Hiroshima Math. J.
36 (2006), 397–403

Disjoint stars and forbidden subgraphs

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(Received August 24, 2005)
(Revised February 6, 2006)

Abstract. Let r, k be integers with $r \geq 3$, $k \geq 2$. We prove that if G is a $K_{1,r}$-free graph of order at least $(k-1)(2r-1)+1$ with $\delta(G) \geq 2$, then G contains k vertex-disjoint copies of $K_{1,2}$. This result is motivated by the problem of characterizing a forbidden subgraph H which satisfies the statement "every H-free graph of sufficiently large order with minimum degree at least t contains k vertex-disjoint copies of a star $K_{1,t}$." In this paper, we also give the answer to this problem.

1. Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges. For a graph G, we denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of G, respectively. For a vertex x of a graph G, the neighborhood of x in G is denoted by $N_G(x)$, and we let $d_G(x) := |N_G(x)|$. For a graph G and a fixed graph H, we say that G is H-free if G does not contain H as an induced subgraph. A graph $K_{1,3}$ is called *claw*, and a $K_{1,3}$-free graph is called a *claw-free* graph.

Our notation is standard except possibly for the following. Let G be a graph. For a subset L of $V(G)$, the subgraph induced by L is denoted by $\langle L \rangle$. For a subset M of $V(G)$, we let $G - M = \langle V(G) - M \rangle$. For subsets L and M of $V(G)$ with $L \cap M = \emptyset$, we let $E(L, M)$ denote the set of edges of G joining a vertex in L and a vertex in M. A vertex x is often identified with the set $\{x\}$. Thus if $x \in V(G)$, then $G - x$ means $G - \{x\}$, and $E(x, M)$ means $E(\{x\}, M)$ for $M \subset V(G - x)$.

In this paper, we are concerned with the existence of vertex-disjoint copies of $K_{1,t}$ and forbidden subgraphs. As for the existence of vertex-disjoint copies of $K_{1,t}$ in general graphs, Ota made the following conjecture in [5].

[1] This work was partially supported by the 21st Century COE Program; Integrative Mathematical Sciences: Progress in Mathematics Motivated by Social and Natural Sciences. Also, this work was partially supported by Japan Society for the Promotion of Science, Grant-in-Aid for Young Scientists (B), 18740059.

2000 *Mathematics Subject Classification*. 05C70, 05C75.

Keywords and phrases. claw-free, star, vertex-disjoint, forbidden subgraphs.

Conjecture 1.1 ([5]). *Let k, t be integers with $k \geq 2$, $t \geq 2$. Let G be a graph of order at least $(t+1)k + t^2 - t$ with $\delta(G) \geq k + t - 1$. Then G contains k vertex-disjoint copies of $K_{1,t}$.*

As is shown in [5], in this conjecture, the condition on the minimum degree of G is sharp in the sense that for any fixed t and k, there exists a graph of arbitrarily large order which has minimum degree $k + t - 2$ but does not contain k vertex-disjoint copies of $K_{1,t}$ and, if k is sufficiently large compared with t, then the condition on the order of G is also sharp in the sense that there exists a graph G with $|V(G)| = (t+1)k + t^2 - t - 1$ and $\delta(G) \geq k + t - 1$ such that G does not contain k vertex-disjoint copies of $K_{1,t}$. Conjecture 1.1 is settled affirmatively for $t = 2$ in [5]. Also, in [1], Egawa and Ota proved that Conjecture 1.1 is true for $t = 3$. As for the case $t \geq 4$ of this conjecture, the author obtained the following partial result in [4]:

Theorem 1.1 ([4]). *Let k, t be integers with $k \geq 2$, $t \geq 4$. Let G be a graph of order at least $(t+1)k + 2t^2 - 4t + 2$ with $\delta(G) \geq k + t - 1$. Then G contains k vertex-disjoint copies of $K_{1,t}$.*

In this paper, we focus on the relationship between the existence of vertex-disjoint copies of $K_{1,t}$ in graphs and forbidden subgraphs. From the structure of $K_{1,t}$, the degree condition "$\delta(G) \geq t$" seems to be natural for a graph to contain $K_{1,t}$. So, now we consider the statement "every H-free graph of sufficiently large order with minimum degree at least t contains k vertex-disjoint copies of $K_{1,t}$." The problem is to determine H that makes the statement true.

Our result is the following:

Theorem 1.2. *Let $k \geq 3$, $t \geq 2$, and let H be a connected graph with $|V(H)| \geq 3$. If there exists a positive integer n_0 such that every H-free graph G with $|V(G)| \geq n_0$ and $\delta(G) \geq t$ contains k vertex-disjoint copies of $K_{1,t}$, then $H \in \{K_{1,r} \mid r \geq 2\}$.*

We see from Theorem 1.2 that a star $K_{1,r}$ is important as a forbidden subgraph for a graph with minimum degree at least t to have k vertex-disjoint copies of $K_{1,t}$. Along this line, we propose the following conjecture:

Conjecture 1.2. *Let r, k, t be integers with $r \geq 3$, $k \geq 2$ and $t \geq 2$. If G is a $K_{1,r}$-free graph of order at least $(k-1)\{t(r-1)+1\}+1$ with $\delta(G) \geq t$, then G contains k vertex-disjoint copies of $K_{1,t}$.*

If the conjecture is true, the bound on $|V(G)|$ is best possible. To see this, let $B_i = K_t$ for each $1 \leq i \leq r-1$, and consider $G = \bigcup_{i=1}^{k-1} A_i$ where $A_i = K_1 + \bigcup_{j=1}^{r-1} B_j$ for each $1 \leq i \leq k-1$. Then G is a $K_{1,r}$-free graph of order

$(k-1)\{t(r-1)+1\}$ with $\delta(G) \geq t$. It is easy to check that G does not contain k vertex-disjoint copies of $K_{1,t}$.

The author proved that Conjecture 1.2 is true for $r = t = 3$ in [3].

Theorem 1.3 ([3]). *Let G be a claw-free graph of order at least $7k-6$ with $\delta(G) \geq 3$. Then G contains k vertex-disjoint claws.*

Also, as for this conjecture, the following theorem is proved in [2]:

Theorem 1.4 ([2]). *Let r, t be integers with $r \geq 3$, $t \geq 2$. Let G be a $K_{1,r}$-free graph of order at least $(t+1)(k-1)\{t(r-1)+1\}+1$ with $\delta(G) \geq t$. Then G contains k vertex-disjoint copies of $K_{1,t}$.*

In this paper, we prove that Conjecture 1.2 is true for $t = 2$.

Theorem 1.5. *Let r, k be integers with $r \geq 3$, $k \geq 2$. If G is a $K_{1,r}$-free graph of order at least $(k-1)(2r-1)+1$ with $\delta(G) \geq 2$, then G contains k vertex-disjoint copies of $K_{1,2}$.*

2. Proof of Theorem 1.2

Let k, t, n_0 be fixed integers as in the assumption of Theorem 1.2. By contradiction, we may assume that H is not isomorphic to a star (i.e., $H \notin \{K_{1,r} \mid r \geq 2\}$). For an integer i with $1 \leq i$, let X_i be a complete balanced bipartite graph of order $2(t-1)$ with partite sets Y_i, Z_i with $|Y_i| = |Z_i| = t-1$. We define G_1, G_2 as follows:

(1) G_1 is a graph with vertex set $V(G_1)$ and edge set $E(G_1)$ as follows:

$$V(G_1) = \{y, z\} \cup \left(\bigcup_{j=1}^{m} V(X_j) \right),$$

$$E(G_1) = \left(\bigcup_{j=1}^{m} E(X_j) \right) \cup \left\{ yp \,\middle|\, p \in \bigcup_{j=1}^{m} Y_j \right\} \cup \left\{ zq \,\middle|\, q \in \bigcup_{j=1}^{m} Z_j \right\}$$

where m is an integer with $2m(t-1)+2 \geq n_0$.

(2) $G_2 = K_1 + nK_t$ where n is an integer with $nt + 1 \geq n_0$.

It is easy to see that $\delta(G_i) \geq t$ and G_i does not contain k vertex-disjoint copies of $K_{1,t}$ for $i = 1, 2$. Hence by the assumption of Theorem 1.2, it follows that both G_1 and G_2 contain H as an induced subgraph. Since G_1 contains H as a(n induced) subgraph, H does not contain K_3. On the other hand, since G_2 contains H as an induced subgraph, this together with $H \notin \{K_{1,r} \mid r \geq 2\}$ implies that H contains K_3 because $|V(H)| \geq 3$. This is a contradiction. This completes the proof of Theorem 1.2. ■

3. Proof of Theorem 1.5

Let G be a $K_{1,r}$-free graph of order at least $(k-1)(2r-1)+1$ with $\delta(G) \geq 2$. Take s vertex-disjoint subgraphs $C_1, C_2, \ldots, C_s$ such that C_i contains $K_{1,2}$ as a spanning subgraph for each i with $1 \leq i \leq s$. Let $C = \langle V(C_1) \cup \cdots \cup V(C_s) \rangle$ and $H = G - C$. We may assume that $C_1, C_2, \ldots, C_s$ are chosen so that

(1) s is maximum, and
(2) subject to the condition (1), $|E(H)|$ is maximum, and
(3) subject to the condition (2), $\sum_{i=1}^{s} |E(C_i)|$ is maximum.

We may assume that $s \leq k-1$. It follows from the maximality of s that H consists of $m+n$ components $P_1, P_2, \ldots, P_m$, $Q_1, Q_2, \ldots, Q_n$ where $P_i \cong K_2$ for $1 \leq i \leq m$ and $Q_j \cong K_1$ for $1 \leq j \leq n$. (Thus $V(H) = V(P_1) \cup \cdots \cup V(P_m) \cup V(Q_1) \cup \cdots \cup V(Q_n)$ where $m \geq 0$, $n \geq 0$.) Note that the condition (2) is equivalent to the statement that "m is maximum." Then

$$\begin{aligned}|V(H)| &\geq (k-1)(2r-1)+1-3s \\ &\geq (k-1)(2r-1)+1-3(k-1) = 2(k-1)(r-2)+1.\end{aligned}$$

For each i with $1 \leq i \leq m$, take $p_i \in V(P_i)$ and fix it. Also, let $V(Q_j) = \{q_j\}$ for each j with $1 \leq j \leq n$. Let $H' = \{p_1, \ldots, p_m, q_1, \ldots, q_n\}$. Then $|H'| = m+n \geq \left\lceil \frac{|V(H)|}{2} \right\rceil \geq (k-1)(r-2)+1$. For each i with $1 \leq i \leq s$, let a_i be a vertex in $V(C_i)$ such that $|E(a_i, V(C_i - a_i))| = 2$, and fix it.

We first prove the following claim.

CLAIM 3.1. *Let i be an integer with $1 \leq i \leq s$. Let x, y be distinct vertices in C_i, and let H_1, H_2 be distinct components of H with $|V(H_1)| \geq |V(H_2)|$. Suppose that $E(x, V(H_1)) \neq \emptyset$ and $E(y, V(H_2)) \neq \emptyset$. Then $x = a_i$, $H_1 \in \{P_1, P_2, \ldots, P_m\}$ and $H_2 \in \{Q_1, Q_2, \ldots, Q_n\}$. Furthermore, $C_i \cong K_{1,2}$ and $E(V(C_i - y), V(H_2)) = \emptyset$.*

PROOF. If $H_1, H_2 \in \{P_1, P_2, \ldots, P_m\}$, then we can find two vertex-disjoint copies of $K_{1,2}$ in $\langle V(H_1) \cup V(H_2) \cup V(C_i) \rangle$, which contradicts the maximality of s. Thus $H_2 \in \{Q_1, Q_2, \ldots, Q_n\}$ holds. Suppose that $H_1 \in Q_1, Q_2, \ldots, Q_n$. Then by the symmetry of the roles of H_1 and H_2, we may assume that $y \neq a_i$. Then by replacing C_i by $\langle V(C_i - y) \cup V(H_1) \rangle$, we get a contradiction to the maximality of m. Thus we have $H_1 \in \{P_1, P_2, \ldots, P_m\}$. Next suppose that $x \neq a_i$. Then we can find two vertex-disjoint copies of $K_{1,2}$ in $\langle V(H_1) \cup V(H_2) \cup V(C_i) \rangle$, which contradicts the maximality of s. Thus $x = a_i$, and it is easy to see that this forces $C_i \cong K_{1,2}$. Now, if $E(V(C_i) - \{x, y\}, V(H_2)) \neq \emptyset$, then $\langle V(H_1) \cup \{x\} \rangle \supset K_{1,2}$ and $\langle V(H_2) \cup V(C_i - x) \rangle \supset K_{1,2}$, a contradiction. Also, if $E(a_i, V(H_2)) \neq \emptyset$, then by replacing C_i by $\langle \{a_i, y\} \cup V(H_2) \rangle$, we get a contradiction to the maximality of $\sum_{i=1}^{s}|E(C_i)|$. Thus $E(a_i, V(H_2)) = \emptyset$. Hence $E(V(C_i - y), V(H_2)) = \emptyset$. ■

We define a family $\mathscr{F}$ of vertex subsets as follows:

$$\mathscr{F} := \{\{x_1, x_2, \ldots, x_s\} \mid x_i \in V(C_i) \text{ for each } i \text{ with } 1 \le i \le s\}$$

Claim 3.2. *There exists $F \in \mathscr{F}$ such that $\bigcup_{x \in F} N_G(x) \supset H'$.*

Proof. Choose $F \in \mathscr{F}$ so that $|(\bigcup_{x \in F} N_G(x)) \cap H'|$ is maximum, and subject to the condition, $|F \cap \{a_1, \ldots, a_s\}|$ is maximum. Put $F = \{x_1, x_2, \ldots, x_s\}$. We may assume that there exists $v \in H'$ such that $v \notin (\bigcup_{x \in F} N_G(x)) \cap H'$. Since $\delta(G) \ge 2$, $E(v, V(C)) \ne \emptyset$. Hence there exists C_i such that $x_i v \notin E(G)$ and $E(v, V(C_i - x_i)) \ne \emptyset$. Let y_i be a vertex in C_i such that $y_i v \in E(G)$. If $E(x_i, H') = \emptyset$, then by replacing F by $(F - x_i) \cup \{y_i\}$, we get a contradiction to the maximality of $|(\bigcup_{x \in F} N_G(x)) \cap H'|$. Hence there exists $u \in H' - v$ such that $x_i u \in E(G)$. Then by Claim 3.1, $C_i \cong K_{1,2}$, $a_i \in \{x_i, y_i\}$ and we may assume that $\{u, v\} = \{p_1, q_1\}$. Suppose that $y_i = a_i$. Then it is easy to see that $E(x_i, H' - u) = \emptyset$. Then by replacing F by $(F - x_i) \cup \{y_i\}$, we get a contradiction to the maximality of $|F \cap \{a_1, \ldots, a_s\}|$. Thus $x_i = a_i$. Then by Claim 3.1, $u = p_1$, $v = q_1$ and $E(V(C_i - y_i), q_1) = \emptyset$. Since $\delta(G) \ge 2$, there exists C_j with $j \ne i$ such that $E(q_1, V(C_j)) \ne \emptyset$. Let y_j be a vertex in C_j such that $q_1 y_j \in E(G)$. Since $v \notin (\bigcup_{x \in F} N_G(x)) \cap H'$, $y_j \notin F$. By the choice of F, we have $E(x_j, H' - q_1) \ne \emptyset$. Then by Claim 3.1, $x_j = a_j$. If there exists $v' \in H' - \{p_1, q_1\}$ such that $a_j v' \in E(G)$, then by Claim 3.1, we may assume $v' \in V(P_2)$, and then by replacing C_i, C_j by $\langle V(P_1) \cup \{a_i\}\rangle$, $\langle V(P_2) \cup \{a_j\}\rangle$, $\langle\{y_i, q_1, y_j\}\rangle$, we get a contradiction to the maximality of s. Thus we have $E(x_j, H') = \{p_1 x_j\}$ by Claim 3.1. Then by replacing F by $(F - \{x_j\}) \cup \{y_j\}$, we get a contradiction to the maximality of $|(\bigcup_{x \in F} N_G(x)) \cap H'|$. ■

By Claim 3.2, we choose $F \in \mathscr{F}$ such that $\bigcup_{x \in F} N_G(x) \supset H'$ and fix it. Since $|H'| \ge (k-1)(r-2)+1$, there exists $x_i \in F$ such that $|E(x_i, H')| \ge r-1$ because $|F| = s \le k-1$. Let $N_G(x_i) \cap H' = \{v_1, v_2, \ldots, v_l\}$ where $l \ge r-1$. Since G is $K_{1,r}$-free, it follows that $l = r-1$ because H' is independent. Also, we see from Claim 3.1 that $x_i \ne a_i$. Hence $C_i \cong K_{1,2}$ and we may assume that $v_1 a_i \in E(G)$ because G is $K_{1,r}$-free. If $r \ge 4$, then by replacing C_i by $\langle V(C_i - x_i) \cup \{v_1\}\rangle$, $\langle\{v_2, v_3, x_i\}\rangle$, we get a contradiction to the maximality of s. Thus we have $r = 3$ and $l = 2$.

Claim 3.3. *Let i be an integer with $1 \le i \le s$, and let w_1, w_2 be distinct vertices in H'. Suppose that $E(w_1, V(C_i)) \ne \emptyset$ and $E(w_2, V(C_i)) \ne \emptyset$. Then there exists C_j with $j \ne i$ such that $E(\{w_1, w_2\}, V(C_j)) \ne \emptyset$ and $E(V(C_j), H') = E(V(C_j), \{w_1, w_2\})$. Further, $E(V(C_i), H') = E(V(C_i), \{w_1, w_2\})$ holds.*

Proof. Since now G is claw-free, $E(V(C_i), \{w_1, w_2\})$ has two independent edges. Let $V(C_i) = \{a_i, b_i, c_i\}$. Then in view of Claim 3.1, we may assume that $C_i \cong K_{1,2}$, $w_1 \in \{p_1, p_2, \dots, p_m\}$, $w_2 \in \{q_1, q_2, \dots, q_n\}$ and $E(V(C_i), \{w_1, w_2\}) = \{a_i w_1, b_i w_1, b_i w_2\}$. Then by the maximality of s, it is easy to see that $E(V(C_i), H') = E(V(C_i), \{w_1, w_2\})$. Also, since $\delta(G) \geq 2$, there exists C_j with $j \neq i$ such that $E(w_2, V(C_j)) \neq \emptyset$. If $C_j \cong K_{1,2}$, then since G is claw-free, we have $E(w_2, V(C_j - a_j)) \neq \emptyset$. Also, if $C_j \cong K_3$, then by symmetry, we may assume that $E(w_2, V(C_j - a_j)) \neq \emptyset$. Thus, in any case, we may assume that there exists $b_j \in V(C_j - a_j)$ such that $b_j w_2 \in E(G)$. Suppose that there exists $w' \in H' - \{w_1, w_2\}$ such that $E(w', V(C_j)) \neq \emptyset$. Then by Claim 3.1, $w' a_j \in E(G)$. Then $\langle \{w'\} \cup V(C_j - b_j) \rangle \supset K_{1,2}$, $\langle \{b_j, w_2, b_i\} \rangle \supset K_{1,2}$, $\langle \{w_1\} \cup V(C_i - b_i) \rangle \supset K_{1,2}$, a contradiction. This implies that $E(V(C_j), H') = E(V(C_j), \{w_1, w_2\})$. Thus the claim holds. ■

We choose $F' \subset F$ with $\bigcup_{x \in F'} N_G(x) \supset H'$ so that $|F'|$ is minimum. Let $F_i := \{x \in F' \mid |E(x, V(H'))| = i\}$ for $i = 1, 2$. Since G is claw-free, this together with the minimality of $|F'|$ implies that $F' = F_1 \cup F_2$.

Claim 3.4. *If $F_2 \neq \emptyset$, then there exists a one-to-one mapping $f : F_2 \to F - F'$.*

Proof. Let $x \in F_2$, and let $N_G(x) \cap H' = \{w_1, w_2\}$. We may assume that $x \in V(C_i)$. Then by Claim 3.3, there exists C_j with $j \neq i$ such that $E(V(C_j), \{w_1, w_2\}) \neq \emptyset$ and $E(V(C_j), H') = E(V(C_j), \{w_1, w_2\})$, and also $E(V(C_i), H') = E(V(C_i), \{w_1, w_2\})$ holds. Then by the minimality of $|F'|$, $V(C_j) \cap F' = \emptyset$, i.e., $V(C_j) \cap (F - F') \neq \emptyset$. This together with

$$E(V(C_i), H') = E(V(C_i), \{w_1, w_2\}), \qquad E(V(C_j), \{w_1, w_2\}) \neq \emptyset$$

and $E(V(C_j), H') = E(V(C_j), \{w_1, w_2\})$ implies that there exists a one-to-one mapping $f : F_2 \to F - F'$. ■

By Claim 3.4, $|F'| + |F_2| \leq |F|$. Consequently, $|H'| \leq |F_1| + 2|F_2| = |F'| + |F_2| \leq |F| = s \leq k - 1$, and hence $|V(H)| = \left\lceil \frac{|V(H)|}{2} \right\rceil + \left\lfloor \frac{|V(H)|}{2} \right\rfloor \leq 2|H'| \leq 2(k-1)$.

On the other hand, since $|V(H)| \geq 2(k-1)(r-2) + 1 = 2(k-1) + 1$, this is a contradiction. This completes the proof of Theorem 1.5. ■

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