omega} e^{av} \varphi \, dx \\ &\le C \exp\left(\frac{a^2}{8\pi} \|\nabla v\|^2_{L^2(B(x_0, R') \cap \Omega)} + C \right) + C \\ &\le C \exp\left(\frac{a^2}{8\pi} \int_\Omega |\nabla v|^2 \varphi \, dx \right) \end{aligned}$$

by (43). This shows (46). A similar calculation gives (47) if $x_0 \in \Omega$. The proof is complete. □

The following lemma is a modification of [21].

Lemma 5.4. *We have*

$$\int_\Omega uv\varphi \, dx \le \int_\Omega (u \log u) \varphi \, dx + M_\varphi \log\left(\int_\Omega e^v \varphi \, dx \right) - M_\varphi \log M_\varphi, \tag{48}$$

where $M_\varphi = \int_\Omega u\varphi \, dx$.

Proof. Since $-\log s$ is convex, Jensen's inequality applies as

$$\begin{aligned} -\log\left(\frac{1}{M_\varphi} \int_\Omega e^v \varphi \, dx \right) &= -\log\left(\int_\Omega \frac{e^v}{u} \frac{u}{M_\varphi} \varphi \, dx \right) \\ &\le \int_\Omega \left\{ -\log\left(\frac{1}{u} e^v \right) \frac{u}{M_\varphi} \varphi \right\} dx \\ &= -\frac{1}{M_\varphi} \int_\Omega \left\{ u \log\left(\frac{e^v}{u} \right) \varphi \right\} dx. \end{aligned}$$

This means (48). □

This implies the following.

Lemma 5.5. *Suppose* $T_{\max} < +\infty$ *and take* $x_0 \in \mathscr{B}_I$, *and* $0 < R' < R \ll 1$. *Then the relation*

$$\lim_{t \uparrow T_{\max}} \int_\Omega u\varphi \, dx \geq m_*$$

follows, where m_* *is the constant in Theorem* 1 *and* $\varphi = \varphi_{x_0, R', R}$.

PROOF. In use of (43), we have

$$\left| \frac{d}{dt} \int_\Omega u\varphi \, dx \right| \leq C$$

similarly to (42), so that $\lim_{t \uparrow T_{\max}} \|u\varphi\|_{L^1(\Omega)}$ exists. Suppose

$$\lim_{t \uparrow T_{\max}} M_\varphi(t) = \lim_{t \uparrow T_{\max}} \|u\varphi\|_{L^1(\Omega)} < m_*. \tag{49}$$

In the case that $x_0 \in \Omega$ we have (47). Inequality (48) implies

$$\begin{aligned}
\frac{1}{2} \int_\Omega (|\nabla v|^2 + v^2)\varphi \, dx &= W_\varphi - \int_\Omega (u \log u - uv)\varphi \, dx \\
&\leq W_\varphi + M_\varphi \log\left(\int_\Omega e^v \varphi \, dx \right) - M_\varphi \log M_\varphi \\
&\leq C + \frac{M_\varphi}{16\pi} \int_\Omega |\nabla v|^2 \varphi \, dx + M_\varphi \log \frac{C}{M_\varphi}
\end{aligned}$$

by (44). It follows that

$$\frac{1}{2}\left(1 - \frac{M_\varphi}{8\pi}\right) \int_\Omega |\nabla v|^2 \varphi \, dx \leq C + M_\varphi \log \frac{C}{M_\varphi}.$$

Therefore, (49) with $m_* = 8\pi$ gives

$$\limsup_{t \uparrow T_{\max}} \int_{B(x_0, R/2) \cap \Omega} |\nabla v|^2 dx \leq \limsup_{t \uparrow T_{\max}} \int_\Omega |\nabla v|^2 \varphi \, dx < +\infty.$$

This contradicts (45) with R replaced by $R/2$.

The case $x_0 \in \partial\Omega$ can be treated similarly and the proof is complete. □

We are able to give the following.

PROOF of THEOREM 1. Let $x_0 \in \mathscr{B}_I$ and $\varphi = \varphi_{x_0, R/2, R}$. From above lemma, the value

$$m(x_0, R) = \lim_{t \uparrow T_{\max}} \int_\Omega u(x,t)\varphi_R(x) dx \geq m_*$$

exists for any $x_0 \in \mathscr{B}_I$ and $0 < R \ll 1$. Moreover,

$$m(x_0, R) - m(x_0, R/2) = \lim_{t \uparrow T_{\max}} \int_\Omega u(x,t)(\varphi_R(x) - \varphi_{R/2}(x)) dx \geq 0$$

and there exists

$$m(x_0) = \lim_{k\to\infty} m(x_0, R/2^k) \geq m_*.$$

Inequality (43) implies

$$\sup_{A(x_0,r,R)\cap\Omega\times[0,T_{\max})} |u_t| < +\infty$$

for $0 < r < R$. Therefore,

$$f(x) = u(x,0) + \int_0^{T_{\max}} u_t(x,t)dt$$
$$= \lim_{t\uparrow T_{\max}} u(x,t) \geq 0 \tag{50}$$

exists for $x \in \overline{B(x_0,R)\cap\Omega}\setminus\{x_0\}$. In use of (43) again, convergence (50) holds in the sense $C(\overline{A(x_0,r,R)\cap\Omega})$, where $r \in (0,R)$. Also $f \in L^1(\Omega)$ follows from (11).

For simplicity we set $E = B(x_0,R)\cap\Omega$. Given $\xi \in C(\bar{E})$, we have

$$\int_E u\xi\,dx - m(x_0)\xi(x_0) - \int_E f\xi\,dx$$
$$= \xi(x_0)\left(\int_E u\varphi_{R/2^k}\,dx - m(x_0)\right) + \int_E (\xi - \xi(x_0))u\varphi_{R/2^k}\,dx$$
$$- \int_E \xi f\varphi_{R/2^k}\,dx + \int_E \xi(u-f)(1-\varphi_{R/2^k})dx$$

for $k = 1,2,3,\ldots$. It follows that

$$\left|\int_E u\xi\,dx - m(x_0)\xi(x_0) - \int_E f\xi\,dx\right|$$
$$\leq \|\xi\|_{L^\infty(E)}\left|\int_E u\varphi_{R/2^k}\,dx - m(x_0)\right| + \|u_0\|_{L^1(\Omega)}\|\xi - \xi(x_0)\|_{L^\infty(B(x_0,R/2^k)\cap\Omega)}$$
$$+ \|\xi\|_{L^\infty(E)}\left(\int_E f\varphi_{R/2^k}\,dx + \|u-f\|_{L^\infty(A(x_0,R/2^{k+1},R/2^k)\cap\Omega)}\right)$$

and hence

$$\limsup_{t\uparrow T_{\max}}\left|\int_E u\xi\,dx - m(x_0)\xi(x_0) - \int_E f\xi\,dx\right|$$
$$\leq \|\xi\|_{L^\infty(E)}|m(x_0,R/2^k) - m(x_0)|$$
$$+ \|u_0\|_{L^1(\Omega)}\|\xi - \xi(x_0)\|_{L^\infty(B(x_0,R/2^k)\cap\Omega)} + \|\xi\|_{L^\infty(E)}\int_E f\varphi_{R/2^k}\,dx.$$

Making $k \to +\infty$, we get

$$\lim_{t \uparrow T_{\max}} \int_E u\xi \, dx = m(x_0)\xi(x_0) + \int_E f\xi \, dx$$

by $f \in L^1(E)$. This means (2) and proof is complete. □

References

[1] R. D. Adams, Sobolev Spaces, Academic Press, New York, 1975.

[2] N. D. Alikakos, L^p bounds of solutions of reaction-diffusion equations, Comm. Partial Differential Equations **4** (1979), 827–868.

[3] P. Biler, Local and global solvability of some parabolic systems modeling chemotaxis, Adv. Math. Sci. Appl. **8** (1998), 715–743.

[4] S. Y. A. Chang and P. C. Yang, Conformal deformation of metrics on S^2, J. Differential Geom. **27** (1988), 259–296.

[5] S. Childress, Nonlinear aspects of chemotaxis, Math. Biosci. **56** (1981), 217–237.

[6] S. Childress and J. K. Percus, Chemotactic collapse in two dimensions, Lecture Notes in Biomath. **55**, Springer, Berlin, 1984, pp. 61–66.

[7] H. Gajewski and K. Zacharias, Global behaviour of a reaction-diffusion system modelling chemotaxis, Math. Nachr. **195** (1998), 77–114.

[8] Y. Giga, A local characterization of blowup points of semilinear heat equations, Comm. Math. Phys. **103** (1986), 415–421.

[9] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Second edition, Springer, Berlin, 1983.

[10] G. Harada, T. Nagai, T. Senba and T. Suzuki, Concentration lemma, Brezis-Merle type inequality, and a parabolic system of chemotaxis, to appear in Adv. Differential Equations.

[11] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math. **840**, Springer, Berlin, 1981.

[12] M. A. Herrero and J. J. L. Velázquez, A blow-up mechanism for a chemotaxis model, Ann. Scuola Norm. Sup. Pisa IV **35** (1997), 633–683.

[13] R. Ikehata and T. Suzuki, Stable and unstable sets for evolution equations of parabolic and hyperbolic type, Hiroshima Math. J. **26** (1996), 475–491.

[14] W. Jäger and S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis, Trans. Amer. Math. Soc. **329** (1992), 819–824.

[15] T. Kato, Perturbation Theory for Linear Operators, Springer, Berlin, 1966.

[16] E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theor. Biol. **26** (1970), 399–415.

[17] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uralt'seva, Linear and Quasi-linear Equations of Parabolic Type, Nauka, Moscow, 1967: English translation: Amer. Math. Soc., Providence, R. I., 1968.

[18] J. Moser, A sharp form of an inequality by N. Trudinger, Indiana Univ. Math. J. **20** (1971), 1077–1092.

[19] T. Nagai, Blow-up of radially symmetric solutions to a chemotaxis system, Adv. Math. Sci. Appl. **5** (1995), 581–601.

[20] T. Nagai, T. Senba and T. Suzuki, Concentration behavior of blow-up solutions for a simplified system of chemotaxis, preprint.

[21] T. Nagai, T. Senba and K. Yoshida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, Funckcial. Ekvac. **40** (1997), 411–433.

[22] V. Nanjundiah, Chemotaxis, signal relaying, and aggregation morphology, J. Theor. Biol. **42** (1973), 63–105.

[23] T. Senba and T. Suzuki, Some structures of the solution set for a stationary system of chemotaxis, Adv. Math. Sci. Appl. **10** (2000), 191–224.

[24] T. Senba and T. Suzuki, Chemotactic collapse in a parabolic—elliptic system of mathematical biology, to appear in Advances in Differential Equations.

[25] T. Senba and T. Suzuki, Local and norm behavior of blowup solutions to a parabolic system of chemotaxis, to appear in J. Korean Math. Soc.

[26] H. Tanabe, Functional Analytic Methods for Partial Differential Equations, Dekker, New York, 1997.

[27] A. Yagi, Norm behavior of solutions to the parabolic system of chemotaxis, Math. Japonica **45** (1997), 241–265.

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