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An index of an enhanced state of a virtual link diagram and Miyazawa polynomials

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Abstract. We construct a polynomial invariant of a virtual magnetic graph diagram by defining an index of an enhanced state. For a virtual link diagram, it equals the Miyazawa polynomial and then the maximal degree on t of the polynomials not only gives a lower bound of the real crossing number but also that of the virtual crossing number. Moreover, by definition we can calculate the polynomial for a link in a thickened surface or a Gauss chord diagram directly without transforming it into a virtual link diagram.

1. Introduction

Virtual knot theory is introduced by Kauffman [9] which is a generalization of knot theory. A *virtual link diagram* is a link diagram in $\mathbf{R}^2$ possibly with some encircled crossings without over/under information, called *virtual crossings*. A *virtual link* is the equivalence class of such a diagram by generalized Reidemeister moves (Reidemeister moves of type I, of type II, of type III and virtual Reidemeister moves of type I, of type II, of type III, and of type IV depicted in Figure 1).

Virtual links are abstracted from link diagrams on a surface. They are stable Reidemeister equivalence classes of link diagrams on closed oriented surfaces [1, 6, 9, 10]. Invariants of virtual links are invariants of links in thickened surfaces.

A lot of invariants of virtual links are defined with disregard to virtual crossings. In [9], Kauffman defined a polynomial invariant $f_L(A) \in \mathbf{Z}[A^2, A^{-2}]$ for a virtual link L, which we call the Jones-Kauffman polynomial. It is defined through Kauffman's bracket in the same manner as that of a classical link, namely considering states by smoothing real crossings of L. For a

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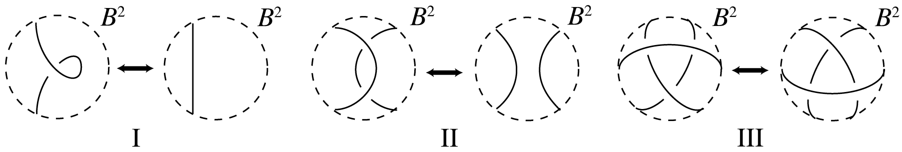
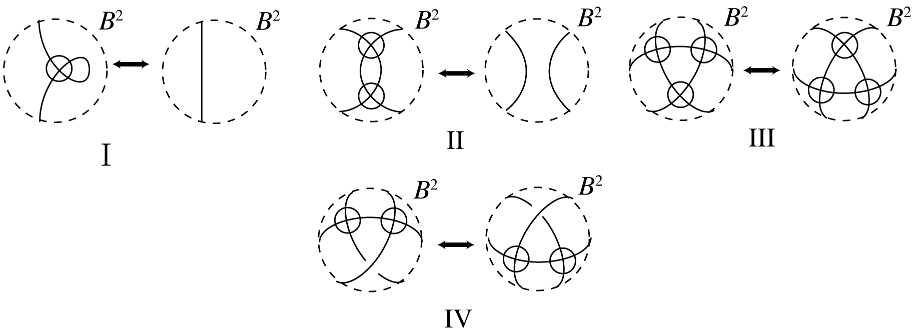


Reidemeister moves



virtual Reidemeister moves

Fig. 1

classical link L, it is the Jones polynomial $V_L(t)$ after substituting $\sqrt{t}$ for A^{-2}. Sawollek [12] applied an invariant of a link in a thickened surface defined by Jaeger, Kauffman and Saleur [5] to a virtual link. It is the determinant of a matrix derived from the information of real crossings. Silver and Williams [13] defined an invariant of a virtual link without any reference to virtual crossings and so do Bartholomew, Budden, Fenn, Jordan and Kauffman [2, 3, 4]. On the other hand, the Miyazawa polynomial is defined by use of virtual writhe obtained from virtual crossings. It is not calculated for a link diagram on a surface unless it is described as a virtual link diagram. The Miyazawa polynomial is introduced as an invariant of a virtual magnetic graph diagram, which is a generalization of a virtual link.

In this paper we define a polynomial invariant, $X_D(A,t)$, of a virtual magnetic graph diagram by introducing an index of an enhanced state, without use of any information on virtual crossings. If the diagram is a virtual link diagram, the index of a state is equal to the virtual writhe in the sense of Miyazawa (Theorem 3). Hence for a virtual link, our invariant coincides with the Miyazawa polynomial. It means that we give an alternative definition of the Miyazawa polynomial of a virtual link. Since our definition does not use virtual crossings, we can calculate the polynomial for a link in a thickened surface or a Gauss chord diagram directly without transforming it into a virtual link diagram. Also we see that the maximal degree on t of the polynomial gives a lower bound of the real crossing number and the virtual crossing number.

In Section 2, we introduce our invariant $X_D(A,t)$, after recalling the Miyazawa polynomial $Y_D(A,t)$ for a virtual magnetic diagram. A relationship between them is given. In Sections 3 and 4, we give proofs of Theorem 2 and Lemma 4, respectively. In Sections 5, we show some features of the invariants. We introduce the notion of a double flype and show that our invariant and the Miyazawa polynomial are preserved under this move.

The author would like to thank Atsushi Ishii for his helpful comment to improve the proof of Lemma 4.

2. The invariants X_D and Y_D

A *magnetic graph* is a 2-valent graph whose edges are oriented alternately around vertices as in Figure 2. It may have some components consisting of closed edges without vertices.

A *virtual magnetic graph diagram*, which is written as VMG diagram for short, is a magnetic graph immersed in $\mathbf{R}^2$ generically such that some crossings have over/under information, called *real crossings*, and the other crossings are encircled without over/under information, called *virtual crossings*. See Figure 3, for example.

If two VMG diagrams are related by a finite sequence of generalized Reidemeister moves (Figure 1), they are said to be *equivalent*. (We do not allow the moves in Figure 4.) Note that virtual link diagrams are VMG diagrams without vertices, and that two virtual link diagrams represent the same virtual link if and only if they are equivalent as VMG diagrams.

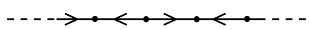


Fig. 2

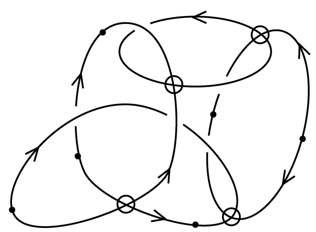


Fig. 3

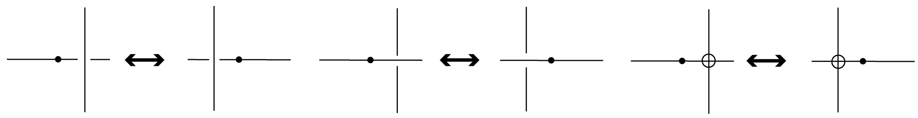


Fig. 4

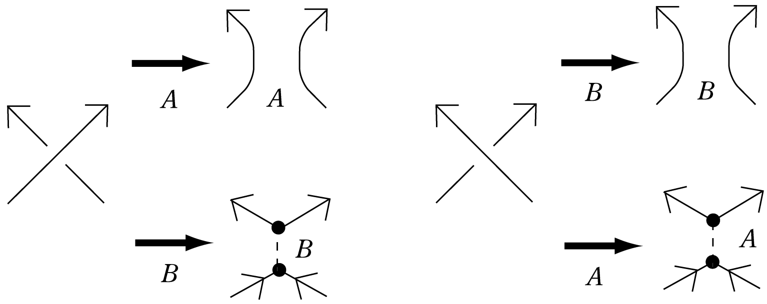


Fig. 5

First we recall the Miyazawa polynomial. Let D be a VMG diagram, and let p be a real crossing. By *applying* A*-splice* (or B*-splice*) at p, we mean the local replacement about p depicted in Figure 5.

A *state* of D is a diagram obtained from D by applying A-splice or B-splice at each real crossing, or an assignment of A-splice or B-splice to each real crossing. It is a VMG diagram without real crossings such that some vertices are connected by dashed arcs. (Our states correspond to *oriented states* in [8].) Let S be a state of a VMG diagram of D. A *weight map* σ of S is a map from the set of edges of S to $\{+1,-1\}$ such that $\sigma(e) \neq \sigma(e')$ for adjacent edges e and e' of S. An *enhanced state* means a pair (S,σ) of a state S and a weight map σ of S. The set of enhanced states of D is denoted by $s(D)$.

Let (S,σ) be an enhanced state of a VMG diagram D. For a virtual crossing v of S on edges e and e' as depicted in Figure 6 (1), the *sign* of v is defined by

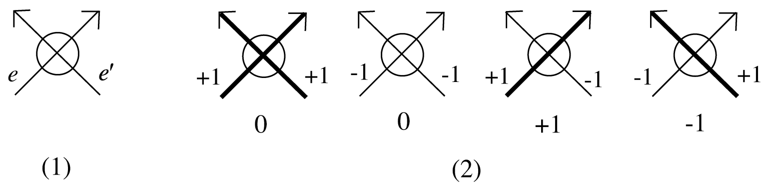


Fig. 6

$$\frac{\sigma(e) - \sigma(e')}{2}.$$

See Figure 6 (2), where edges assigned $+1$ by the weight map σ are drawn thickly. The *virtual writhe*, $\omega^v(S, \sigma)$, of (S, σ) is defined to be the sum of signs of all virtual crossings.

For a VMG diagram D, we define $\langle\langle D \rangle\rangle_Y \in \mathbf{Q}[A, A^{-1}, t, t^{-1}]$ by

$$\langle\langle D \rangle\rangle_Y = \sum_{(S,\sigma) \in s(D)} A^{\natural S} \left(\frac{-A^2 - A^{-2}}{2} \right)^{\# S} t^{\omega^v(S,\sigma)},$$

where $\natural S$ is the number of A-splices minus that of B-splices used to obtain S from D, and $\# S$ is the number of components of S (ignoring the dashed arcs). The *Miyazawa polynomial* of D, which we denote by $Y_D(A, t)$, is defined by

$$Y_D(A, t) = (-A^3)^{-\omega(D)} \langle\langle D \rangle\rangle_Y,$$

where $\omega(D)$, called the *writhe*, is the number of positive crossings minus that of negative crossings of D.

Theorem 1 (Y. Miyazawa [11]). *The Miyazawa polynomial $Y_D(A, t)$ is an invariant of the equivalence class of a VMG diagram.*

Remark. (1) The Miyazawa polynomial $Y_D(A, t)$ is a generalization of the invariant introduced in [7]. Refer to [11] for details. (2) Let $\langle D \rangle$ and $f_D(A)$ be the bracket polynomial and the Jones-Kauffman polynomial defined in [9] for a virtual link diagram D. Then by definition we have $\langle D \rangle = (-A^2 - A^{-2})^{-1} \langle\langle D \rangle\rangle_Y |_{t=1}$ and $f_D(A) = (-A^2 - A^{-2})^{-1} Y_D(A, 1)$.

Let D be a VMG diagram and (S, σ) an enhanced state of D. If two vertices of S originate from a real crossing of D, a pair of them is called a *c-pair* of S. They are connected by a dashed arc in the state. Let c be a c-pair, and let e_k $(k = 1, 2, 3, 4)$ be the edges around c as depicted in Figure 7 (1). The *sign* of c of S with respect to a weight map σ is defined as

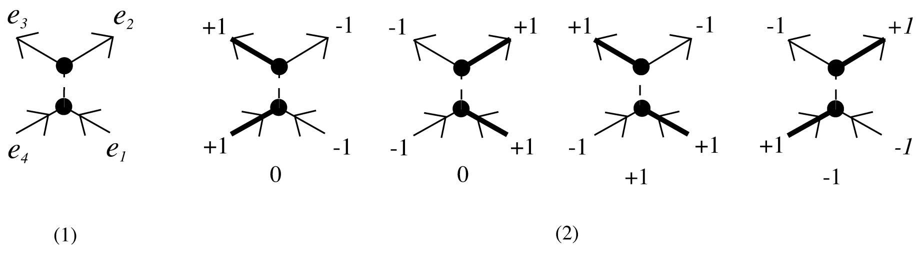
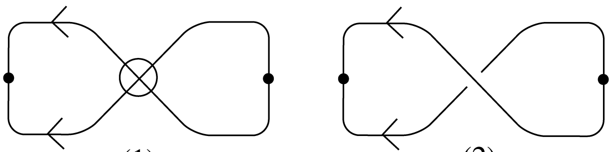


Fig. 7



(1) (2)

Fig. 8

$$\frac{\sigma(e_1)-\sigma(e_2)}{2} \qquad \left(\text{or } \frac{\sigma(e_3)-\sigma(e_4)}{2}\right).$$

See Figure 7 (2), where edges assigned $+1$ by the weight map σ, are drawn thickly.

The *index*, $\iota(S,\sigma)$, of an enhanced state (S,σ) is the sum of signs of all c-pairs. For a VMG diagram D, we define $\langle\!\langle D\rangle\!\rangle_X \in \mathbf{Q}[A,A^{-1},t,t^{-1}]$ by

$$\langle\!\langle D\rangle\!\rangle_X = \sum_{(S,\sigma)\in s(D)} A^{\natural S}\left(\frac{-A^2-A^{-2}}{2}\right)^{\# S} t^{\iota(S,\sigma)},$$

and define $X_D(A,t)$ by $X_D(A,t) = (-A^3)^{-\omega(D)}\langle\!\langle D\rangle\!\rangle_X$.

THEOREM 2. *The polynomial $X_D(A,t)$ is an invariant of the equivalence class of a VMG diagram.*

For VMG diagrams, our invariants are not equal to the Miyazawa polynomials in general. For example, let D be the diagram in Figure 8 (1). Then $X_D(A,t) = -A^2-A^{-2}$ and $Y_D(A,t) = (-A^2-A^{-2})(t^1+t^{-1})/2$. Let D be the diagram in Figure 8 (2). Then $X_D(A,t) = -A^{-3}(-A^2-A^{-2})\cdot\{2(A-A^{-3})-(A+A^{-3})(t+t^{-1})\}/4$ and $Y_D(A,t) = A^{-6}(-A^2-A^{-2})$.

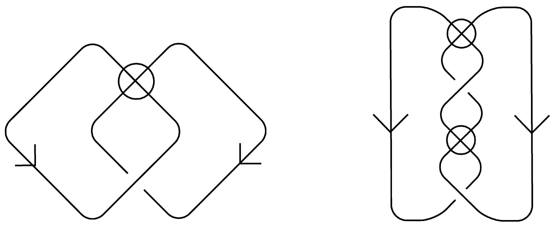
THEOREM 3. *If D is a virtual link diagram, then $X_D(A,t)$ coincides with $Y_D(A,t)$.*

This theorem shows that the definition of our invariant $X_D(A,t)$ is an alternate definition of the Miyazawa polynomial $Y_D(A,t)$ for a virtual link diagram D. It comes from the following lemma, which is our key lemma.

LEMMA 4. *Let D be a virtual link diagram, and let (S,σ) be an enhanced state of D. Then*

$$\omega^v(S,\sigma) = \iota(S,\sigma).$$

We prove Theorem 2 and Lemma 4 in Sections 3 and 4, respectively.



(1) (2)

Fig. 9

For a virtual link L, the *real crossing number* and the *virtual crossing number* of L mean the minimal number of real crossings and that of virtual crossings, respectively, among all diagrams representing L. By the definition of $Y_D(A,t)$ we have the following.

Proposition 5 (Y. Miyazawa [11]). *Let L be a virtual link, represented by a diagram D. The virtual crossing number of L is equal to or greater than the maximal degree on t of $Y_D(A,t)$ $(= X_D(A,t))$.*

By the definition of $X_D(A,t)$, we have the following.

Proposition 6. *Let L be a virtual link, represented by a diagram D. The real crossing number of L is equal to or greater than the maximal degree on t of $X_D(A,t)$ $(= Y_D(A,t))$.*

For the virtual link diagram D in Figure 9 (1), $X_D(A,t)$ is $(-A^2 - A^{-2})(-A^{-2} + A^{-4}(t+t^{-1})/2)$. Thus we see that the real crossing number and the virtual crossing number of the virtual link represented by D are one. For the virtual link diagram D in Figure 9 (2), $X_D(A,t)$ is $(-A^2 - A^{-2})^2(t+t^{-1})^2/4$. Thus the real crossing number and the virtual crossing number of the virtual link represented by D are two.

For a VMG diagram D, we denote by $D^{\#}$ the VMG diagram obtained from D by replacing all positive (or negative) crossings of D with negative (or positive) ones, and by D^* the VMG diagram which is symmetric to D with respect to a line in $\mathbf{R}^2$.

Lemma 7. *For a VMG diagram D, we have*

$$X_D(A,t) = X_{D^{\#}}(A^{-1},t) = X_{D^*}(A^{-1},t) \qquad \textit{and}$$

$$Y_D(A,t) = Y_{D^{\#}}(A^{-1},t) = Y_{D^*}(A^{-1},t).$$

Proof. This is obvious from the definition. $\square$

3. Proof of Theorem 2

Let B^2 be a 2-disk in $\mathbf{R}^2$ and let E be $\mathbf{R}^2 \backslash B^2$. Let D and D' be VMG diagrams such that $D \cap E = D' \cap E$. For enhanced states (S, σ) and (S', σ') of D and D' respectively, we denote by $(S, \sigma)|_E = (S', \sigma')|_E$ if $S \cap E = S' \cap E$ and the restriction of σ to $S \cap E$ is equal to that of σ' to $S' \cap E$.

PROOF OF THEOREM 2. Let D and D' be VMG diagrams such that they are related by a Reidemeister move of type I in a 2-disk B^2, as depicted in Figure 10. (The other cases of a Reidemeister move of type I follow from this case by Lemma 7.)

There is a three-to-one correspondence, $\{(S_1, \sigma_1), (S_2, \sigma_2), (S_3, \sigma_3)\} \leftrightarrow (S', \sigma')$, between $s(D)$ and $s(D')$ such that $(S_i, \sigma_i)|_E = (S', \sigma')|_E$ for $i = 1, 2, 3$. We assume that $S_1 = S_2$ and they are A-splices at the real crossing of $D \cap B^2$ as depicted in Figure 11.

Then

$$\natural S_1 = \natural S_2 = \natural S' + 1, \qquad \natural S_3 = \natural S' - 1, \qquad \# S_1 = \# S_2 = \# S' + 1,$$

$$\# S_3 = \# S', \qquad \text{and} \qquad \imath(S_i, \sigma_i) = \imath(S', \sigma') \qquad \text{for } i = 1, 2, 3.$$

Thus we have

$$\sum_{i=1}^{3} A^{\natural S_i} \left(\frac{-A^2 - A^{-2}}{2} \right)^{\# S_i} t^{\imath(S_i, \sigma_i)} = (-A^3) \times A^{\natural S'} \left(\frac{-A^2 - A^{-2}}{2} \right)^{\# S'} t^{\imath(S', \sigma')}.$$

Hence we have $X_D(A, t) = X_{D'}(A, t)$, since $\omega(D) = \omega(D') + 1$.

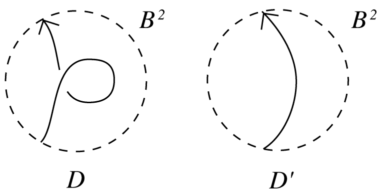


Fig. 10

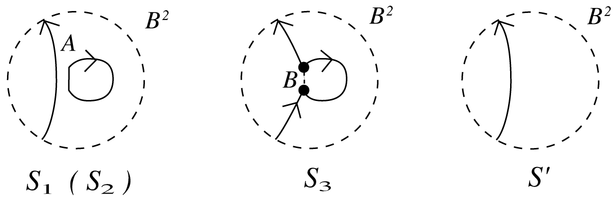


Fig. 11

Let D and D' be VMG diagrams such that they are related by a Reidemeister move of type II in a 2-disk B^2 depicted in Figure 12 (1) (or (2)). (The other cases of a Reidemeister move of type II follow from these cases by Lemma 7.)

For $\lambda_1, \lambda_2 \in \{A, B\}$, let $s_{\lambda_1\lambda_2}(D)$ be the subset of $s(D)$ consisting of enhanced states such that the assignment of the splices at the real crossings of D in B^2 are λ_1 and λ_2 as depicted in Figure 13 or Figure 14, when D and D' are as in Figure 12 (1) or in Figure 12 (2), respectively.

Note that $s(D) = s_{AB}(D) \amalg s_{AA}(D) \amalg s_{BB}(D) \amalg s_{BA}(D)$. First we show

$$(1) \qquad \sum_{(S,\sigma)\in s_{AA}(D)\amalg s_{BB}(D)\amalg s_{BA}(D)} A^{\natural S}\left(\frac{-A^2 - A^{-2}}{2}\right)^{\#S} t^{\iota(S,\sigma)} = 0.$$

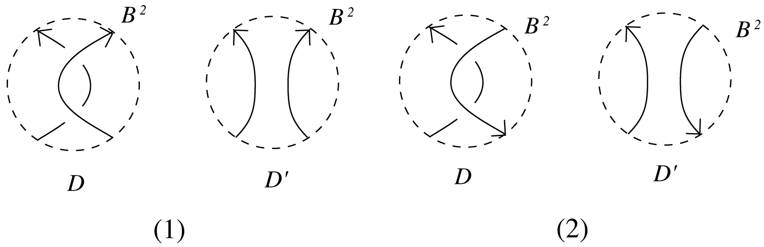


Fig. 12

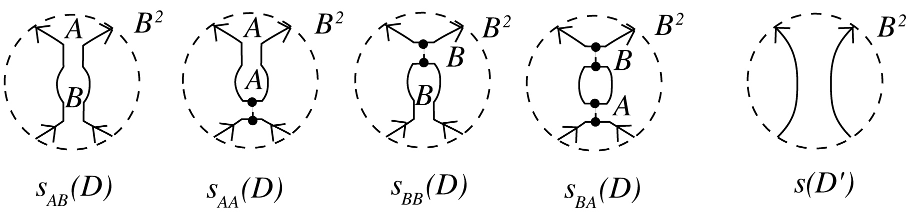


Fig. 13

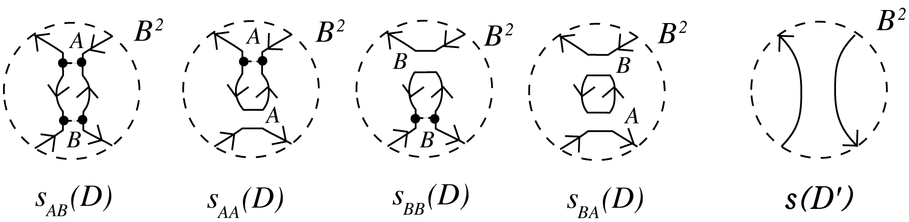


Fig. 14

There is a one-to-one-to-two correspondence, $(S_1, \sigma_1) \leftrightarrow (S_2, \sigma_2) \leftrightarrow \{(S_3, \sigma_3), (S_4, \sigma_4)\}$, among $s_{AA}(D)$, $s_{BB}(D)$, and $s_{BA}(D)$ such that $(S_i, \sigma_i)|_E = (S_j, \sigma_j)|_E$ for $i, j = 1, 2, 3, 4$, where $(S_1, \sigma_1) \in s_{AA}(D)$, $(S_2, \sigma_2) \in s_{BB}(D)$, and $(S_3, \sigma_3), (S_4, \sigma_4) \in s_{BA}(D)$.

Then

$$\natural S_2 = \natural S_1 - 4, \qquad \natural S_3 = \natural S_4 = \natural S_1 - 2, \qquad \# S_2 = \# S_1,$$

$$\# S_3 = \# S_4 = \# S_1 + 1, \qquad \text{and} \qquad \imath(S_i, \sigma_i) = \imath(S_j, \sigma_j) \qquad \text{for } i, j = 1, 2, 3, 4.$$

So we have

$$\sum_{i=1}^{4} A^{\natural S_i} \left(\frac{-A^2 - A^{-2}}{2} \right)^{\# S_i} t^{\imath(S_i, \sigma_i)} = 0,$$

which implies the equality (1).

On the other hand, it is easily seen that there is a one-to-one correspondence between $s_{AB}(D)$ and $s(D')$, and this correspondence preserves $\natural$, $\#$, and $\imath$. Thus we have $X_D(A, t) = X_{D'}(A, t)$.

Let D and D' be VMG diagrams such that they are related by a Reidemeister move type of III in a 2-disk B^2 as depicted in Figure 15. (The other cases of a Reidemeister move of type III follow from this case by Lemma 7 and by Reidemeister moves of type II.)

For $\lambda_1, \lambda_2, \lambda_3 \in \{A, B\}$, let $s_{\lambda_1 \lambda_2 \lambda_3}(D)$ (or $s_{\lambda_1 \lambda_2 \lambda_3}(D')$) be the subset of $s(D)$ (or $s(D')$) consisting of enhanced states such that the assignment of the splices at the real crossings of D (or D') in B^2, are λ_1, λ_2 and λ_3 as depicted in Figure 16.

First we show

$$(2) \qquad \sum_{(S, \sigma) \in s_{AAB}(D) \amalg s_{ABB}(D) \amalg s_{BBB}(D)} A^{\natural S} \left(\frac{-A^2 - A^{-2}}{2} \right)^{\# S} t^{\imath(S, \sigma)} = 0.$$

There is a one-to-two-to-one correspondence, $(S_1, \sigma_1) \leftrightarrow \{(S_2, \sigma_2), (S_3, \sigma_3)\} \leftrightarrow (S_4, \sigma_4)$, among $s_{AAB}(D)$, $s_{ABB}(D)$, and $s_{BBB}(D)$ such that $(S_i, \sigma_i)|_E =$

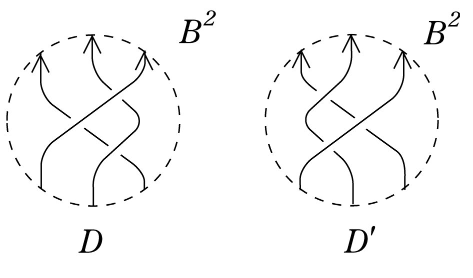


Fig. 15

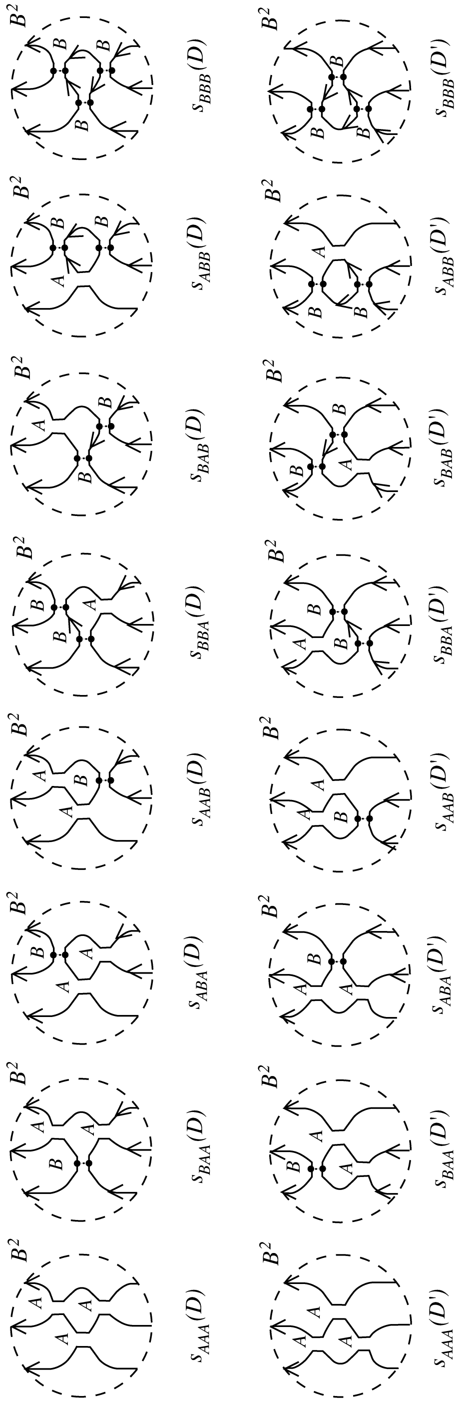

$s_{BBB}(D)$
$s_{ABB}(D)$
$s_{BAB}(D)$
$s_{BBA}(D)$
$s_{AAB}(D)$
$s_{ABA}(D)$
$s_{BAA}(D)$
$s_{AAA}(D)$
$s_{BBB}(D')$
$s_{ABB}(D')$
$s_{BAB}(D')$
$s_{BBA}(D')$
$s_{AAB}(D')$
$s_{ABA}(D')$
$s_{BAA}(D')$
$s_{AAA}(D')$
B^2
A
B

Fig. 16

$(S_j, \sigma_j)|_E$ for $i, j = 1, 2, 3, 4$, where $(S_1, \sigma_1) \in s_{AAB}(D)$, $(S_2, \sigma_2), (S_3, \sigma_3) \in s_{ABB}(D)$, and $(S_4, \sigma_4) \in s_{BBB}(D)$.

Then

$$\natural S_2 = \natural S_3 = \natural S_1 - 2, \qquad \natural S_4 = \natural S_1 - 4, \qquad \# S_2 = \# S_3 = \# S_1 + 1,$$
$$\# S_4 = \# S_1, \qquad \text{and} \qquad \imath(S_i, \sigma_i) = \imath(S_j, \sigma_j) \qquad \text{for } i, j = 1, 2, 3, 4.$$

So we have

$$\sum_{i=1}^{4} A^{\natural S_i} \left(\frac{-A^2 - A^{-2}}{2} \right)^{\# S_i} t^{\imath(S_i, \sigma_i)} = 0,$$

which implies the equality (2).

Similarly we have

$$\sum_{(S', \sigma') \in s_{AAB}(D') \amalg s_{ABB}(D') \amalg s_{BBB}(D')} A^{\natural S'} \left(\frac{-A^2 - A^{-2}}{2} \right)^{\# S'} t^{\imath(S', \sigma')} = 0.$$

On the other hand, it is easily seen that, for each $(\lambda_1, \lambda_2, \lambda_3) = (A, A, A), (B, A, A), (A, B, A), (B, B, A)$ and (B, A, B), there is a one-to-one correspondence between $s_{\lambda_1 \lambda_2 \lambda_3}(D)$ and $s_{\lambda_1 \lambda_2 \lambda_3}(D')$, and this correspondence preserves $\natural$, $\#$, and $\imath$. Thus we have $X_D(A, t) = X_{D'}(A, t)$.

If D and D' are VMG diagrams such that they are related by one of virtual Reidemeister moves of type I, of type II and of type III, it is obvious that $X_D(A, t) = X_{D'}(A, t)$.

Let D and D' be VMG diagrams such that they are related by a virtual Reidemeister move of type IV in a 2-disk B^2 as depicted in Figure 17. (The other cases of a virtual Reidemeister move of type IV follow from this case by Lemma 7 and by virtual Reidemeister moves of type II.)

For $\lambda \in \{A, B\}$, let $s_\lambda(D)$ (or $s_\lambda(D')$) be the subset of $s(D)$ (or $s(D')$) consisting of enhanced states such that the assignment of the splice at the real crossing of D (or D') in B^2 is λ (see Figure 18).

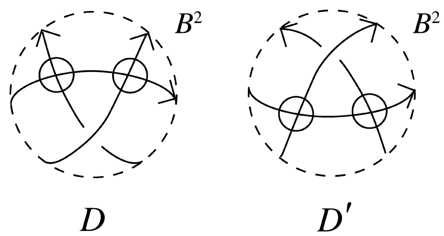


Fig. 17

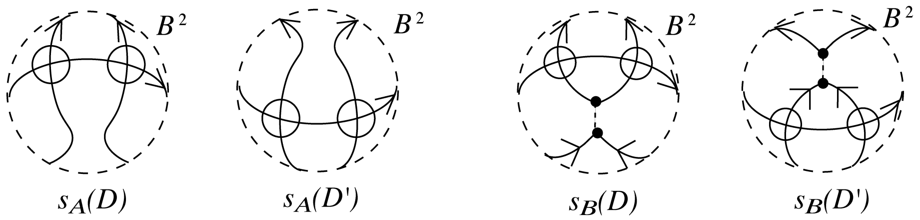


Fig. 18

For $\lambda \in \{A, B\}$, there is a one-to-one correspondence between $s_\lambda(D)$ and $s_\lambda(D')$ which preserves $\natural$, $\#$ and ι. Thus we have $\langle\!\langle D \rangle\!\rangle_X = \langle\!\langle D' \rangle\!\rangle_X$, and $X_D(A, t) = X_{D'}(A, t)$ since $\omega(D) = \omega(D')$. $\square$

4. Proof of Lemma 4

Let D be a virtual link diagram and let (S, σ) be an enhanced state. Remove all c-pairs of S by replacing them as in Figure 19, where edges assigned $+1$ by the weight map σ, are drawn thickly. Let D' be the result.

The diagram D' consists of immersed circles in $\mathbf{R}^2$, with each circle assigned $+1$ or -1. Let D'_+ and D'_- be the union of circles of D' whose signs are $+1$ and -1, respectively.

Let a_1, a_2, a_3 and a_4 be the numbers of crossings of D' as in Figure 20 (1), (2), (3) and (4), respectively. Since the algebraic intersection number of D'_+

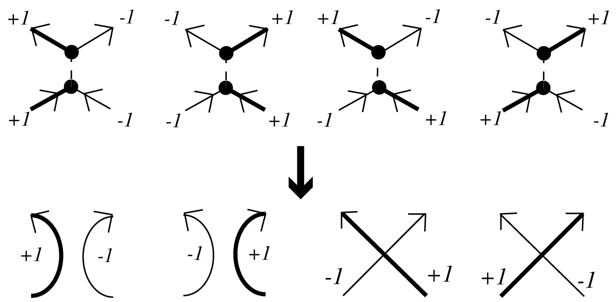


Fig. 19

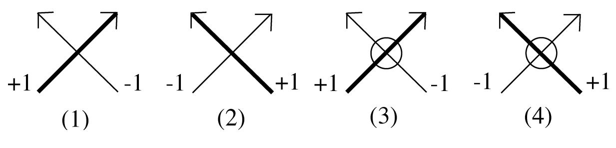


Fig. 20

and D'_- is zero, we see that $a_1 - a_2 + a_3 - a_4 = 0$. Noting that the crossings as in Figure 20 (1) and (2) come from c-pairs as in Figure 19, by definition of $\iota(S,\sigma)$, we have

$$a_1 - a_2 = -\iota(S,\sigma).$$

On the other hand, by definition of $\omega^v(S,\sigma)$, we have

$$a_3 - a_4 = \omega^v(S,\sigma).$$

Therefore we have

$$\iota(S,\sigma) = \omega^v(S,\sigma).$$

5. Some features of the invariants

The operation on virtual link diagrams depicted in Figure 21 (1) is called *Kauffman's flype*. Jones-Kauffman polynomials are preserved under Kauffman's flype. Our invariants and Miyazawa polynomials are not preserved under Kauffman's flype. (For example, Kishino's knot has a non-trivial Miyazawa polynomial [11], although it turns into a trivial diagram by Kauffman's flype.)

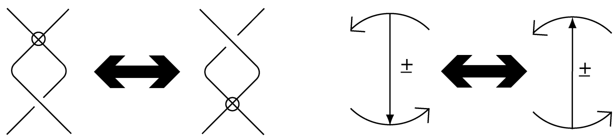
We introduce an operation which preserves our invariant and the Miyazawa polynomial.

Let us call the operations depicted in Figure 22 (1) and (2) *double flypes*.

In terms of Gauss chord diagrams, Kauffman's flype is expressed as in Figure 21 (2), and the double flypes are as in Figure 22 (3) and (4). This is the reason why we call them double flypes.

Theorem 8. *For VMG diagrams, double flypes preserve $X_D(A,t)$ and $Y_D(A,t)$.*

Proof. Let D and D' be VMG diagrams which differ as in Figure 22 (1). For $\lambda_1, \lambda_2 \in \{A, B\}$, let $s_{\lambda_1\lambda_2}(D)$ (or $s_{\lambda_1\lambda_2}(D')$) be the subset of $s(D)$ (or $s(D')$) consisting of enhanced states such that the assignment of the splices at two real crossings in B^2 are λ_1 and λ_2 as in Figure 23.



(1) (2)

Fig. 21

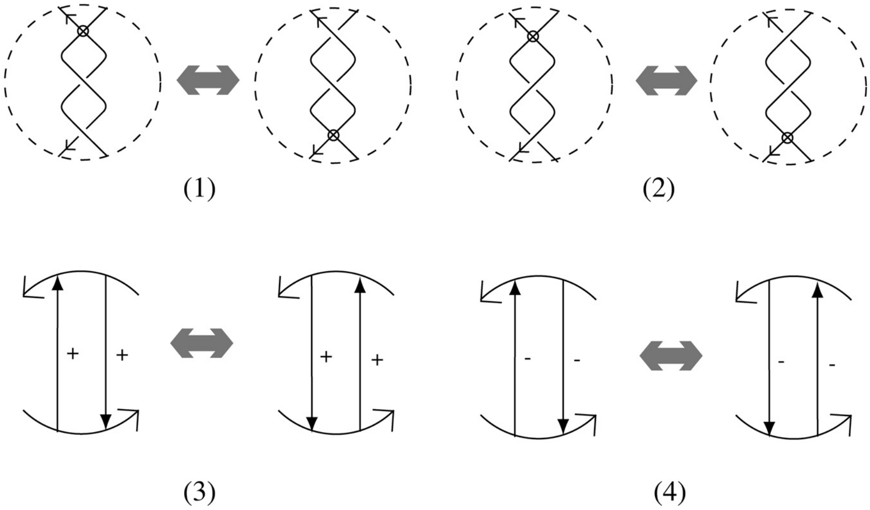


Fig. 22

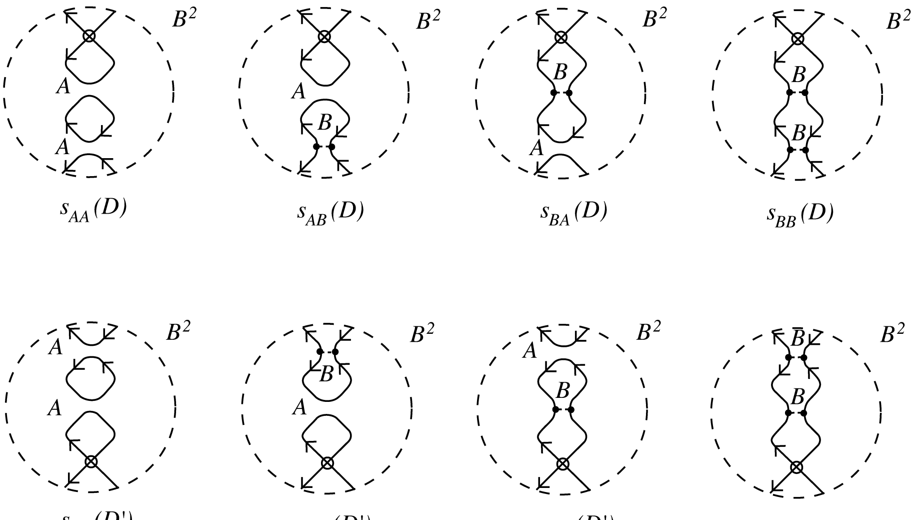


Fig. 23

For each λ_1 and λ_2 ($\lambda_1, \lambda_2 \in \{A, B\}$), there is a one-to-one correspondence, $(S, \sigma) \leftrightarrow (S', \sigma')$, between $s_{\lambda_1\lambda_2}(D)$ and $s_{\lambda_1\lambda_2}(D')$ such that $(S, \sigma)|_E = (S', \sigma')|_E$ and

$$\natural S = \natural S', \qquad \# S = \# S', \qquad \iota(S, \sigma) = \iota(S', \sigma') \qquad \text{and} \qquad \omega^v(S, \sigma) = \omega^v(S', \sigma').$$

Hence we have $X_D(A, t) = X_{D'}(A, t)$ and $Y_D(A, t) = Y_{D'}(A, t)$.

If D and D' differ as in Figure 22 (2), we have the conclusion by Lemma 7. □

The two virtual link diagrams in Figure 24 are related by a double flype. Their Miyazawa polynomials are $((A^{-6} - A^{-10} + A^{-14} - A^{-18}) \cdot (t + t^{-1})/2 + A^{-4} - A^{-8} + A^{-12})(-A^2 - A^{-2})$. They are not equivalent since the invariant defined in Sawollek [12] of the virtual link diagram on the left is $y^{-1}(x-1)(x+1)(y+1)(x+y)$ and that on the right is $2y^{-1}(x-1)(y+1)(x+y)$.

We do not call the operations in Figure 25 (1) and (2) double flypes. These operations do not preserve $X_D(A,t)$ and $Y_D(A,t)$.

The two virtual link diagrams in Figure 26 are related by the operation of Figure 25 (1). The polynomial $X_D(A,t)$ $(= Y_D(A,t))$ of the virtual link

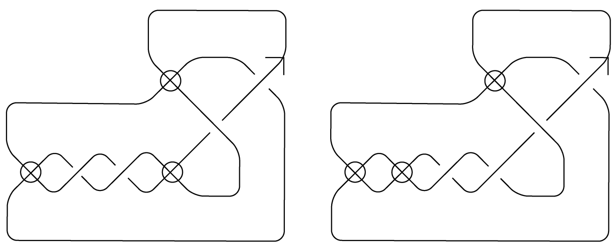


Fig. 24

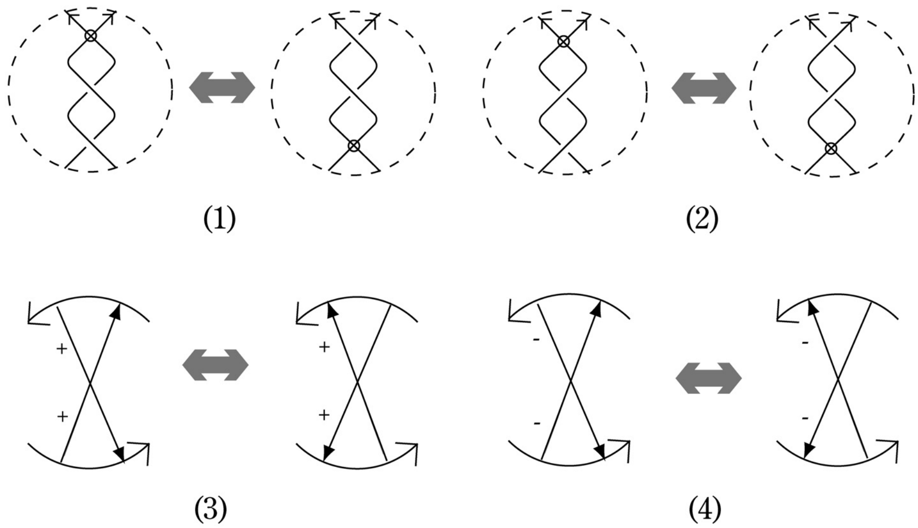


Fig. 25

diagram on the left is $((A^{-12} - A^{-16})(t^2 + t^{-2})/2 + (A^{-10} - A^{-14})(t + t^{-1})/2 + A^{-8})(-A^2 - A^{-2})$ and that on the right is $((A^{-10} - A^{-14})(t + t^{-1})/2 + A^{-8} + A^{-12} - A^{-16})(-A^2 - A^{-2})$.

Theorem 9. *Let D^1, D^2, D^3, D^+, D^0 and D^v be VMG diagrams such that they differ as in Figure 27 in B^2 and they are identical outside B^2. Then we have*

$$\begin{aligned} &2A^6(X_{D^1} + X_{D^3} - (A^2 - A^{-2})A^{-2}X_{D^2}) \\ &\quad = \delta((A^2 - A^{-2})A^{-1}(t + t^{-1})(AX_{D^+} + A^{-1}X_{D^0}) - 2X_{D^v}), \end{aligned}$$

where $\delta = -A^2 - A^{-2}$, and the same equality holds for Y_D's.

In what follows, let δ denote $-A^2 - A^{-2}$. For a VMG diagram D and a set of enhanced states of D, say s', we define $X_D|_{s'}$ by

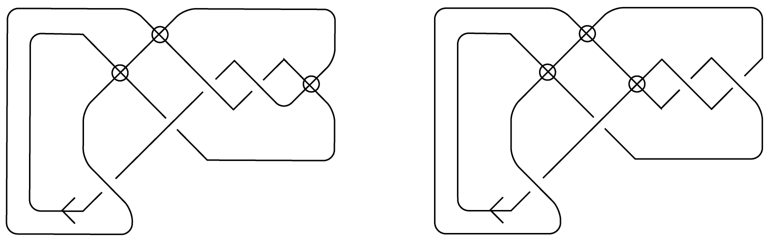


Fig. 26

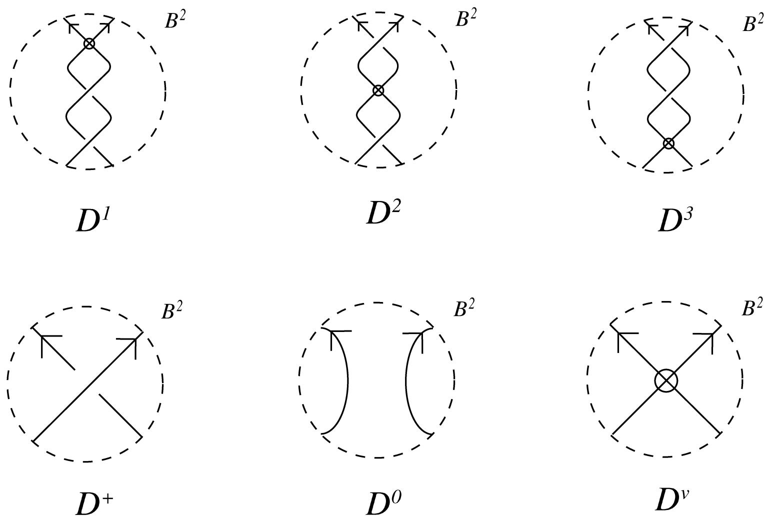


Fig. 27

$$X_D|_{s'} = (-A^3)^{-\omega(D)} \sum_{(S,\sigma)\in s'} A^{\natural S} \left(\frac{\delta}{2}\right)^{\#S} t^{\imath(S,\sigma)}.$$

It is obvious that $X_D|_{s'} = X_D(A,t)$ if $s' = s(D)$, the set of all enhanced states of D.

PROOF. For $\lambda_1, \lambda_2 \in \{A, B\}$ and $l \in \{1,2,3\}$, let $s_{\lambda_1\lambda_2}(D^l)$ be the subset of $s(D^l)$ consisting of enhanced states such that the assignment of the splices at the real crossings of D^l in B^2 are λ_1 and λ_2 as in Figure 28. For $\lambda \in \{A, B\}$, let $s_\lambda(D^+)$ be the subset of $s(D^+)$ consisting of enhanced states such that the assignment of the splice at the real crossing of D^+ in B^2 is λ as in Figure 28.

All enhanced states for D^1, D^2, D^3, D^+, D^0 and D^v belong to the following families;

- $s_{AA}(D^1)$, $s_{AA}(D^2)$, $s_{AA}(D^3)$ and $s(D^v)$
- $s_{AB}(D^1)$, $s_{BA}(D^1)$, $s_{AB}(D^2)$ and $s_{BB}(D^1)$
- $s_{AB}(D^3)$, $s_{BA}(D^3)$, $s_{BA}(D^2)$ and $s_{BB}(D^3)$
- $s_{BB}(D^2)$ and $s_B(D^+)$
- $s(D^0)$ and $s_A(D^+)$

There is a one-to-one-to-one-to-one correspondence, $(S_1,\sigma_1) \leftrightarrow (S_2,\sigma_2) \leftrightarrow (S_3,\sigma_3) \leftrightarrow (S^v,\sigma^v)$, among $s_{AA}(D^1)$, $s_{AA}(D^2)$, $s_{AA}(D^3)$ and $s(D^v)$ such that $(S_i,\sigma_i)|_E = (S^v,\sigma^v)|_E$ for $i = 1,2,3$. Then

$$\natural S_i = \natural S^v + 2, \qquad \#S_i = \#S^v, \qquad \text{and} \qquad \imath(S_i,\sigma_i) = \imath(S^v,\sigma^v),$$

for $i = 1,2,3$. Hence we have

$$\begin{aligned} &A^{\natural S_1}\left(\frac{\delta}{2}\right)^{\#S_1} t^{\imath(S_1,\sigma_1)} + A^{\natural S_3}\left(\frac{\delta}{2}\right)^{\#S_3} t^{\imath(S_3,\sigma_3)} \\ &\qquad - (A^2 - A^{-2})A^{-2}A^{\natural S_2}\left(\frac{\delta}{2}\right)^{\#S_2} t^{\imath(S_2,\sigma_2)} = -\delta A^{\natural S_v}\left(\frac{\delta}{2}\right)^{\#S_v} t^{\imath(S_v,\sigma_v)}. \end{aligned}$$

This implies that

$$2A^6(X_{D^1}|_{s_{AA}(D^1)} + X_{D^3}|_{s_{AA}(D^3)} - (A^2 - A^{-2})A^{-2}X_{D^2}|_{s_{AA}(D^2)}) = -2\delta X_{D^v}.$$

There is a one-to-one-to-two-to-one correspondence, $(S_4,\sigma_4) \leftrightarrow (S_5,\sigma_5) \leftrightarrow \{(S_6,\sigma_6),(S_7,\sigma_7)\} \leftrightarrow (S_8,\sigma_8)$, among $s_{AB}(D^1)$, $s_{BA}(D^1)$, $s_{BB}(D^1)$ and $s_{AB}(D^2)$ such that $(S_i,\sigma_i)|_E = (S_j,\sigma_j)|_E$ for $i,j = 4,5,6,7,8$. Then

$$\natural S_5 = \natural S_4, \qquad \natural S_6 = \natural S_7 = \natural S_4 - 2, \qquad \natural S_8 = \natural S_4,$$

$$\#S_5 = \#S_4, \qquad \#S_6 = \#S_7 = \#S_4 + 1, \qquad \#S_8 = \#S_4, \qquad \text{and}$$

$$\imath(S_i,\sigma_i) = \imath(S_j,\sigma_j), \qquad \text{for } i,j = 4,5,6,7,8.$$

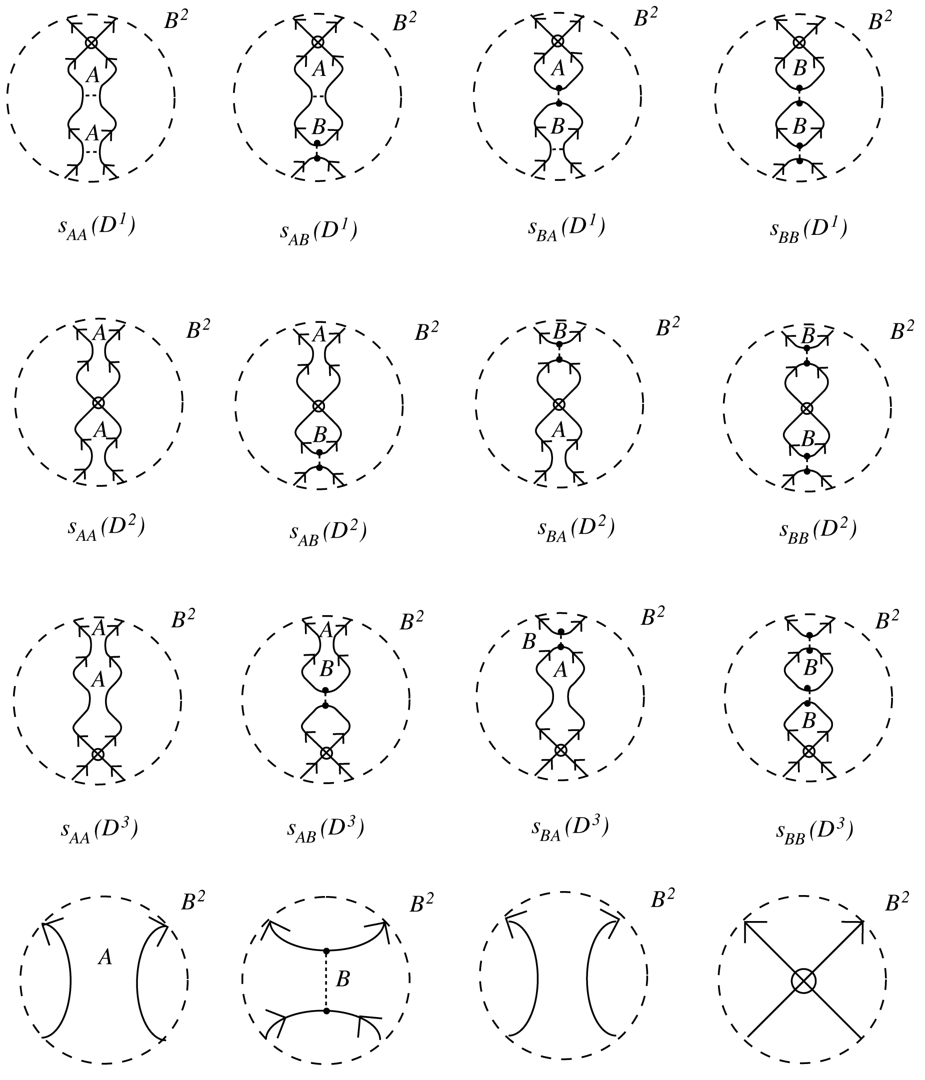


Fig. 28

Hence we obtain

$$\sum_{i=4}^{7} A^{\natural S_i}\left(\frac{\delta}{2}\right)^{\# S_i} t^{\imath(S_i,\sigma_i)} - (A^2 - A^{-2})A^{-2}A^{\natural S_8}\left(\frac{\delta}{2}\right)^{\# S_8} t^{\imath(S_8,\sigma_8)} = 0.$$

This implies that

$$2A^6(X_{D^1}|_{s_{AB}(D^1)\amalg s_{BA}(D^1)\amalg s_{BB}(D^1)} - (A^2 - A^{-2})A^{-2}X_{D^2}|_{s_{AB}(D^2)}) = 0.$$

Similarly we have

$$2A^6(X_{D^3}|_{s_{AB}(D^3)\amalg s_{BA}(D^3)\amalg s_{BB}(D^3)} - (A^2 - A^{-2})A^{-2}X_{D^2}|_{s_{BA}(D^2)}) = 0.$$

There is a two-to-one correspondence, $\{(S_9, \sigma_9), (S_{10}, \sigma_{10})\} \leftrightarrow (S^+, \sigma^+)$, between $s_{BB}(D^2)$ and $s_B(D^+)$ such that $(S_i, \sigma_i)|_E = (S^+, \sigma^+)|_E$ for $i = 9, 10$. Then

$$\natural S_i = \natural S^+ - 1, \qquad \# S_i = \# S^+ + 1 \qquad \text{and} \qquad \imath(S_i, \sigma_i) = \imath(S^+, \sigma^+) + (-1)^i,$$

for $i = 9, 10$. Thus we have

$$\sum_{i=9}^{10} A^{\natural S_i}\left(\frac{\delta}{2}\right)^{\# S_i} t^{\imath(S_i,\sigma_i)} = \frac{t + t^{-1}}{2}\delta A^{-1}A^{\natural S^+}\left(\frac{\delta}{2}\right)^{\# S^+} t^{\imath(S^+,\sigma^+)}.$$

This implies that

$$2A^4(A^2 - A^{-2})X_{D^2}|_{s_{BB}(D^2)} = -(t + t^{-1})(A^2 - A^{-2})\delta X_{D^+}|_{s_B(D^+)}.$$

There is a one-to-one correspondence, $(S^0, \sigma^0) \leftrightarrow (S^+, \sigma^+)$, between $s(D^0)$ and $s_A(D^+)$ such that $(S^0, \sigma^0)|_E = (S^+, \sigma^+)|_E$. Then we have

$$A^{\natural S^+}\left(\frac{\delta}{2}\right)^{\# S^+} t^{\imath(S^+,\sigma^+)} = AA^{\natural S^0}\left(\frac{\delta}{2}\right)^{\# S^0} t^{\imath(S^0,\sigma^0)}.$$

This implies that

$$X_{D^+}|_{s_A(D^+)} = -A^{-2}X_{D^0}.$$

So we have $X_{D^+}|_{s_B(D^+)} = X_{D^+} - X_{D^+}|_{s_A(D^+)} = X_{D^+} + A^{-2}X_{D^0}$. Therefore we have the desired equality for X_D's. Similarly we have the conclusion for Y_D's. □

By Lemma 7, we have

$$\begin{aligned} &2A^{-6}(X_{D^1\#} + X_{D^3\#} - (A^{-2} - A^2)A^2X_{D^2\#}) \\ &\quad = \delta((A^{-2} - A^2)A(t + t^{-1})(A^{-1}X_{D^+\#} + AX_{D^0\#}) - 2X_{D^v\#}). \end{aligned}$$

Similarly we have the same equality for Y_D's.

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