Hiroshima Math. J.
36 (2006), 289–329

Tilings from non-Pisot unimodular matrices

Maki Furukado

(Received July 5, 2005)
(Revised November 8, 2005)

Abstract. Using the unimodular Pisot substitution of the free monoid on d letters, the existence of graph-directed self-similar sets $\{X_i\}_{i=1,2,\dots,d}$ satisfying the set equation (0.0.1) with the positive measure on the A-invariant contracting plane P is well-known, where A is the incidence matrix of the substitution. Moreover, under some conditions, the set $\{X_i\}_{i=1,2,\dots,d}$ is the prototile of the quasi-periodic tiling of P (see Figure 1). In this paper, even in the case of non-Pisot matrix A, the generating method of graph-directed self-similar sets and quasi-periodic tilings is proposed under the "blocking condition".

0. Introduction

The following fact is well-known: using the unimodular Pisot substitution σ of the free monoid on d letters, we obtain the prototiles $\{X_i\}_{i=1,2,\dots,d}$ with fractal boundary of the A-invariant contracting plane P, satisfying the set equation:

$$A^{-1}X_i = \bigcup_{j=1}^{l_i}(\boldsymbol{v}_j^{(i)} + X_j) \qquad \text{(non-overlapping)} \tag{0.0.1}$$

where the transformation A is the incidence matrix of the substitution σ and vectors $\boldsymbol{v}_j^{(i)} \in P$, $1 \le j \le l_i$ are some translations. Moreover, under the super coincidence condition in [14], we see that the prototiles $\{X_i\}_{i=1,2,\dots,d}$ give us a graph directed self-similar tiling of P (see Figure 1). The prototiles from the substitution have been studied first by Rauzy in [20]. Since Rauzy (see Figure 1), several properties of prototiles have been studied by many authors. For example, basic properties of $\{X_i\}_{i=1,2,\dots,d}$ have been studied in [16], [4], [10], [21] and [2], the estimation of the Hausdorff dimention of ∂X_i in [10], topological properties of X_i in [22], [1], the relation with the Markov partition generated by $\{X_i\}_{i=1,2,\dots,d}$ in [4], [18], the relation with the algebraic β-expansion in [15], [14], Diophantine approximation in [13], quasi-periodic tiling in [14], [17], etc. In fact, we know that to study the structure of $\{X_i\}_{i=1,2,\dots,d}$ is useful and

2000 *Mathematics Subject Classfication*. 52C23, 37A45, 28A80.
Key words and phrases. non-Pisot unimodular substitutions, graph-directed self-affine tilings.

important to research of fractal geometry, number theory, tiling theory, ergodic theory and dynamical systems. However, the study of the case that the matrix A is unimodular and non-Pisot is very few (see [11]). The purpose of this paper is to give a sufficient condition of the existence of prototiles $\{X_i\}_{i=1,2,\dots,\mathrm{K}}$ which satisfies the set equation (0.0.1) and generates a quasi-periodic tiling of the contracting eigenspace P, starting from the 4×4 non-Pisot unimodular hyperbolic integer matrix A.

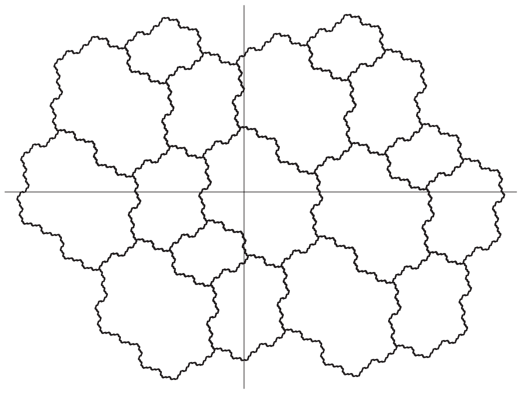


Fig. 1. Rauzy fractal tiling.

1. Definition and notations

1.1. non-Pisot matrix

In this paper, we consider that the integer matrix A satisfies the following conditions.

ASSUMPTION 1.1. *Let us assume that*
(1) *the eigenvalues* λ_i, $i = 1, 2, 3, 4$ *of the matrix* A *satisfy*

$$|\lambda_1| \geq |\lambda_2| > 1 > |\lambda_3| \geq |\lambda_4| \qquad (\textit{hyperbolic non-Pisot condition});$$

(2) $\det A = \pm 1$ *(unimodular condition);*
(3) *the matrix* A *has the standard position property which is mentioned later.*

For eigenvalues λ_i, $i = 1, 2, 3, 4$ of the matrix A, let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ be the corresponding basis of $\mathbf{R}^4$ generated by eigenvectors, that is, we consider that the 2-dimensional contracting eigenspace P_c of the linear transformation A is spanned by $\{\boldsymbol{v}_3, \boldsymbol{v}_4\}$ and that the 2-dimesional expanding eigenspace P_e is spanned by $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$. And let $\pi_c(x\boldsymbol{v}_1 + y\boldsymbol{v}_2 + z\boldsymbol{v}_3 + w\boldsymbol{v}_4) = z\boldsymbol{v}_3 + w\boldsymbol{v}_4$ and

$\pi_e(x\boldsymbol{v}_1 + y\boldsymbol{v}_2 + z\boldsymbol{v}_3 + w\boldsymbol{v}_4) = x\boldsymbol{v}_1 + y\boldsymbol{v}_2$ be the projections to P_c and P_e respectively. Then, following relations $P_c \circ A = A \circ P_c$ and $P_e \circ A = A \circ P_e$ hold.

Using the representation by

$$[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4] = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4]\begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{21} & x_{22} & x_{23} & x_{24} \\ x_{31} & x_{32} & x_{33} & x_{34} \\ x_{41} & x_{42} & x_{43} & x_{44} \end{bmatrix},$$

the projected vectors $\pi_c\boldsymbol{e}_i \in P_c$ and $\pi_e\boldsymbol{e}_i \in P_e$ of the canonical basis $\{\boldsymbol{e}_i \mid i = 1, 2, 3, 4\}$ are given by

$$\pi_c\boldsymbol{e}_i = x_{3i}\boldsymbol{v}_3 + x_{4i}\boldsymbol{v}_4 \simeq [x_{3i}, x_{4i}]^t,$$

$$\pi_e\boldsymbol{e}_i = x_{1i}\boldsymbol{v}_1 + x_{2i}\boldsymbol{v}_2 \simeq [x_{1i}, x_{2i}]^t$$

respectively.

We say that $\pi_c\boldsymbol{e}_i$ and $\pi_c\boldsymbol{e}_j$ are in *standard position* for i, j $(i \neq j)$ if $\pi_c\boldsymbol{e}_i$ is not parallel to $\pi_c\boldsymbol{e}_j$. And we say that the matrix A has the *standard position property* if any pair of $\pi_c\boldsymbol{e}_i$, $i = 1, 2, 3, 4$ are in standard position.

For easy undrstanding of several definitions and properties, we introduce an example at the end of each section.

Example. *Let us consider the following matrix* $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$. *The characteristic polynomial of* A *is given by* $\Phi_A(x) = x^4 - 3x^3 + x^2 + x + 1$ *and it is irreducible, moreover* λ_i, $1 \le i \le 4$ *of* A *satisfy* $\lambda_1 > \lambda_2 > 1 > |\lambda_3| = |\lambda_4|$. *The basis* $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ *is chosen as* $\boldsymbol{v}_1 := \boldsymbol{u}_1$, $\boldsymbol{v}_2 := \boldsymbol{u}_2$, $\boldsymbol{v}_3 := \frac{\boldsymbol{u}_3 + \boldsymbol{u}_4}{2}$, $\boldsymbol{v}_4 := \frac{\boldsymbol{u}_3 - \boldsymbol{u}_4}{2i}$ *where* $\boldsymbol{u}_j$ *is the eigenvector of* λ_j, $1 \le j \le 4$. *In this example, vectors* $\pi_c\boldsymbol{e}_i$, $i = 1, 2, 3, 4$ *are represented by the following figure (see Figure* 2*):*

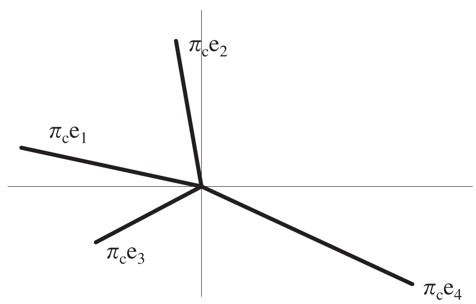


Fig. 2. Vectors $\pi_c\boldsymbol{e}_i$ $(i = 1, 2, 3, 4)$ in Example.

It is clear that the matrix A *in Example satisfies Assumption* 1.1 (1) (2) (3).

1.2. Parallelograms and segments

From now on, we denote π instead of the projection π_c and P instead of the plane P_c for simplicity.

For $i, j \in \{1,2,3,4\}$, let $i \wedge j$ be the symbolic parallelogram generated by vectors $\pi \boldsymbol{e}_i$ and $\pi \boldsymbol{e}_j$ where $i \wedge j$ $(i \neq j)$ is chosen if the counterclockwise angle α between $\pi \boldsymbol{e}_i$ and $\pi \boldsymbol{e}_j$ satisfies $0 < \alpha < \pi$. We write the set of symbolic parallelograms as

$$V_2 := \left\{ i \wedge j \,\middle|\, \begin{array}{l} i, j \in \{1,2,3,4\}, i \neq j, \\ \text{the angle } \alpha \text{ between } \pi \boldsymbol{e}_i \text{ and } \pi \boldsymbol{e}_j \text{ is chosen by } 0 < \alpha < \pi \end{array} \right\}.$$

It is clear that the cardinality of V_2 is equal to $6(={}_4C_2)$ from the standard position property.

For $i \in \{1,2,3,4\}$, let i be the symbolic segment generated by $\pi \boldsymbol{e}_i$. We write the set of symbolic segments as

$$V_1 := \{ i \mid i \in \{1,2,3,4\}\}.$$

A pair $(\boldsymbol{x}, i \wedge j) \in \pi \mathbf{Z}^4 \times V_2$ means geometrically the positive oriented parallelogram $i \wedge j$ with the base-point $\boldsymbol{x}$ of P, that is,

$$(\boldsymbol{x}, i \wedge j) := \{ \boldsymbol{x} + \mu \pi \boldsymbol{e}_i + \nu \pi \boldsymbol{e}_j \mid 0 \leq \mu, \nu \leq 1 \}$$

(see Figure 3).

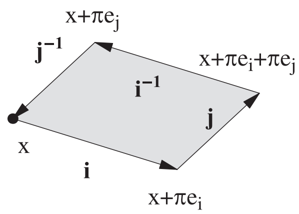


Fig. 3. $(\boldsymbol{x}, i \wedge j)$.

Let us define the set of all of the finite formal sum of the parallelogram with the base-point $\lambda \in \pi \mathbf{Z}^4 \times V_2$ as follows:

$$\mathscr{G}_2 := \left\{ \sum_{\lambda \in \pi \mathbf{Z}^4 \times V_2} m_\lambda \lambda \mid m_\lambda \in \mathbf{Z}, \#\{\lambda \mid m_\lambda \neq 0\} < +\infty \right\}.$$

We call an element of $\mathscr{G}_2$ a *patch*. For patches γ and δ such that $\gamma = \sum_{\lambda \in \pi \mathbf{Z}^4 \times V_2} m_\lambda \lambda$, $\delta = \sum_{\lambda \in \pi \mathbf{Z}^4 \times V_2} n_\lambda \lambda$, we define the sum by

$$\gamma + \delta = \sum_{\lambda \in \pi\mathbf{Z}^4 \times V_2} m_\lambda \lambda + \sum_{\lambda \in \pi\mathbf{Z}^4 \times V_2} n_\lambda \lambda := \sum_{\lambda \in \pi\mathbf{Z}^4 \times V_2} (m_\lambda + n_\lambda)\lambda.$$

Then we know that $\mathscr{G}_2$ is a free $\mathbf{Z}$-module. On the notation $\Lambda_\delta^{(+)} = \{n_\lambda \lambda \mid n_\lambda \neq 0, n_\lambda \in \mathbf{N}\}$ and $\Lambda_\delta^{(-)} = \{n_\lambda \lambda \mid n_\lambda \neq 0, n_\lambda \in -\mathbf{N}\}$ for the pach $\delta = \sum_{\lambda \in \mathbf{Z}^4 \times V_2} n_\lambda \lambda$, we say that δ is the *subpatch* of γ if $\Lambda_\gamma^{(+)} \supset \Lambda_\delta^{(+)}$, $\Lambda_\gamma^{(-)} \supset \Lambda_\delta^{(-)}$, and denote $\gamma \succ \delta$. (see Figure 4).

We continue to define the symbolic segment and the set of all of the finite formal sum of the segments analogously.

A pair $(\boldsymbol{x}, i) \in \pi\mathbf{Z}^4 \times V_1$ means geometrically the positive segment i with the base-point $\boldsymbol{x}$ of P, that is,

$$(\boldsymbol{x}, i) := \{\boldsymbol{x} + \mu\pi\boldsymbol{e}_i \mid 0 \leq \mu \leq 1\}.$$

Fig. 4. Patch δ is the subpach of γ.

Let us define the set of all of the finite formal sum of the segment with the base-point $\lambda \in \pi\mathbf{Z}^4 \times V_1$ as follows:

$$\mathscr{G}_1 := \left\{ \sum_{\lambda \in \pi\mathbf{Z}^4 \times V_1} m_\lambda \lambda \mid m_\lambda \in \mathbf{Z}, \#\{\lambda \mid m_\lambda \neq 0\} < \infty \right\}.$$

Then, $\mathscr{G}_1$ is the $\mathbf{Z}$-module analogously.

In Example, the set of symbolic parallelograms V_2 *is chosen as*

$$V_2 = \{2 \wedge 1, 1 \wedge 3, 1 \wedge 4, 2 \wedge 3, 4 \wedge 2, 3 \wedge 4\}$$

(see Figure 2*).*

1.3. The maps $E_2(\theta)$ and $E_1(\theta)$ generated by an automorphism θ of the free group $F\langle 1, 2, 3, 4 \rangle$

In this section, we consider the covering of P by parallelograms with base-points. The map $E_i(\theta)$ $(0 \leq i \leq d)$ is defined in [7] and has led many results in the Pisot case. From this fact, we introduce maps $E_2(\theta) : \mathscr{G}_2 \to \mathscr{G}_2$ and $E_1(\theta) : \mathscr{G}_1 \to \mathscr{G}_1$ for making the covering rule of P by parallelograms with base-points in the non-Pisot case.

Let $F\langle 1,2,3,4\rangle$ be the free group on the alphabet $\{1,2,3,4\}$ and let σ be an automorphism of $F\langle 1,2,3,4\rangle$.

For an automorphism σ, we define the automorphism θ as the mirror image of σ^{-1} and we denote

$$\begin{aligned}\theta(i) &:= w_1^{(i)} w_2^{(i)} \dots w_{k-1}^{(i)} \boxed{w_k^{(i)}} w_{k+1}^{(i)} \dots w_{l_i}^{(i)} \\ &= P_k^{(i)} w_k^{(i)} S_k^{(i)}\end{aligned}$$

where l_i is the length of $\theta(i)$, $P_1^{(i)} = \varepsilon$ (the empty word) and $S_{l_i}^{(i)} = \varepsilon$ for any $i \in \{1,2,3,4\}$. We call $P_k^{(i)}$ the k-prefix and $S_k^{(i)}$ the k-suffix of the element $\theta(i)$ of the free group $F\langle 1,2,3,4\rangle$ respectively (see [7]).

REMARK. θ is the mirror image of σ^{-1}, that is,

$\theta(i) = w_1^{(i)} w_2^{(i)} \dots w_{l_i-1}^{(i)} w_{l_i}^{(i)}$ is given by $\sigma^{-1}(i) = w_{l_i}^{(i)} w_{l_i-1}^{(i)} \dots w_2^{(i)} w_1^{(i)}$.

The natural homomorphism $f : F\langle 1,2,3,4\rangle \to \mathbf{Z}^4$ is defined by $f(i^a) = a\boldsymbol{e}_i$ for $a \in \mathbf{Z}$ and $f(w_1 w_2) = f(w_1) + f(w_1)$.

For an automorphism σ, the corresponding linear representation (or incidence matrix) of σ is given by

$$L_\sigma := [f(\sigma(1)), f(\sigma(2)), f(\sigma(3)), f(\sigma(4))].$$

Then, the commutative relation

$$L_\sigma \circ f = f \circ \sigma$$

holds.

From now on, we assume that the incidence matrix L_σ of σ coincides with A.

REMARK. For any unimodular matrix A, there exists an automorphism σ of the free group $F\langle 1,2,3,4\rangle$ such that $L_\sigma = A$ by Theorem 7.3.4 in [12].

Let us define the map $E_2(\theta)$ on $\mathscr{G}_2$ as follows:

$$\begin{aligned} E_2(\theta)(\mathbf{0}, i \wedge j) &:= (\mathbf{0}, \theta(i) \wedge \theta(j)) \\ &= \sum_{\substack{1 \le k \le l_i \\ 1 \le l \le l_j}} (f(P_k^{(i)}) + f(P_l^{(j)}), w_k^{(i)} \wedge w_l^{(j)}) \qquad (\mathrm{mod}\ V_2) \\ E_2(\theta)(\boldsymbol{x}, i \wedge j) &:= A^{-1}\boldsymbol{x} + E_2(\theta)(\mathbf{0}, i \wedge j) \end{aligned} \tag{1.3.2}$$

$$E_2(\theta) \sum_{\lambda \in \pi\mathbf{Z}^4 \times V_2} m_\lambda \lambda := \sum_{\lambda \in \pi\mathbf{Z}^4 \times V_2} m_\lambda E_2(\theta)\lambda.$$

Remark. For two *oriented* parallelograms $(\boldsymbol{x}, p \wedge q)$ and $(\boldsymbol{y}, r \wedge s)$, $p, q, r, s \in \{1^{\pm 1}, 2^{\pm 1}, 3^{\pm 1}, 4^{\pm 1}\}$, we say

$$(\boldsymbol{x}, p \wedge q) \equiv (\boldsymbol{y}, r \wedge s) \qquad (\mathrm{mod}\ V_2)$$

if two oriented parallelograms coincide each other including the orientation. The formula (1.3.2) means that the elements of $(\mathbf{0}, \theta(i) \wedge \theta(j))$ are rewritten by $\pm(\boldsymbol{x}, k \wedge l)$, $k \wedge l \in V_2$. For example,

$$(\mathbf{0}, i \wedge j) \equiv (\mathbf{0}, i \wedge j) \qquad (\mathrm{mod}\ V_2)$$

$$(\mathbf{0}, i^{-1} \wedge j) \equiv -(-\boldsymbol{e}_i, i \wedge j) \qquad (\mathrm{mod}\ V_2)$$

$$(\mathbf{0}, i \wedge j^{-1}) \equiv -(-\boldsymbol{e}_j, i \wedge j) \qquad (\mathrm{mod}\ V_2)$$

$$(\mathbf{0}, i^{-1} \wedge j^{-1}) \equiv (-\boldsymbol{e}_i - \boldsymbol{e}_j, i \wedge j) \qquad (\mathrm{mod}\ V_2)$$

(see Figure 5).

By the way, for the positive oriented parallelogram $(\mathbf{0}, i \wedge j)$, in most cases, the patch $E_2(\theta)(\mathbf{0}, i \wedge j)$ includes the negative oriented parallelograms. To clarify this fact, we introduce the concept of the matrix A^* as follows.

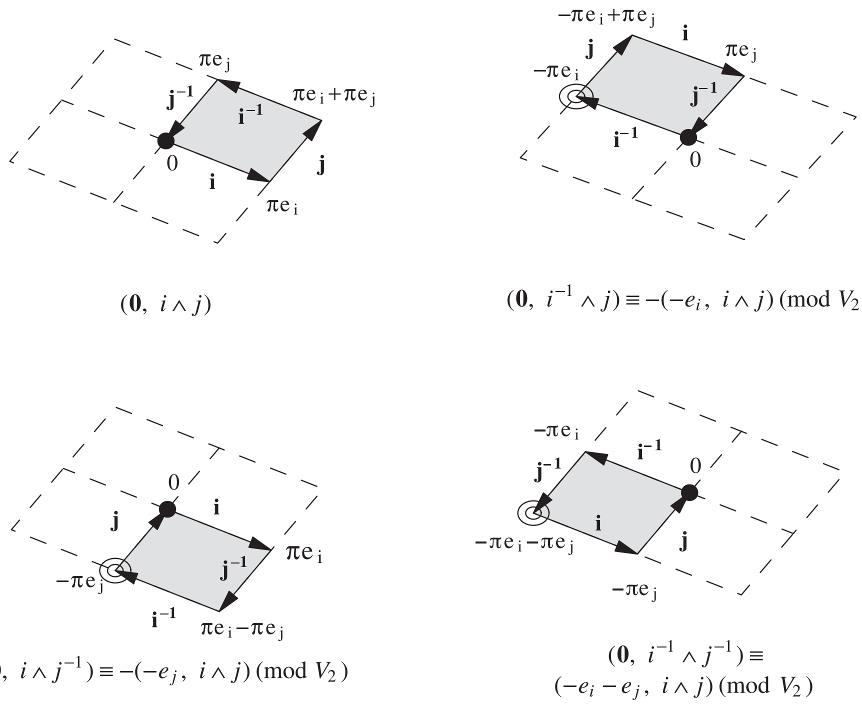


Fig. 5.

Let us denote $A^{-1} = [a_{ij}]_{1 \le i,j \le 4}$ and define the matrix $A^* := [a^*_{i\wedge j, k\wedge l}]_{i\wedge j, k\wedge l \in V_2}$ by

$$\begin{aligned} a^*_{i\wedge j,k\wedge l} &:= \det\begin{bmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{bmatrix} \\ &:= \text{“the number of the positive parallelogram } i\wedge j\text{” − “the number} \\ &\qquad \text{of the negative parallelogram } j\wedge i\text{” in the patch } E_2(\theta)(\mathbf{0}, k\wedge l) \end{aligned}$$

(see [11]). Then we know that A^* is not always $A^* \ge O$ and that A^* must be positive if all elements of $E_2(\theta)(\mathbf{0}, i\wedge j)$ are positive.

Let us define the map $E_1(\theta)$ on $\mathscr{G}_1$ analogously,

$$E_1(\theta)(\mathbf{0}, i) := (\mathbf{0}, \theta(i)) = \sum_{1\le k\le l_i} (f(P_k^{(i)}), w_k^{(i)}) \qquad (\mathrm{mod}\ V_1)$$

$$E_1(\theta)(\boldsymbol{x}, i) := A^{-1}\pi\boldsymbol{x} + E_1(\theta)(\mathbf{0}, i)$$

$$E_1(\theta)\left(\sum_{\lambda\in\pi\mathbf{Z}^4\times V_1} m_\lambda \lambda\right) := \sum_{\lambda\in\pi\mathbf{Z}^4\times V_1} m_\lambda E_1(\theta)\lambda.$$

For two *oriented* segments $(\boldsymbol{x}, p)$ and $(\boldsymbol{y}, r)$, $p, r \in \{1^{\pm 1}, 2^{\pm 1}, 3^{\pm 1}, 4^{\pm 1}\}$, we say

$$(\boldsymbol{x}, p) \equiv (\boldsymbol{y}, r) \qquad (\mathrm{mod}\ V_1)$$

analogously.

Let us define the boundary map $\partial : \mathscr{G}_2 \to \mathscr{G}_1$

$$\partial(\mathbf{0}, i\wedge j) := (\mathbf{0}, i) + (\boldsymbol{e}_i, j) - (\boldsymbol{e}_j, i) - (\mathbf{0}, j)$$

$$\partial(\boldsymbol{x}, i\wedge j) := \pi\boldsymbol{x} + \partial(\mathbf{0}, i\wedge j).$$

Then, we know the following lemma.

Lemma 1.1 ([7]). *The commutative diagram holds:*

$$\begin{array}{ccc} \mathscr{G}_2 & \xrightarrow{E_2(\theta)} & \mathscr{G}_2 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\ \mathscr{G}_1 & \xrightarrow{E_1(\theta)} & \mathscr{G}_1 \end{array}.$$

In Example, from the matrix $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ *let us choose the automorphism* $\sigma : \begin{cases} 1 \to 14 \\ 2 \to 3 \\ 3 \to 423 \\ 4 \to 142 \end{cases}$, *then the mirror image* θ *of* σ^{-1} *is determined by*

$$\sigma^{-1} : \begin{cases} 1 \to 423^{-1} \\ 2 \to 1^{-1}4 \\ 3 \to 2 \\ 4 \to 32^{-1}4^{-1}1 \end{cases}, \qquad \theta : \begin{cases} 1 \to 3^{-1}24 \\ 2 \to 41^{-1} \\ 3 \to 2 \\ 4 \to 14^{-1}2^{-1}3 \end{cases}.$$

Then, the map $E_2(\theta)$ *is given by*

$$\begin{aligned} E_2(\theta)(\mathbf{0}, 2 \wedge 1) &= (\mathbf{0}, \theta(2) \wedge \theta(1)) = (\mathbf{0}, 41^{-1} \wedge 3^{-1}24) \\ &= (\mathbf{0}, 4 \wedge 3^{-1}) + (f(3^{-1}), 4 \wedge 2) + (f(3^{-1}2), 4 \wedge 4) \\ &\quad + (f(4), 1^{-1} \wedge 3^{-1}) + (f(4) + f(3^{-1}), 1^{-1} \wedge 2) \\ &\quad + (f(4) + f(3^{-1}2), 1^{-1} \wedge 4) \qquad (\text{mod } V_2) \\ &= (-\boldsymbol{e}_3, 3 \wedge 4) + (-\boldsymbol{e}_3, 4 \wedge 2) + (\boldsymbol{e}_4 - \boldsymbol{e}_1 - \boldsymbol{e}_3, 1 \wedge 3) \\ &\quad + (\boldsymbol{e}_4 - \boldsymbol{e}_3 - \boldsymbol{e}_1, 2 \wedge 1) - (\boldsymbol{e}_4 - \boldsymbol{e}_3 + \boldsymbol{e}_2 - \boldsymbol{e}_1, 1 \wedge 4). \end{aligned}$$

Analogously, we obtain

$$\begin{aligned} E_2(\theta)(\mathbf{0}, 1 \wedge 3) &= (-\boldsymbol{e}_3, 2 \wedge 3) + (\boldsymbol{e}_2 - \boldsymbol{e}_3, 4 \wedge 2) \\ E_2(\theta)(\mathbf{0}, 1 \wedge 4) &= (-\boldsymbol{e}_3, 1 \wedge 3) + (-\boldsymbol{e}_3, 2 \wedge 1) - (\boldsymbol{e}_2 - \boldsymbol{e}_3, 1 \wedge 4) \\ E_2(\theta)(\mathbf{0}, 2 \wedge 3) &= (\mathbf{0}, 4 \wedge 2) + (-\boldsymbol{e}_1 + \boldsymbol{e}_4, 2 \wedge 1) \\ E_2(\theta)(\mathbf{0}, 4 \wedge 2) &= (\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_4, 4 \wedge 2) + (-\boldsymbol{e}_2, 2 \wedge 1) + (\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_4, 3 \wedge 4) \\ &\quad + (-\boldsymbol{e}_2, 1 \wedge 3) \\ E_2(\theta)(\mathbf{0}, 3 \wedge 4) &= (\mathbf{0}, 2 \wedge 1) + (\boldsymbol{e}_1 - \boldsymbol{e}_4, 4 \wedge 2) + (\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_4, 2 \wedge 3) \end{aligned}$$

(see Figure 6*).*

The colors of the positive and negative oriented parallelograms are gray and black respectively in this paper.

From the previous calculation for $E_2(\theta)(\mathbf{0}, i \wedge j)$, $i \wedge j \in V_2$, A^* *is given by*

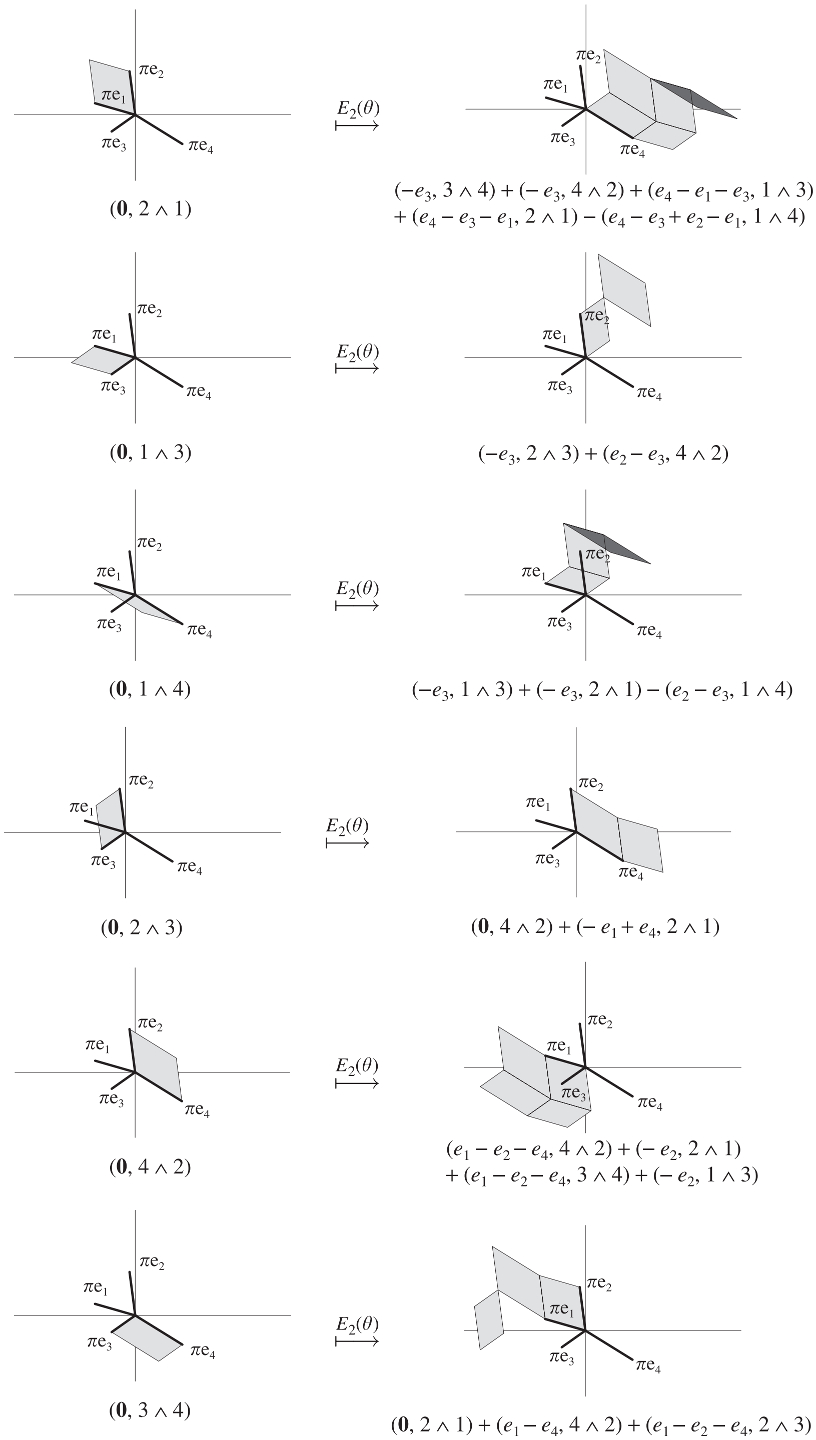


Fig. 6. $(\boldsymbol{0}, i \wedge j)$ and $E_2(\theta)(\boldsymbol{0}, i \wedge j)$, $i \wedge j \in V_2$ in Example.

$$A^* = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Moreover, the map $E_1(\theta)$ *is given by*

$$\begin{aligned} E_1(\theta)(\mathbf{0},1) &= (\mathbf{0},\theta(1)) = (\mathbf{0},3^{-1}24) \\ &= (\mathbf{0},3^{-1}) + (f(3^{-1}),2) + (f(3^{-1}2),4) \qquad (\mathrm{mod}\ V_1) \\ &= -(-\boldsymbol{e}_3,3) + (-\boldsymbol{e}_3,2) + (-\boldsymbol{e}_3+\boldsymbol{e}_2,4). \end{aligned}$$

Analogously, we obtain

$$E_1(\theta)(\mathbf{0},2) = (\mathbf{0},4) - (\boldsymbol{e}_4-\boldsymbol{e}_1,1)$$

$$E_1(\theta)(\mathbf{0},3) = (\mathbf{0},2)$$

$$E_1(\theta)(\mathbf{0},4) = (\mathbf{0},1) - (\boldsymbol{e}_1-\boldsymbol{e}_4,4) - (\boldsymbol{e}_1-\boldsymbol{e}_4-\boldsymbol{e}_2,2) + (\boldsymbol{e}_1-\boldsymbol{e}_4-\boldsymbol{e}_2,3)$$

(see Figure 7).

2. The seed $\mathscr{U}$ of $E_2(\theta)$ and the covering substitution

The patch $\gamma = \sum_{k=1}^{L} m_k(\boldsymbol{x}_k,\delta_k) \in \mathscr{G}_2$, $m_k \neq 0$, that is, the formal finite sum of oriented parallelograms, can be considered as the compact set $\gamma = \bigcup_{k=1}^{L}(\boldsymbol{x}_k,\delta_k) \subset P$. We should take care that

the boundary of the patch $\gamma \neq$ the boundary of the compact set γ

(see Figure 8).

In this section, we consider the topological property of compact sets

$$X_{i\wedge j} := \lim_{n\to\infty} A^n E_2^n(\theta)(\mathbf{0}, i\wedge j), \qquad i\wedge j \in V_2.$$

Definition 2.1. *If we can find the patch* $\mathscr{U} := \sum_{k=1}^{M} m_k(\boldsymbol{x}_k,\delta_k) \in \mathscr{G}_2$, $m_k = \pm 1$ *which satisfies the following conditions, we say that* $\mathscr{U}$ *is the* seed *of* $E_2(\theta)$ *and that* $E_2(\theta)$ *is a* covering substitution *of* P*:*

(a) $\mathscr{U}^\circ \ni \mathbf{0}$, *where* $\mathscr{U}^\circ$ *means the interior of the compact set* $\mathscr{U} = \bigcup_{k=1}^{M} m_k(\boldsymbol{x}_k,\delta_k)$*;*

(b) *there exists* $N \in \mathbf{N}$ *such that* $E_2^N(\theta)\mathscr{U} \succ \mathscr{U}$*;*

(c) $d(\partial(E_2^n(\theta)\mathscr{U}),\mathbf{0}) \to \infty \ (n\to\infty)$.

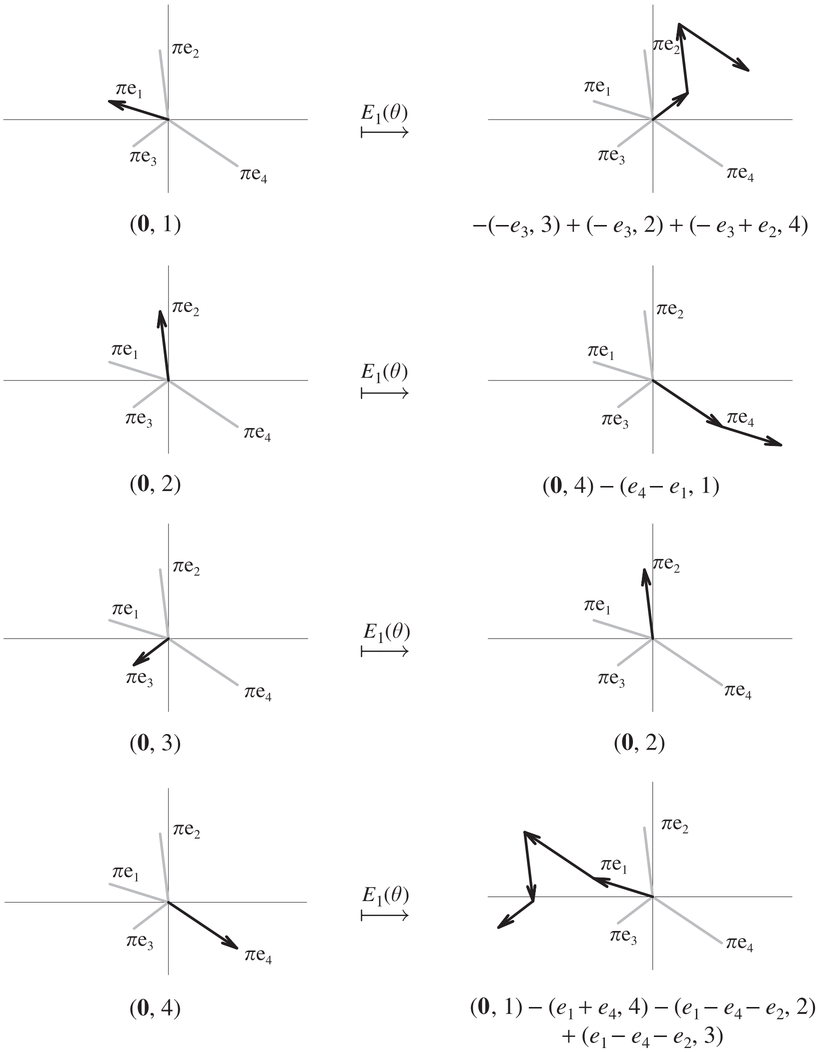


Fig. 7. $(\boldsymbol{0}, i)$ and $E_1(\theta)(\boldsymbol{0}, i)$, $i \in V_1$ in Example.

Remark.

(1) Even if there exists a seed $\mathcal{U}$ of $E_2(\theta)$ such that $E_2^N(\theta)\mathcal{U} \succ \mathcal{U}$, we can not say that $E_2^{nN}(\theta)\mathcal{U} \succ E_2^{(n-1)N}(\theta)\mathcal{U}$ for all n, because of the influence that $E_2(\theta)^{nN}\mathcal{U}$ has often negative parallelograms which bring about cancellations.

(2) Our conjecture is that there exists a fixed point $\hat{F}$, which is the infinite sum of parallelograms, that is, $\exists \hat{F} : \hat{F} = E_2(\theta)\hat{F}$ and $\hat{F}$ might be given by

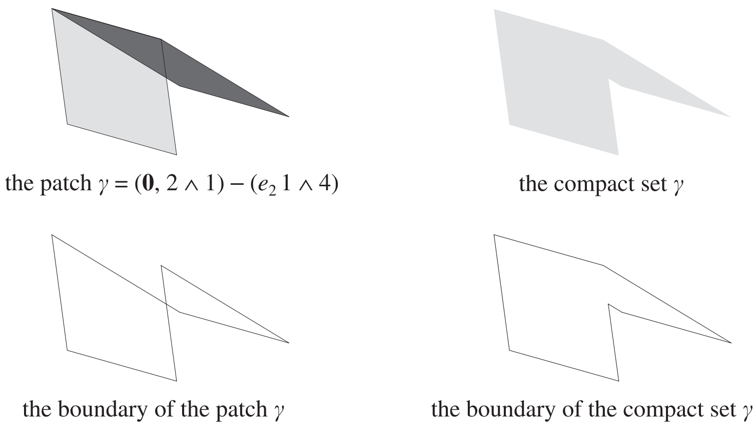


Fig. 8.

$\lim_{n\to\infty} E_2^n(\theta)\mathscr{U}$. But we leave it at the moment and we claim that we can set up the fractal tiling in this paper.

Lemma 2.1. *The limit set* $X_{i\wedge j} := \lim_{n\to\infty} A^n E_2^n(\theta)(\mathbf{0}, i\wedge j)$, $i \wedge j \in V_2$ *exists in the sense of the Hausdorff metric on* P.

Proof. We put

$$d := \max_{i\wedge j \in V_2} \left\{ \max_{\substack{\gamma: \\ \gamma \prec E_2(\theta)(\mathbf{0}, i\wedge j)}} \{D(A^{-1}(\mathbf{0}, i\wedge j), \gamma)\} \right\}$$

where D is the Hausdorff metric on P. On the notation $E_2^n(\theta)(\mathbf{0}, i\wedge j) = \sum_{k=1}^{L(n)} (\boldsymbol{x}_k^{(n)}, \delta_k^{(n)})$ and the fundamental property of the Hausdorff metric such that $D(A_1 \cup A_2, B_1 \cup B_2) \le \max\{D(A_1, B_1), D(A_2, B_2)\}$ (see [5]), we see that

$$\begin{aligned}
&D(A^{-1}E_2^n(\theta)(\mathbf{0}, i\wedge j), E_2^{n+1}(\theta)(\mathbf{0}, i\wedge j)) \\
&\quad = D\left(A^{-1} \bigcup_{k=1}^{L(n)} (\boldsymbol{x}_k^{(n)}, \delta_k^{(n)}), E_2(\theta)\left(\sum_{k=1}^{L(n)} (\boldsymbol{x}_k^{(n)}, \delta_k^{(n)}) \right) \right) \\
&\quad = D\left(\bigcup_{k=1}^{L(n)} A^{-1}(\boldsymbol{x}_k^{(n)}, \delta_k^{(n)}), \bigcup_{k=1}^{L(n)} (E_2(\theta)(\boldsymbol{x}_k^{(n)}, \delta_k^{(n)}) \cap E_2^{n+1}(\mathbf{0}, i\wedge j)) \right) \\
&\quad \le \max_{1\le k\le L(n)} \{D(A^{-1}(\boldsymbol{x}_k^{(n)}, \delta_k^{(n)}), E_2(\theta)(\boldsymbol{x}_k^{(n)}, \delta_k^{(n)}) \cap E_2^{n+1}(\theta)(\mathbf{0}, i\wedge j))\} \le d
\end{aligned}$$

where $\gamma \cap \delta = \{(\boldsymbol{x}, i\wedge j) \mid (\boldsymbol{x}, i\wedge j) \in \gamma \text{ and } (\boldsymbol{x}, i\wedge j) \in \delta\}$. By the operation of A^{n+1}, we have

$$D(A^n E_2^n(\theta)(\mathbf{0}, i \wedge j), A^{n+1} E_2^{n+1}(\theta)(\mathbf{0}, i \wedge j)) \le \lambda^{n+1} d$$

where $\lambda = \max\{|\lambda_3|, |\lambda_4|\} < 1$. Therefore, the limit set $X_{i \wedge j}$ exists. □

Corollary 2.1. *Let $\mathscr{U} = \sum_{k=1}^{M} m_k(\boldsymbol{x}_k, \delta_k)$ be a seed of $E_2(\theta)$, then the limit set X as*

$$X := \lim_{n\to\infty} A^n E_2^n(\theta)\mathscr{U}$$

exists and

$$X = \bigcup_{k=1}^{M} m_k(\boldsymbol{x}_k + X_{\delta_k})$$

where $X_{\delta_k} = \lim_{n\to\infty} A^n E_2^n(\theta)(\mathbf{0}, \delta_k)$.

The proof is obtained analogously with Lemma 2.1.

Lemma 2.2. *The compact set X satisfies $X^\circ \neq \varnothing$.*

Proof. We put C_i the 'fractal' curve generated by $E_1(\theta)$ from the segment $(\mathbf{0}, i)$, r_i the Hausdorff metric D between C_i and $(\mathbf{0}, i)$, and E_i the r_i-neighbors of the segment of $(\mathbf{0}, i)$, that is,

$$C_i := \lim_{n\to\infty} A^n E_1^n(\theta)(\mathbf{0}, i),$$

$$r_i := D(C_i, (\mathbf{0}, i)),$$

$$E_i := \{z \mid d(z, (\mathbf{0}, i)) \le r_i\},$$

and moreover we put $r := \max_{1\le i\le 4}\{r_i\}$. Then, it is clear that $C_i \subset E_i$. Let $C := \lim_{n\to\infty} A^n E_1^n(\theta)\partial\mathscr{U}$, then from the fact that A is continuous, we know

$$A^{-m}C = \lim_{n\to\infty} A^{n-m} E_1^{n-m}(\theta) E_1^m(\theta)\partial\mathscr{U} \qquad \text{for any } m. \tag{2.0.3}$$

On the notation

$$E_1^m(\theta)\partial\mathscr{U} = \sum_{k=1}^{R(m)} (\boldsymbol{x}_k^{(m)}, i_k^{(m)}),$$

we can write the relation (2.0.3) by

$$A^{-m}C \subset \bigcup_{k=1}^{R(m)} \lim_{n\to\infty} A^{n-m}(E_1^{n-m}(\theta)(\boldsymbol{x}_k^{(m)}, i_k^{(m)})).$$

Therefore, we have

$$A^{-m}C \subset \bigcup_{(\boldsymbol{x}_k^{(m)}, i_k^{(m)}) \in E_1^m(\theta)\partial\mathcal{U}} (\boldsymbol{x}_k^{(m)} + E_{i_k^{(m)}})$$

and

$$d(A^{-m}C, \mathbf{0}) \geq d\left(\bigcup_{(\boldsymbol{x}_k^{(m)}, i_k^{(m)}) \in E_1^m(\theta)\partial\mathcal{U}} (\boldsymbol{x}_k^{(m)} + E_{i_k^{(m)}}), \mathbf{0} \right) \geq d(E_1^m(\theta)\partial\mathcal{U}, \mathbf{0}) - r.$$

From the assumption of Definition 2.1 (c), if we take m satisfying

$$d(E_1^m(\theta)\partial\mathcal{U}, \mathbf{0}) \geq 2r,$$

then

$$d(A^{-m}C, \mathbf{0}) \geq r.$$

Therefore, there exists N such that $n \geq N$ implies

$$d(A^{-m+n}E_1^n(\theta)\partial\mathcal{U}, \mathbf{0}) > \frac{r}{2}.$$

This means that the compact set $A^{-m}A^nE_2^n(\theta)\mathcal{U}$ satisfies $A^{-m}A^nE_2^n(\theta)\mathcal{U} \supset B\big(\mathbf{0}, \frac{r}{2}\big)$ for all $n \geq N$, where $B\big(\mathbf{0}, \frac{r}{2}\big)$ is a ball of the center $\mathbf{0}$ and the radius $\frac{r}{2}$. Therefore, we have $A^{-m}X \supset B\big(\mathbf{0}, \frac{r}{2}\big)$ and $X \supset A^mB\big(\mathbf{0}, \frac{r}{2}\big)$. □

In Example, let us choose the patch $\mathcal{U}$ on the plane P by

$$\begin{aligned}\mathcal{U} := {} & (\boldsymbol{e}_1 - \boldsymbol{e}_2 - 2\boldsymbol{e}_3 + \boldsymbol{e}_4, 2 \wedge 1) + (\boldsymbol{e}_1 - \boldsymbol{e}_2 - 2\boldsymbol{e}_3 + \boldsymbol{e}_4, 1 \wedge 3) \\ & - (\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3, 1 \wedge 4) + (2\boldsymbol{e}_1 - \boldsymbol{e}_2 - 2\boldsymbol{e}_3, 2 \wedge 3) + (2\boldsymbol{e}_1 - \boldsymbol{e}_2 - 2\boldsymbol{e}_3, 4 \wedge 2) \\ & + (2\boldsymbol{e}_1 - \boldsymbol{e}_2 - 2\boldsymbol{e}_3, 3 \wedge 4) + (\boldsymbol{e}_1 - 2\boldsymbol{e}_2 - \boldsymbol{e}_3, 2 \wedge 1).\end{aligned}$$

Then, we can see that the patch $\mathcal{U}$ satisfies the seed condition of $E_2(\theta)$ in Definition 2.1 (a), (b) *as $N = 1$,* (c) *(see Figure* 9*).*

3. Blocking

3.1. Blocking patch

To treat the map $E_2(\theta)$ which generates not only positive orientated parallelograms but also negative ones from the positive parallelogram, we use the new idea "blocking" in this section.

Definition 3.1. *Let B be the family of the finite number of patches γ_p,*

$$B := \{\gamma_p \mid 1 \leq p \leq \mathrm{K}, \gamma_p \in \mathcal{G}_2\}.$$

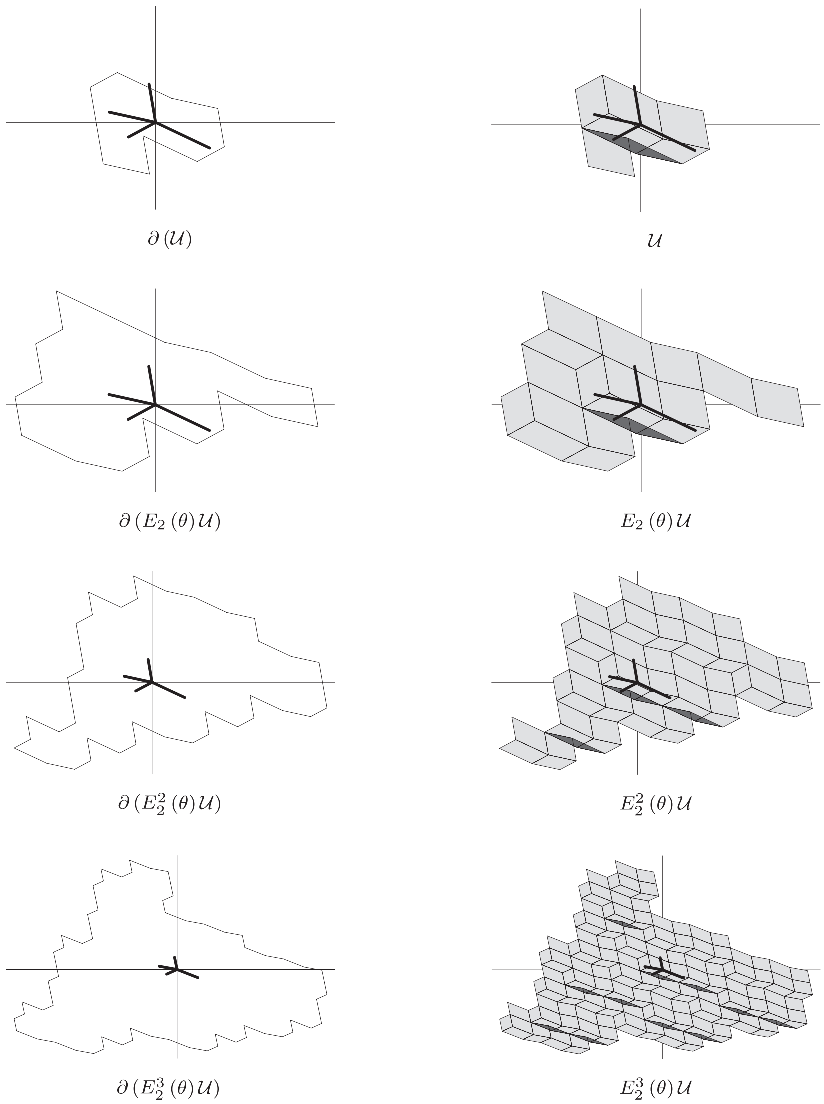


Fig. 9. $\partial(E_2^n(\theta)\mathscr{U})$ and $E_2^n(\theta)\mathscr{U}$, $n = 0, 1, 2, 3$ in Example.

If B satisfies the following conditions, we call B a family of blocking patches associated with σ *and we say σ satisfies the blocking condition.*

(1) *For each p, there exist a translation vectors $\boldsymbol{x}_k^{(p)} \in \pi\mathbf{Z}^4$ and the patch $\gamma_{V_k^{(p)}} \in B$ such that*

$$E_2(\theta)\gamma_p = \sum_{k=1}^{L_p} (\boldsymbol{x}_k^{(p)} + \gamma_{V_k^{(p)}}) \subset P,$$

that is, the patch $E_2(\theta)\gamma_p$ *can be decomposed by the translation of patches* $\{\gamma_{V_k^{(p)}}\}_{1\le k\le L_p}$, $\gamma_{V_k^{(p)}} \in B$;

(2) *Let* $\hat{M} = [\hat{m}_{pq}]_{1\le p,q\le \mathrm{K}}$ *be a* $\mathrm{K}\times\mathrm{K}$ *non-negative integer matrix given by*

$$\hat{m}_{pq} := \#\left\{ k \mid \gamma_{V_k^{(q)}} = \gamma_p, E_2(\theta)\gamma_q = \sum_{k=1}^{L_q} (\boldsymbol{x}_k^{(q)} + \gamma_{V_k^{(q)}}) \right\}.$$

We assume that $\hat{M}$ *is primitive, i.e.,* $\exists M_1 : \hat{M}^{M_1} > O$ *and that the maximal eigenvalue of* $\hat{M}$ *coincides with* $|\lambda_1|\cdot|\lambda_2|$ *where* $|\lambda_1|$ *and* $|\lambda_2|$ *are the absolute values of the eingevalues of* A *satisfying* $|\lambda_1| \ge |\lambda_2| > 1 > |\lambda_3| \ge |\lambda_4|$. *We call* $\hat{M}$ *the* incidence matrix *of* $E_2(\theta)$ *in the sense of blocking;*

(3) *There exists a seed* $\mathscr{U}$ *of* $E_2(\theta)$ *given by Definition* 2.1 *such that*

 (a) $\mathscr{U}$ *is decomposed in the sense of blocking, i.e.,* $\exists i_1, i_2, \ldots, i_L \in \{1, 2, \ldots, \mathrm{K}\} : \gamma_{i_k} \in B$, $\mathscr{U} = \sum_{k=1}^{L} (\boldsymbol{x}_{i_k} + \gamma_{i_k})$. *To distinguish* $\mathscr{U} = \sum_{k=1}^{M} m_k(\boldsymbol{x}_k, \delta_k)$ *which is constructed by the parallelogram, we denote* $\bar{\mathscr{U}} = \sum_{k=1}^{L} (\boldsymbol{x}_{i_k} + \gamma_{i_k})$;

 (b) *there exist* $\gamma_p \in B$, $N' \in \mathbf{N}$ *and* $\boldsymbol{z} \in \pi\mathbf{Z}^4$ *such that*

$$E_2^{N'}(\theta)\gamma_p \succ \boldsymbol{z} + \bar{\mathscr{U}} \qquad (\textit{in the sense of blocking}).$$

Remark. About the condition (2), if the seed $\mathscr{U}$ is constructed by 6 pieces parallelograms $(\boldsymbol{x}_{i\wedge j}, i\wedge j)$, $i\wedge j \in V_2$ and moreover all the elements of $E_2(\theta)(\mathbf{0}, i\wedge j)$, $i\wedge j \in V_2$ are positive, then the matrix $\hat{M}$ coincides with A^* and the maximal eigenvalue of $\hat{M}$ is equal to $|\lambda_1|\cdot|\lambda_2|$.

In Example, let us introduce the family of blocking patches $B = \{\gamma_i\}_{1\le i\le 6}$ *associated with* σ *by*

$$\begin{cases} \gamma_1 := (\mathbf{0}, 2\wedge 1) \\ \gamma_2 := (\mathbf{0}, 1\wedge 3) \\ \gamma_3 := -(\mathbf{0}, 1\wedge 4) + (-\boldsymbol{e}_2, 2\wedge 1) \\ \gamma_4 := (\mathbf{0}, 2\wedge 3) \\ \gamma_5 := (\mathbf{0}, 4\wedge 2) \\ \gamma_6 := (\mathbf{0}, 3\wedge 4) \end{cases}.$$

Then, the covering substitution $E_2(\theta)$ *for* γ_i, $1 \le i \le 6$ *is represented by the following in the sense of blocking:*

$$\begin{aligned} E_2(\theta)\gamma_1 &:= (-\boldsymbol{e}_3 + \gamma_6) + (-\boldsymbol{e}_3 + \gamma_5) + (\boldsymbol{e}_4 - \boldsymbol{e}_1 - \boldsymbol{e}_3 + \gamma_2) \\ &\quad + (\boldsymbol{e}_4 - \boldsymbol{e}_3 + \boldsymbol{e}_2 - \boldsymbol{e}_1 + \gamma_3) \\ E_2(\theta)\gamma_2 &:= (-\boldsymbol{e}_3 + \gamma_4) + (\boldsymbol{e}_2 - \boldsymbol{e}_3 + \gamma_5) \\ E_2(\theta)\gamma_3 &:= (\boldsymbol{e}_1 - \boldsymbol{e}_3 - \boldsymbol{e}_4 + \gamma_6) + (\boldsymbol{e}_1 - \boldsymbol{e}_3 - \boldsymbol{e}_4 + \gamma_5) \end{aligned} \tag{3.1.4}$$

$$\begin{aligned}
E_2(\theta)\gamma_4 &:= \gamma_5 + (-\boldsymbol{e}_1 + \boldsymbol{e}_4 + \gamma_1)\\
E_2(\theta)\gamma_5 &:= (\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_4 + \gamma_5) + (-\boldsymbol{e}_2 + \gamma_1)\\
&\qquad + (\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_4 + \gamma_6) + (-\boldsymbol{e}_2 + \gamma_2)\\
E_2(\theta)\gamma_6 &:= \gamma_1 + (\boldsymbol{e}_1 - \boldsymbol{e}_4 + \gamma_5) + (\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_4 + \gamma_4)
\end{aligned}$$

(see Figure 10*).*

Therefore, the incidence matrix $\hat{M}$ of $E_2(\theta)$ in the sense of blocking is given by

$$\hat{M} = \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

and we know the maximal eigenvalue $\lambda_{\hat{M}}$ of $\hat{M}$ coincides with $\lambda_{\hat{M}} = |\lambda_1| \cdot |\lambda_2| =$ $3.18\ldots$.

Let us consider the family of patches $\bar{\mathscr{U}}$ instead of $\mathscr{U}$ by

$$\bar{\mathscr{U}} := \left\{ \begin{array}{c} (\boldsymbol{e}_1 - \boldsymbol{e}_2 - 2\boldsymbol{e}_3 + \boldsymbol{e}_4 + \gamma_1), (\boldsymbol{e}_1 - \boldsymbol{e}_2 - 2\boldsymbol{e}_3 + \boldsymbol{e}_4 + \gamma_2), (\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_3 + \gamma_3), \\ (2\boldsymbol{e}_1 - \boldsymbol{e}_2 - 2\boldsymbol{e}_3 + \gamma_4), (2\boldsymbol{e}_1 - \boldsymbol{e}_2 - 2\boldsymbol{e}_3 + \gamma_5), (2\boldsymbol{e}_1 - \boldsymbol{e}_2 - 2\boldsymbol{e}_3 + \gamma_6) \end{array} \right\}.$$

Then, we can see that B satisfies Definition 3.1 (1), (2), (3) (a)(b) *as $N' = 1$ (see Figure* 11*).*

3.2. Graph of the blocking

From the formula $E_2(\theta)\gamma_p = \sum_{k=1}^{L_p} (\boldsymbol{x}_k^{(p)} + \gamma_{V_k^{(p)}})$, we put two finite sets $\mathscr{V}$ called vertices and $\mathscr{E}$ called edges, and two functions $i : \mathscr{E} \to \mathscr{V}$ and $t : \mathscr{E} \to \mathscr{V}$ as follows;

$$\mathscr{V} := \{\gamma_1, \gamma_2, \ldots, \gamma_{\mathrm{K}}\} \simeq \{1, 2, \ldots, \mathrm{K}\}$$

$$\mathscr{E} := \left\{ \begin{pmatrix} p \\ k \end{pmatrix} \mid 1 \le p \le \mathrm{K}, V_k^{(p)} \in \{1, 2, \ldots, \mathrm{K}\}, 1 \le k \le L_p \right\}$$

$$i\begin{pmatrix} p \\ k \end{pmatrix} := p, \qquad t\begin{pmatrix} p \\ k \end{pmatrix} := V_k^{(p)}.$$

Then, we obtain the directed graph $G := \{\mathscr{V}, \mathscr{E}, i, t\}$ and the set of its admissible sequence as

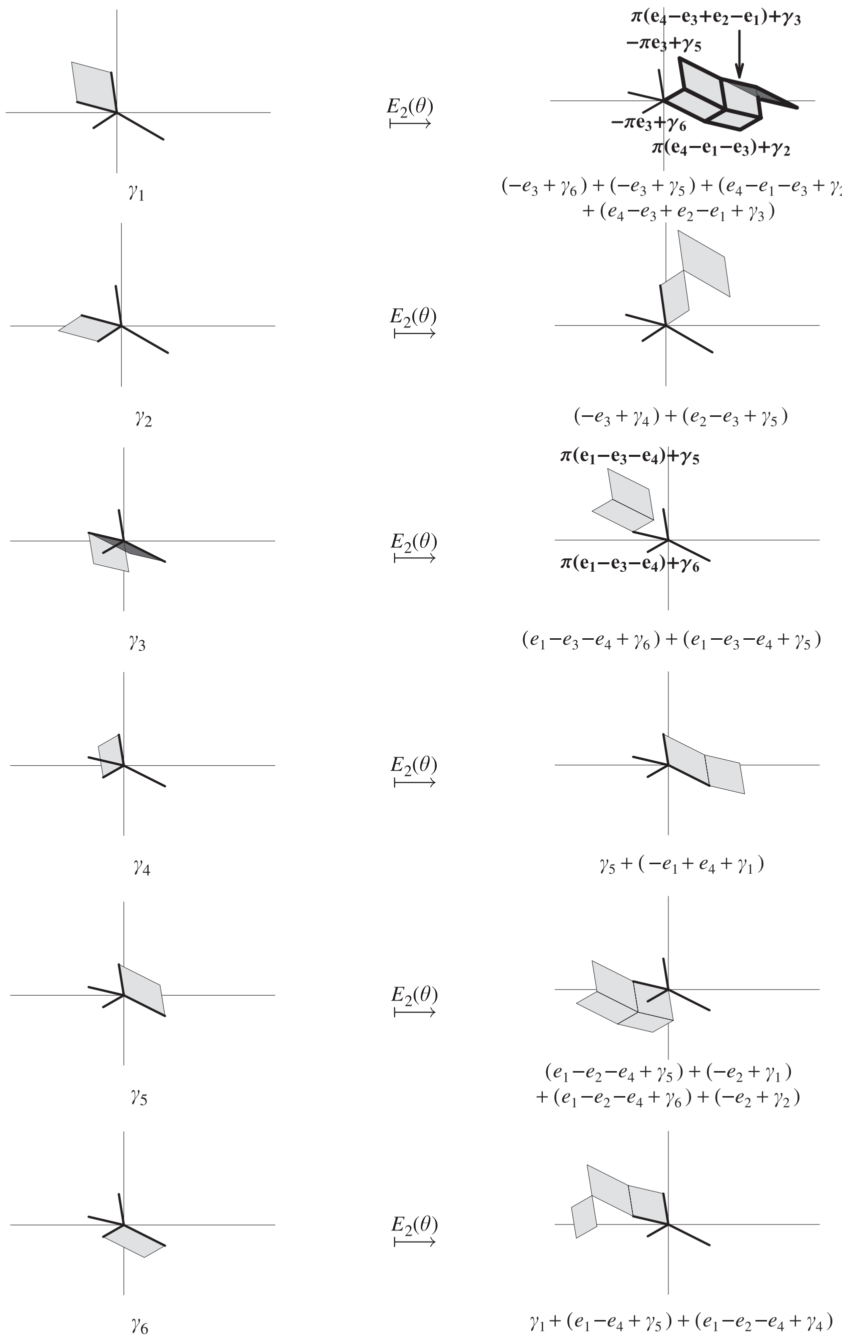


Fig. 10. γ_i and $E_2(\theta)\gamma_i$, $i = 1, 2, \ldots, 6$ in Example.

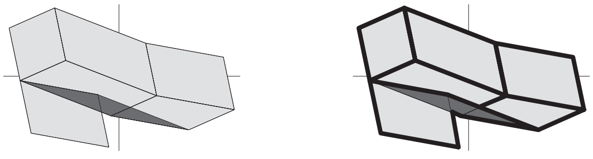


Fig. 11. $\mathscr{U}$ and $\bar{\mathscr{U}}$ in Example.

$$\sum\nolimits_G := \left\{ \begin{pmatrix} p_{j_1} & p_{j_2} & \cdots \\ k_{j_1} & k_{j_2} & \cdots \end{pmatrix} \middle| \begin{pmatrix} p_{j_s} \\ k_{j_s} \end{pmatrix} \in \mathscr{E}, t\begin{pmatrix} p_{j_s} \\ k_{j_s} \end{pmatrix} = i\begin{pmatrix} p_{j_{s+1}} \\ k_{j_{s+1}} \end{pmatrix} \right\}.$$

And from the formula $E_2(\theta)\gamma_p = \sum_{k=1}^{L_p}(\boldsymbol{x}_k^{(p)} + \gamma_{V_k^{(p)}})$, let us give the label function $\mathscr{L}$,

$$\begin{array}{cccc} \mathscr{L}\colon & \mathscr{E} & \longrightarrow & \pi\mathbf{Z}^4 \\ & \Psi & & \Psi \\ & \begin{pmatrix} p \\ k \end{pmatrix} & \longmapsto & \boldsymbol{x}_k^{(p)}, \end{array}$$

then, we have the labeled graph $G_{\mathscr{L}}$ and its admissible labeled sequence of $G_{\mathscr{L}}$. We put the set of the admissible labeled sequence of $G_{\mathscr{L}}$ whose initial vertex is p as

$$\Omega_p := \left\{ \left(\mathscr{L}\begin{pmatrix} p_{j_1} \\ k_{j_1} \end{pmatrix}, \mathscr{L}\begin{pmatrix} p_{j_2} \\ k_{j_2} \end{pmatrix}, \ldots \right) \middle| \begin{pmatrix} p_{j_1} & p_{j_2} & \cdots \\ k_{j_1} & k_{j_2} & \cdots \end{pmatrix} \in \sum\nolimits_G, i\begin{pmatrix} p_{j_1} \\ k_{j_1} \end{pmatrix} = p \right\}$$

and for simplicity, we write

$$\Omega_p = \{(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \ldots) \mid G_{\mathscr{L}}\text{-admissible and } i(\boldsymbol{x}_{j_1}) = p\}.$$

From

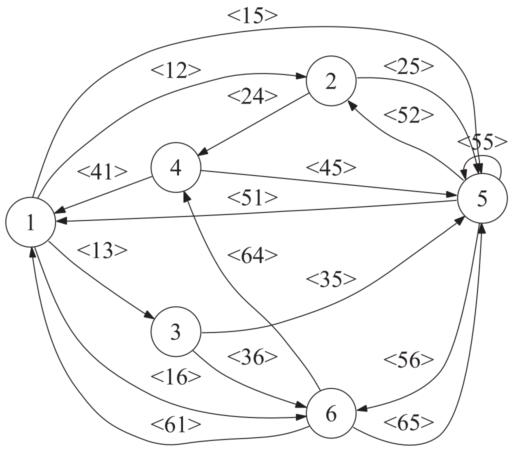
$$\begin{aligned} E_2(\theta)\gamma_p &= \sum_{k=1}^{L_p}(\boldsymbol{x}_k^{(p)} + \gamma_{V_k^{(p)}}) = \sum_{\boldsymbol{x}_{j_1}: i(\boldsymbol{x}_{j_1})=p} (\boldsymbol{x}_{j_1} + \gamma_{t(\boldsymbol{x}_{j_1})}), \\ E_2^2(\theta)\gamma_p &= \sum_{\boldsymbol{x}_{j_1}: i(\boldsymbol{x}_{j_1})=p} (A^{-1}\boldsymbol{x}_{j_1} + E_2(\theta)\gamma_{t(\boldsymbol{x}_{j_1})}) \\ &= \sum_{\boldsymbol{x}_{j_1}: i(\boldsymbol{x}_{j_1})=p} \left(A^{-1}\boldsymbol{x}_{j_1} + \sum_{\boldsymbol{x}_{j_2}: i(\boldsymbol{x}_{j_2})=t(\boldsymbol{x}_{j_1})} (\boldsymbol{x}_{j_2} + \gamma_{t(\boldsymbol{x}_{j_2})}) \right) \\ &= \sum_{(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}): i(\boldsymbol{x}_{j_1})=p} (A^{-1}\boldsymbol{x}_{j_1} + \boldsymbol{x}_{j_2} + \gamma_{t(\boldsymbol{x}_{j_2})}), \end{aligned}$$

we have the following formula in the sense of blocking

$$E_2^n(\theta)\gamma_p = \sum_{(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \dots, \boldsymbol{x}_{j_n}): i(\boldsymbol{x}_{j_1}) = p} (A^{-(n-1)}\boldsymbol{x}_{j_1} + A^{-(n-2)}\boldsymbol{x}_{j_2} + \cdots + \boldsymbol{x}_{j_n} + \gamma_{t(\boldsymbol{x}_{j_n})}) \quad (3.2.5)$$

where $(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \dots, \boldsymbol{x}_{j_n})$ is the finite length path of the $G_{\mathscr{L}}$-admissible sequence.

In Example, we obtain the following labeled graph from (3.1.4) *(see Figure* 12*):*



$<16>=$	$-\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4$	$<15>=$	$-\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4$	$<12>=$	$-\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4$
$<13>=$	$\boldsymbol{0}$	$<24>=$	$-\boldsymbol{e}_2+\boldsymbol{e}_3-\boldsymbol{e}_4$	$<25>=$	$\boldsymbol{e}_3-\boldsymbol{e}_4$
$<36>=$	$\boldsymbol{e}_2-\boldsymbol{e}_4$	$<35>=$	$\boldsymbol{e}_2-\boldsymbol{e}_4$	$<45>=$	$-\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_4$
$<41>=$	$-\boldsymbol{e}_1+\boldsymbol{e}_4+\boldsymbol{e}_2$	$<55>=$	$\boldsymbol{0}$	$<51>=$	$\boldsymbol{0}$
$<56>=$	$\boldsymbol{0}$	$<52>=$	$\boldsymbol{0}$	$<61>=$	$\boldsymbol{e}_2$
$<65>=$	$\boldsymbol{e}_2$	$<64>=$	$\boldsymbol{0}$		

Fig. 12. Labeled graph $G_{\mathscr{L}}$ in Example.

4. Main Theorem

LEMMA 4.1. *We assume that $E_2(\theta)$ has the family of blocking patches B associated with σ given by Definition* 3.1 *and we put*

$$X_p := \lim_{n\to\infty} A^n E_2^n(\theta)\gamma_p, \qquad \gamma_p \in B.$$

Then, the following set equation holds:

$$A^{-1}X_p = \bigcup_{k=1}^{L_p} (\boldsymbol{x}_k^{(p)} + X_{V_k^{(p)}}).$$

Proof. From $E_2(\theta)\gamma_p = \sum_{k=1}^{L_p}(\boldsymbol{x}_k^{(p)} + \gamma_{V_k^{(p)}})$, we get

$$E_2^n(\theta)\gamma_p = \sum_{k=1}^{L_p}(A^{-(n-1)}\boldsymbol{x}_k^{(p)} + E_2^{n-1}(\theta)\gamma_{V_k^{(p)}}).$$

Operate A^{n-1} and $n \to \infty$, then we have the following set equation:

$$A^{-1}X_p = \bigcup_{k=1}^{L_p}(\boldsymbol{x}_k^{(p)} + X_{V_k^{(p)}})$$

where the existence of limit sets X_p can be discussed analogously in the proof of Lemma 2.1. □

By analogous discussion and (3.2.5), we obtain the corollary.

Corollary 4.1. *For any $n \in \mathbf{N}$, the following set equation holds:*

$$A^{-n}X_p = \bigcup_{\substack{(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \dots, \boldsymbol{x}_{j_n}): \\ i(\boldsymbol{x}_{j_1}) = p}} (A^{-(n-1)}\boldsymbol{x}_{j_1} + A^{-(n-2)}\boldsymbol{x}_{j_2} + \cdots + \boldsymbol{x}_{j_n} + X_{t(\boldsymbol{x}_{j_n})})$$

where $(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \dots, \boldsymbol{x}_{j_n})$ is the finite length path of the $G_{\mathscr{L}}$-admissible sequence.

Lemma 4.2. *$X_p^\circ \neq \emptyset$ for all p.*

Proof. From Definition 3.1(3)(b) and Lemma 2.2, there exist $\gamma_p \in B$, $N' \in \mathbf{N}$ and $\boldsymbol{z} \in P$ such that

$$E_2^{N'}(\theta)\gamma_p \succ \boldsymbol{z} + \bar{\mathscr{U}} \qquad \text{(in the sense of the blocking)}$$

and Corollary 4.1, we have

$$\begin{aligned} A^{-N'}X_p &= \bigcup_{\substack{(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \dots, \boldsymbol{x}_{j_{N'}}): \\ i(\boldsymbol{x}_{j_1}) = p}} (A^{-(N'-1)}\boldsymbol{x}_{j_1} + A^{-(N'-2)}\boldsymbol{x}_{j_2} + \cdots + \boldsymbol{x}_{j_{N'}} + X_{t(\boldsymbol{x}_{j_n})}) \\ &\supset \bigcup_{k=1}^{L}(\boldsymbol{z} + (\boldsymbol{x}_{i_k} + X_{i_k})) = \boldsymbol{z} + X \supset \boldsymbol{z} + X^\circ \neq \emptyset \qquad \text{(by Lemma 2.2).} \end{aligned}$$

From Corollary 4.1 and the primitivity of $\hat{M}$, we have $X_p^\circ \neq \emptyset$ for all p. □

Proposition 4.1. *The set equations*

$$A^{-1}X_p = \bigcup_{k=1}^{L_p}(\boldsymbol{x}_k^{(p)} + X_{V_k^{(p)}})$$

$$A^{-n}X_p = \bigcup_{\substack{(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \dots, \boldsymbol{x}_{j_n}): \\ i(\boldsymbol{x}_{j_1}) = p}} (A^{-(n-1)}\boldsymbol{x}_{j_1} + A^{-(n-2)}\boldsymbol{x}_{j_2} + \cdots + \boldsymbol{x}_{j_n} + X_{t(\boldsymbol{x}_{j_n})})$$

are non-overlapping. In particular,

$$X = \bigcup_{k=1}^{L} (\boldsymbol{x}_{i_k} + X_{i_k}) \qquad (\textit{non-overlapping})$$

where $X = \lim_{n\to\infty} A^n E_2^n(\theta)\mathscr{U} = \lim_{n\to\infty} A^n E_2^n(\theta)\bar{\mathscr{U}}$, $\bar{\mathscr{U}} = \sum_{k=1}^{L}(\boldsymbol{x}_{i_k} + \gamma_{i_k})$.

Proof. From the first equation, we get for the volumes $\{|X_i| \mid i = 1, 2, \ldots, \mathrm{K}\}$,

$$(|A^{-1}X_1|, |A^{-1}X_2|, \ldots, |A^{-1}X_{\mathrm{K}}|) \leq (|X_1|, |X_2|, \ldots, |X_{\mathrm{K}}|)\hat{M},$$

$$(|A^{-1}X_1|, |A^{-1}X_2|, \ldots, |A^{-1}X_{\mathrm{K}}|) = |\lambda_1| \cdot |\lambda_2|(|X_1|, |X_2|, \ldots, |X_{\mathrm{K}}|).$$

And from Lemma 4.2, we know $X_p^\circ \neq \emptyset$ for all p. In particular, $|X_p| > 0$ for all p. Therefore by Lemma of [19] or [4], we know that $(|X_1|, |X_2|, \ldots, |X_{\mathrm{K}}|)$ must be the eigenvector of $\hat{M}$ and that the inequality must be the equality. In particular, we know that the set equation is non-overlapping. Analogously, we obtain that the second set equation is non-overlapping. The final non-overlappingness is from Definition 3.1 (3)(b). □

Proposition 4.2. *X and X_p satisfy*

$$\overline{X^\circ} = X \qquad \textit{and} \qquad \overline{X_p^\circ} = X_p.$$

Proof. From Proposition 4.1, we have for any n

$$X = \bigcup_{\substack{(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \ldots, \boldsymbol{x}_{j_n}):\\ i(\boldsymbol{x}_{j_1}) \in \{i_1, i_2, \ldots, i_L\}}} (A\boldsymbol{x}_{j_1} + A^2\boldsymbol{x}_{j_2} + \cdots + A^n\boldsymbol{x}_{j_n} + A^n X_{t(\boldsymbol{x}_{j_n})})$$

$$(\text{non-overlapping}) \tag{4.0.6}$$

where $(\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \ldots, \boldsymbol{x}_{j_n})$ is $G_{\mathscr{L}}$-admissible. Therefore, for any $\boldsymbol{x} \in X$ and $\delta > 0$, let $B_{\boldsymbol{x}}(\delta)$ be the ball with the center $\boldsymbol{x}$ and the radius δ on P, then by the above set equation, there exists m and $\boldsymbol{z} = A\boldsymbol{x}_{j_1} + A^2\boldsymbol{x}_{j_2} + \cdots + A^m\boldsymbol{x}_{j_m}$ such that $B_{\boldsymbol{x}}(\delta) \supset A^m X_{t(\boldsymbol{x}_{j_n})} - \boldsymbol{z}$ and $A^m X_{t(\boldsymbol{x}_{j_n})}^\circ \neq \emptyset$. This means that $X = \overline{X^\circ}$. We see that $X_p = \overline{X_p^\circ}$ analogously. □

The analogous discussion can be found in [10].

Lemma 4.3. *Let us define the set τ,*

$$\tau = \left\{ A^{-(Nn-1)}\boldsymbol{x}_{j_1} + A^{-(Nn-2)}\boldsymbol{x}_{j_2} + \cdots + \boldsymbol{x}_{j_{Nn}} + X_{t(\boldsymbol{x}_{j_n})} \;\middle|\; \begin{array}{l} n \in \mathbf{N}, \\ i(\boldsymbol{x}_{i_1}) \in \{i_1, i_2, \ldots, i_{\mathrm{K}}\}, \\ (\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \ldots, \boldsymbol{x}_{j_{Nn}}) \\ \textit{is } G_{\mathscr{L}}\textit{-admissible} \end{array} \right\},$$

then τ is a quasi-periodic tiling of P.

PROOF. From Definition 2.1 (b) $E_2^N(\theta)\bar{\mathscr{U}} \succ \bar{\mathscr{U}}$, we see that $A^{-N}X \supset X$ ($\supset X^\circ \ni \mathbf{0}$). By the fact that A^{-1} is expanding on P, we know

$$\bigcup_{n=0}^{\infty} A^{-nN}X = P.$$

Therefore, from the non-overlappingness of (4.0.6), the set τ,

$$\tau = \left\{ A^{-(Nn-1)}\boldsymbol{x}_{j_1} + A^{-(Nn-2)}\boldsymbol{x}_{j_2} + \cdots + \boldsymbol{x}_{j_{Nn}} + X_{t(\boldsymbol{x}_{j_n})} \;\middle|\; \begin{array}{l} n \in \mathbf{N}, \\ i(\boldsymbol{x}_{j_1}) \in \{i_1, i_2, \ldots, i_{\mathrm{K}}\}, \\ (\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \ldots, \boldsymbol{x}_{j_{Nn}}) \\ \text{is } G_{\mathscr{L}}\text{-admissible} \end{array} \right\},$$

is a tiling of P_c. The quasi-periodicity of the tiling τ can be seen from the presentation formula of tiles

$$A^{-(Nn-1)}\boldsymbol{x}_{j_1} + A^{-(Nn-2)}\boldsymbol{x}_{j_2} + \cdots + \boldsymbol{x}_{j_{Nn}} + X_{t(\boldsymbol{x}_{j_n})}$$

by $G_{\mathscr{L}}$-admissible sequence by analogous discussion can be found in [9]. $\square$

We call τ the graph-directed self-similar tiling and for simplicity we write *GDSS tiling*.

Summing up the propositions and lemmas, we obtain the following main theorem.

MAIN THEOREM. *Let σ be an automorphism satisfying Assumption* 1.1 (1) *non-Pisot,* (2) *unimodular,* (3) *standard position, moreover there exists a family of blocking patches B associated with σ and the seed $\mathscr{U}$ can be blocked $\mathscr{U} = \bar{\mathscr{U}} = \sum_{k=1}^{L}(\boldsymbol{x}_{i_k} + \gamma_{i_k})$. Put*

$$X_p := \lim_{n\to\infty} A^n E_2^n(\theta)\gamma_p$$

$$X_{i_k} := \lim_{n\to\infty} A^n E_2^n(\theta)\gamma_{i_k}$$

$$X := \bigcup_{k=1}^{L}(\boldsymbol{x}_{i_k} + X_{i_k}),$$

then the sets, X, $\{X_p\}$, $\{(\boldsymbol{x}_{i_k} + X_{i_k})\}$, satisfy

(0) $X^\circ \ni \mathbf{0}$

(1) $X = \bigcup_{k=1}^{L}(\boldsymbol{x}_{i_k} + X_{i_k})$ *(non-overlapping);*

(2) $\overline{X^\circ} = X$;

(3) $A^{-1}X_p = \bigcup_{k=1}^{L_p}(\boldsymbol{x}_k^{(p)} + X_{V_k^{(p)}})$ *(non-overlapping);*

(4) *put $|X_p| :=$ volume of X_p, then $(|X_1|, |X_2|, \ldots, |X_{\mathrm{K}}|)$ is an eigenvector of $\hat{M}$, that is,*

$$\lambda_{\hat{M}}(|X_1|, |X_2|, \ldots, |X_{\mathrm{K}}|) = (|X_1|, |X_2|, \ldots, |X_{\mathrm{K}}|)\hat{M}$$

where $\lambda_{\hat{M}} = |\lambda_1| \cdot |\lambda_2|$;

(5) *let us define the sets* τ,

$$\tau = \left\{ A^{-(Nn-1)}\boldsymbol{x}_{j_1} + A^{-(Nn-2)}\boldsymbol{x}_{j_2} + \cdots + \boldsymbol{x}_{j_{Nn}}^{N} + X_{t(\boldsymbol{x}_{j_n})} \;\middle|\; \begin{array}{l} n \in \mathbf{N}, \\ i(\boldsymbol{x}_{j_1}) \in \{i_1, i_2, \ldots, i_{\mathrm{K}}\}, \\ (\boldsymbol{x}_{j_1}, \boldsymbol{x}_{j_2}, \ldots, \boldsymbol{x}_{j_{Nn}}) \\ \textit{is } G_{\mathscr{P}}\textit{-admissible} \end{array} \right\},$$

then τ *is a quasi-periodic GDSS tiling of* P *generated by* $\{X_p\}_{1 \le p \le \mathrm{K}}$, *where* N *is chosen as* $E_2^N(\theta)\bar{\mathscr{U}} \succ \bar{\mathscr{U}}$.

Corollary 4.2. *Let* σ *be an automorphism satisfying Assumption* 1.1 (1) *non-Pisot,* (2) *unimodular,* (3) *standard property, moreover,* (4) *we can find the special seed* $\mathscr{U} = \sum_{i\wedge j \in V_2} (\boldsymbol{x}_{i\wedge j}, i \wedge j) \in \mathscr{G}_2$ *and all the elements of* $E_2(\theta)(\boldsymbol{x}_{i\wedge j}, i \wedge j)$ *are positive,* (5) *there exist* $(\boldsymbol{x}_{i\wedge j}, i \wedge j) \in \mathscr{U}$, $N \in \mathbf{N}$ *and* $\boldsymbol{z} \in \pi\mathbf{Z}^4$ *such that*

$$E_2^N(\theta)(\boldsymbol{x}_{i\wedge j}, i \wedge j) \succ \boldsymbol{z} + \mathscr{U}.$$

Then, the sets $X := \lim_{n\to\infty} A^n E_2^n(\theta)\mathscr{U}$, $X_{i\wedge j} := \lim_{n\to\infty} A^n E_2^n(\theta)(\mathbf{0}, i \wedge j)$ *satisfies*

(0) $X^\circ \ni \mathbf{0}$;

(1) $X = \bigcup_{i\wedge j \in V_2} (\boldsymbol{x}_{i\wedge j} + X_{i\wedge j})$;

(2) $\overline{X^\circ} = X$, $\overline{X_{i\wedge j}^\circ} = X_{i\wedge j}$;

(3) *from the following notation* $E_2(\theta)(\mathbf{0}, i \wedge j) = \sum_{k=1}^{L(i\wedge j)} (\boldsymbol{x}_k^{i\wedge j}, \delta_k^{i\wedge j})$,

$$A^{-1}X_{i\wedge j} = \bigcup_{k=1}^{L(i\wedge j)} (\boldsymbol{x}_k^{i\wedge j} + X_{\delta_k^{i\wedge j}}) \qquad (\textit{non-overlapping});$$

(4) *the vector from the elements* $|X_{i\wedge j}|$, $i \wedge j \in V_2$ *is the maximal eigenvector of* A^*;

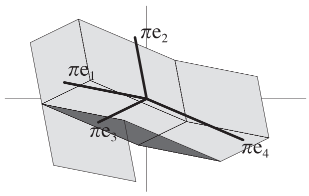
(5) *we obtain the quasi-periodic GDSS tiling* τ *of* P *of the prototiles* $\{X_{i\wedge j}\}_{i\wedge j \in V_2}$.

Proof. Let us define the family of blocking patches B associated with σ,

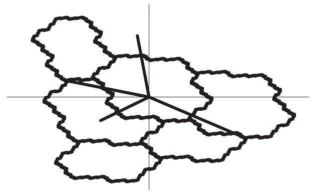
$$B = \left\{ (\boldsymbol{x}_{i\wedge j}, i \wedge j) \mid \mathscr{U} = \sum_{i\wedge j \in V_2} (\boldsymbol{x}_{i\wedge j}, i \wedge j) \right\},$$

then, it is easy to see that B satisfies (1) and (3) in Definition 3.1. For (2): from the assumption [1], the element $a^*_{i\wedge j, k\wedge l}$ of the matrix A^* is given by the number of $(\boldsymbol{x}, k \wedge l)$ in $E_2(\theta)(\mathbf{0}, i \wedge j)$ and A^* is the 6×6 non-negative integer matrix. Thereofore the Perron-Frobenius eigenvalue of A^* is given by $(|\lambda_3|, |\lambda_4|)^{-1} = |\lambda_1|\,|\lambda_2|$. Therefore B satisfies the all of the condition in Definition 3.1. □

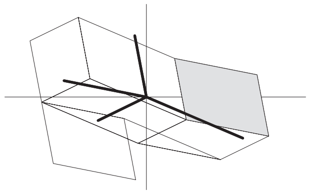
In Example, we know $X_p = X_{i_p}$, $p = 1, 2, \ldots, 6$. *We show* X_p, $\boldsymbol{x}_p + \gamma_p$, $p = 1, 2, \ldots, 6$ *as Figure* 13 *and the tiling* τ *as Figure* 15.



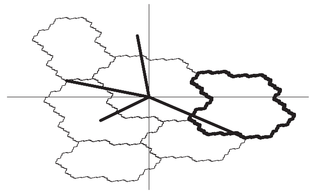
$\overline{\mathcal{U}}$



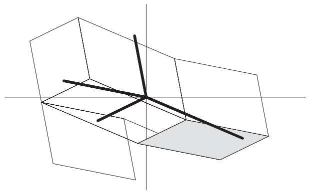
$X = \bigcup_{k=1}^{L} (x_{i_k} + X_{i_k})$ where
$\overline{\mathcal{U}} = \bigcup_{k=1}^{L} (x_{i_k} + \gamma_{i_k})$, $L = 6$



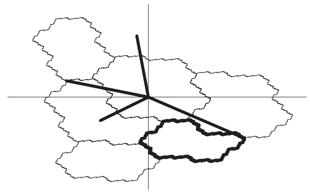
$x_{i_1} + \gamma_{i_1}$



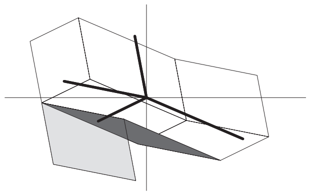
$x_{i_1} + X_{i_1}$



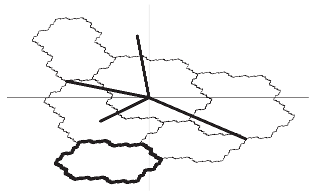
$x_{i_2} + \gamma_{i_2}$



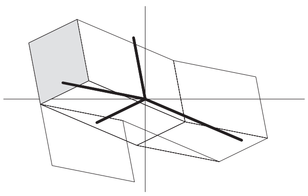
$x_{i_2} + X_{i_2}$



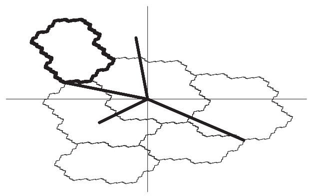
$x_{i_3} + \gamma_{i_3}$



$x_{i_3} + X_{i_3}$



$x_{i_4} + \gamma_{i_4}$



$x_{i_4} + X_{i_4}$

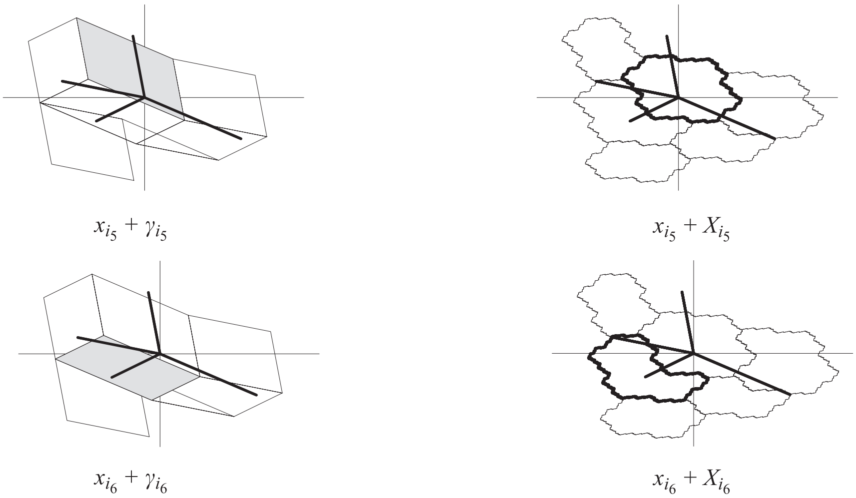


Fig. 13.

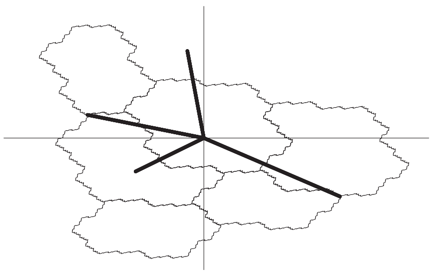


Fig. 14. $X = \bigcup_{k=1}^{L}(\boldsymbol{x}_{i_k} + X_{i_k})$ where $\bar{\mathscr{U}} = \bigcup_{k=1}^{L}(\boldsymbol{x}_{i_k} + \gamma_{i_k})$, $L = 6$ in Example.

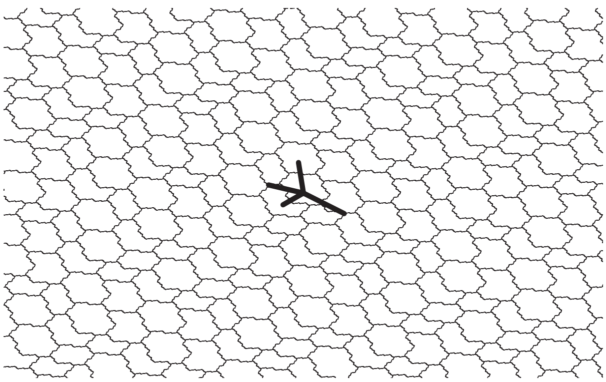


Fig. 15. Quasi-periodic GDSS tiling τ in Main Theorem (5).

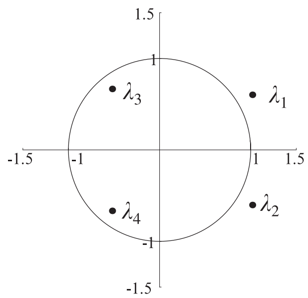
5. Examples

The simple example which satisfies the assumption in Corollary 4.2 can be found as follows:

Example 5.1. *This is the example discussed in* [3]. *Let us consider the following matrix A:*

$$A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then the characteristic polynomial of A is $\Phi_A(x) = x^4 - x^3 + 1$.



The set of symbolic parallelograms is chosen by

$$V_2 := \{2 \wedge 1, 1 \wedge 3, 4 \wedge 1, 3 \wedge 2, 2 \wedge 4, 4 \wedge 3\}.$$

We will choose the automorphism σ *and the mirror image* θ *of* σ^{-1} *is detemined by*

$$\sigma : \begin{cases} 1 \to 12 \\ 2 \to 3 \\ 3 \to 4 \\ 4 \to 1^{-1} \end{cases}, \qquad \theta : \begin{cases} 1 \to 4^{-1} \\ 2 \to 14 \\ 3 \to 2 \\ 4 \to 3 \end{cases}.$$

Then, the covering substitution $E_2(\theta)$ *keeps the positive orientation (see Figure* 16*).*

In this example, the seed $\mathscr{U}$ *is chosen by*

$$\begin{aligned} \mathscr{U} := {} & (-\boldsymbol{e}_1 - \boldsymbol{e}_2 - \boldsymbol{e}_4, 2 \wedge 1) + (-\boldsymbol{e}_1 - \boldsymbol{e}_3 - \boldsymbol{e}_4, 1 \wedge 3) + (-\boldsymbol{e}_1 - \boldsymbol{e}_3 - \boldsymbol{e}_4, 4 \wedge 1) \\ & + (-\boldsymbol{e}_2 - \boldsymbol{e}_3, 3 \wedge 2) + (-\boldsymbol{e}_2 - \boldsymbol{e}_4, 2 \wedge 4) + (-\boldsymbol{e}_3 - \boldsymbol{e}_4, 4 \wedge 3) \end{aligned}$$

(see Figure 17*).*

Then, $\mathscr{U}$ *satisfies* $E_2(\theta)\mathscr{U} \succ \mathscr{U}$ *and* $d(\partial E_2(\theta)^n \mathscr{U}, \mathbf{0}) \to \infty$ *(see Figure* 18*).*

We obtain the quasi-periodic GDSS tiling τ *(see Figure* 20*).*

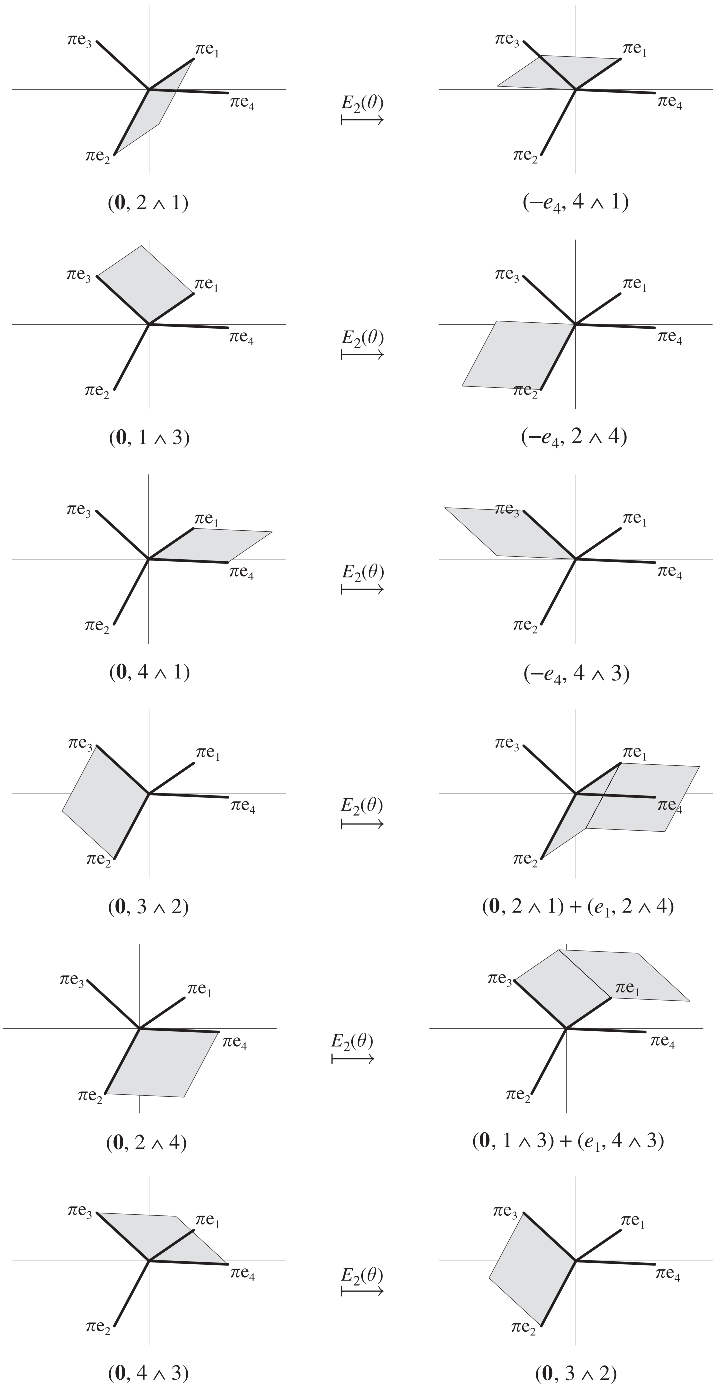


Fig. 16. $(\boldsymbol{0}, i \wedge j)$ and $E_2(\theta)(\boldsymbol{0}, i \wedge j)$, $i \wedge j \in V_2$ in Example 5.1.

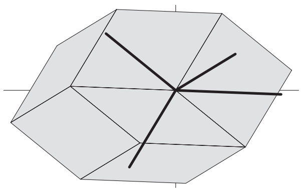


Fig. 17. The seed $\mathscr{U}$ in Example 5.1.

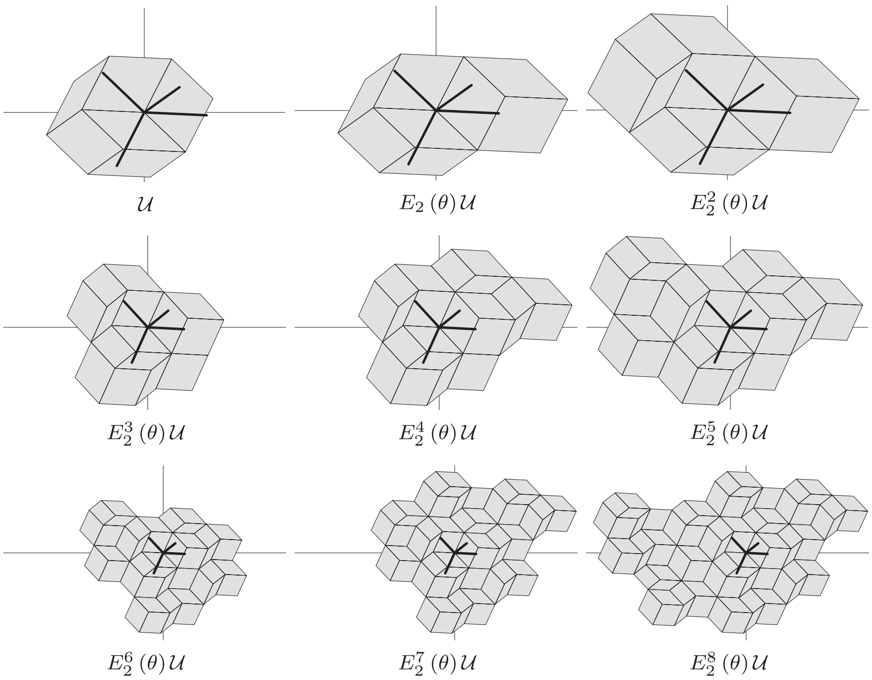


Fig. 18. $E_2^n(\theta)\mathscr{U}$, $n = 0, 1, \ldots, 8$ in Example 5.1.

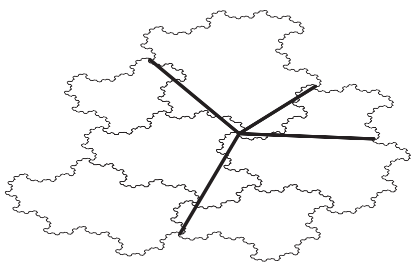


Fig. 19. $\{\boldsymbol{x}_{i\wedge j} + X_{i\wedge j} \mid i \wedge j \in V_2\}$ in Example 5.1.

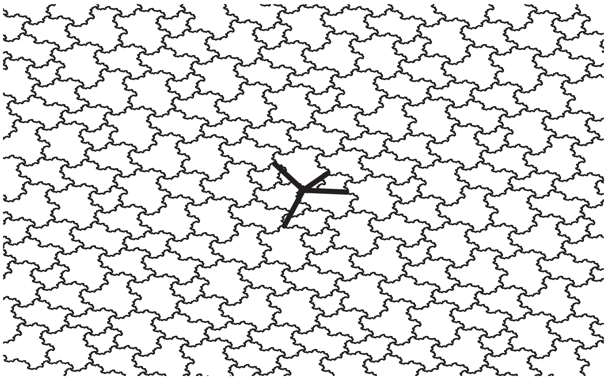


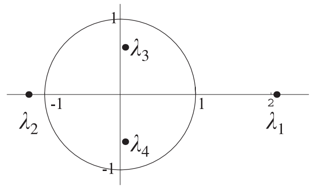
Fig. 20. Quasi-periodic GDSS tiling τ in Example 5.1.

The cardinality of the family of blocking patches B associated with σ is usually different from the cardinality of the parallelograms constructing the seed $\mathscr{U}$ of $E_2(\theta)$. We propose such an example as Example 5.2.

Example 5.2. *Let us consider the following matrix A:*

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then the characteristic polynomial of A is $\Phi_A(x) = x^4 - x^3 - 2x^2 - 1$.



The set of symbolic parallelograms is chosen by

$$V_2 := \{1 \wedge 2, 3 \wedge 1, 4 \wedge 1, 2 \wedge 3, 2 \wedge 4, 3 \wedge 4\}.$$

Let us choose the automorphism σ (invertible substitution) and the mirror image θ of σ^{-1} is determined by

$$\sigma : \begin{cases} 1 \to 12 \\ 2 \to 113 \\ 3 \to 4 \\ 4 \to 1 \end{cases}, \qquad \theta : \begin{cases} 1 \to 4 \\ 2 \to 14^{-1} \\ 3 \to 24^{-1}4^{-1} \\ 4 \to 3 \end{cases}.$$

In this example, the covering substitution $E_2(\theta)$ sometimes produce the negative orientated parallelograms from the positive one (see Figure 21).

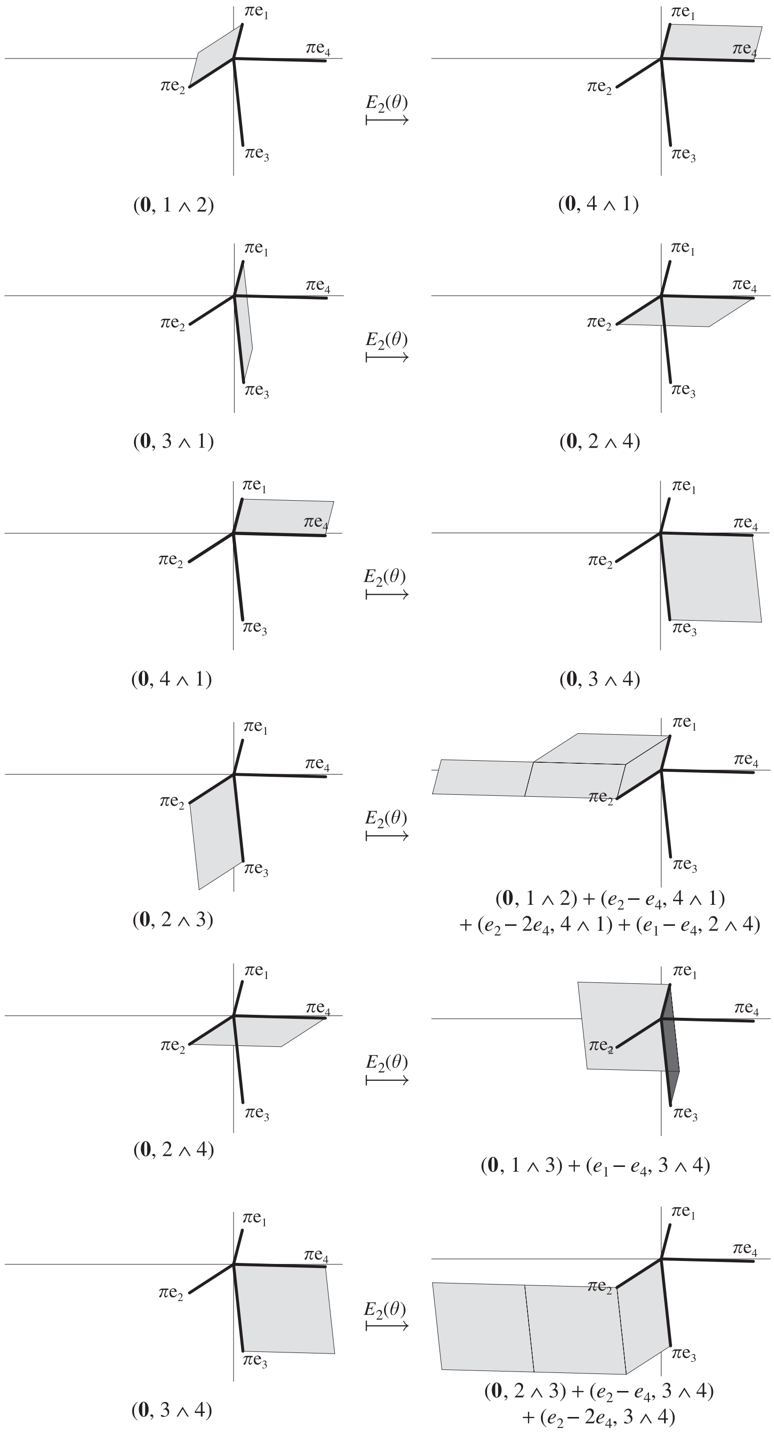


Fig. 21. $(\boldsymbol{0}, i \wedge j)$ and $E_2(\theta)(\boldsymbol{0}, i \wedge j)$, $i \wedge j \in V_2$ in Example 5.2.

If $\mathscr{U}$ is chosen as

$$\mathscr{U} := (\mathbf{0}, 1 \wedge 2) + (\mathbf{0}, 3 \wedge 1) + (\boldsymbol{e}_2 - \boldsymbol{e}_4, 4 \wedge 1) + (\mathbf{0}, 2 \wedge 3) + (\boldsymbol{e}_1 - \boldsymbol{e}_4, 2 \wedge 4) + (\boldsymbol{e}_2 - \boldsymbol{e}_4, 3 \wedge 4),$$

then we see that
(1) $E_2^4(\theta)\mathscr{U} \succ \mathscr{U}$;
(2) $d(\partial E_2^{4n}(\theta)\mathscr{U}, \mathbf{0}) \to \infty \ (n \to \infty)$
(see Figure 22).

Let us define the family of blocking patches B associated with σ by

$$\begin{cases} \gamma_1 := -(\mathbf{0}, 3 \wedge 1) + (\boldsymbol{e}_1 - \boldsymbol{e}_4, 3 \wedge 4) \\ \gamma_2 := -(\mathbf{0}, 2 \wedge 4) + (\boldsymbol{e}_2 - \boldsymbol{e}_3, 3 \wedge 4) \\ \gamma_3 := (\mathbf{0}, 1 \wedge 2) \\ \gamma_4 := (\mathbf{0}, 3 \wedge 1) \\ \gamma_5 := (\mathbf{0}, 4 \wedge 1) \\ \gamma_6 := (\mathbf{0}, 2 \wedge 3) \\ \gamma_7 := (\mathbf{0}, 2 \wedge 4) \\ \gamma_8 := (\mathbf{0}, 3 \wedge 4) \end{cases}$$

(see Figure 23). Then, we see that the incidence matrix $\hat{M}$ of $E_2(\theta)$ can be given by

$$\hat{M} = \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \\ \gamma_7 \\ \gamma_8 \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

and we know it's maximal eigenvalue $\lambda_{\hat{M}}$ of $\hat{M}$ coincides with $|\lambda_1| \cdot |\lambda_2|$ where λ_1, λ_2 are eigenvalues of A where $|\lambda_1| > |\lambda_2| > 1 > |\lambda_3| = |\lambda_4|$. (c.f. The characteristic polynomial of $\hat{M}$ is given by $(x^6 - 2x^5 + x^4 - 5x^3 - x^2 - 2x - 1) \cdot (x+1)(x-1)$ and the first polynomial coincides with the characteristic polynomial of A^*). Let us consider the family of patches $\bar{\mathscr{U}}$ instead of the seed $\mathscr{U}$ by

$$\bar{\mathscr{U}} := \{\gamma_3, \gamma_4, \boldsymbol{e}_2 - \boldsymbol{e}_4 + \gamma_5, \gamma_6, \boldsymbol{e}_1 - \boldsymbol{e}_4 + \gamma_7, \boldsymbol{e}_2 - \boldsymbol{e}_4 + \gamma_8\}.$$

Then, we can see that B satisfies Definition 3.1 (1), (2), (3) (a)(b) as $N' = 4$. We show the figures of the quasi-periodic GDSS tiling τ (see Figure 25).

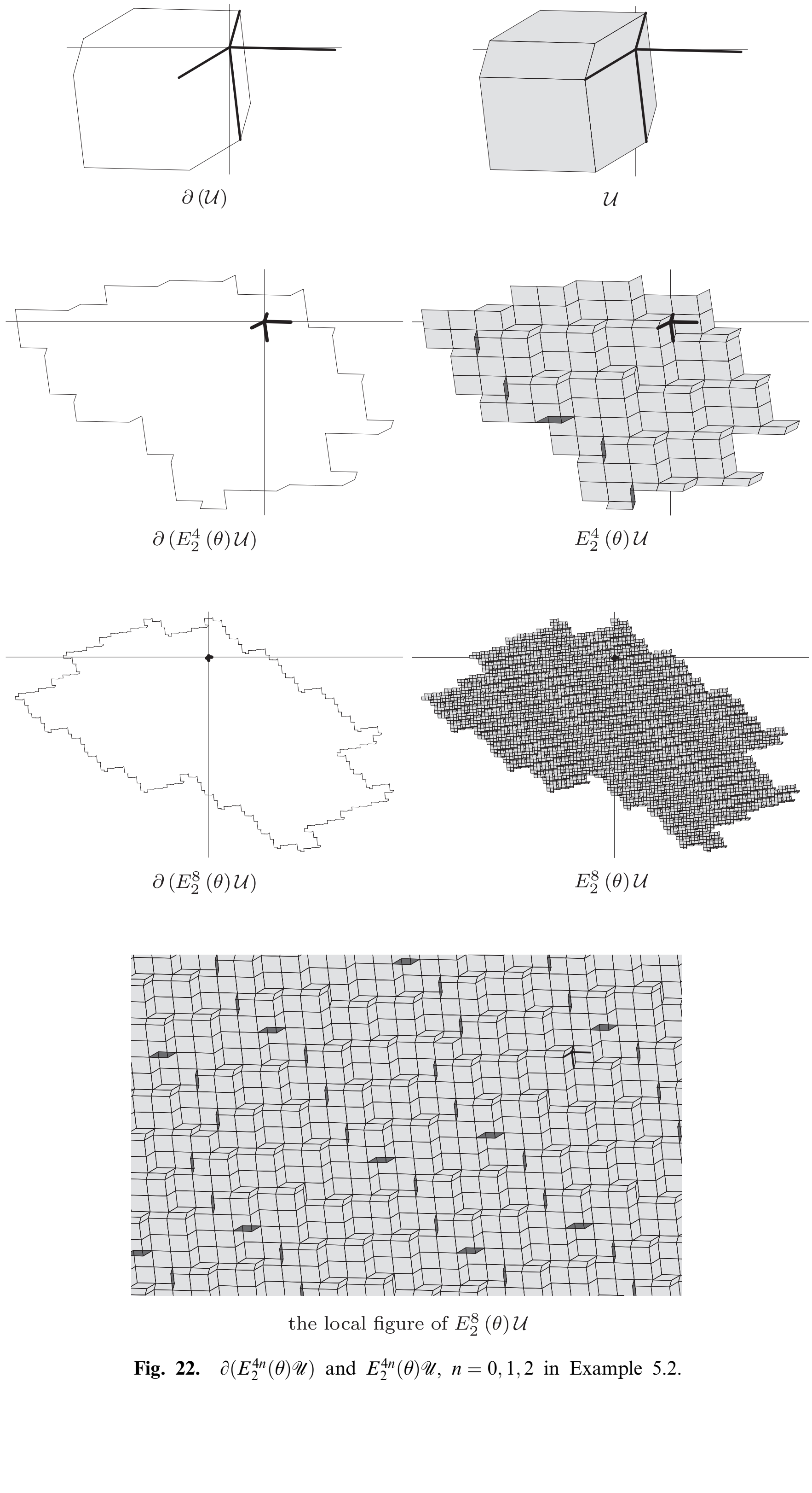


Fig. 22. $\partial(E_2^{4n}(\theta)\mathscr{U})$ and $E_2^{4n}(\theta)\mathscr{U}$, $n = 0, 1, 2$ in Example 5.2.

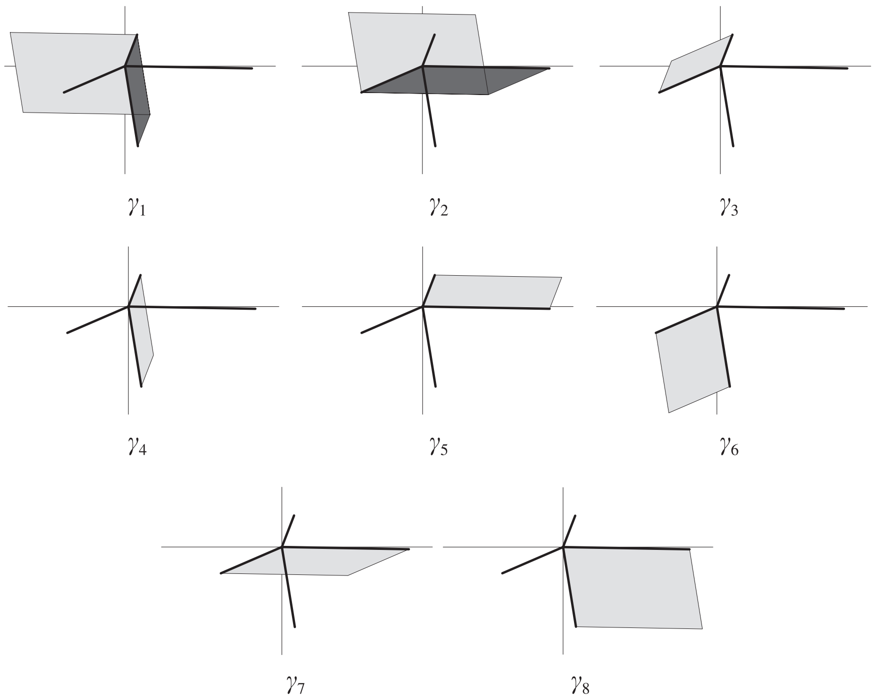


Fig. 23. Elements of the family of blocking patches B associated with σ in Example 5.2.

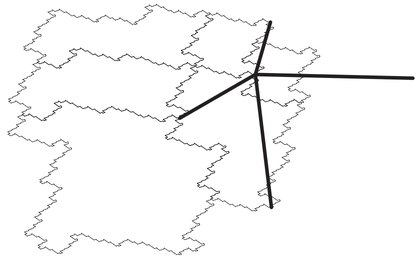


Fig. 24. $X = \bigcup_{k=1}^{L}(\boldsymbol{x}_{i_k} + X_{i_k})$ where $\bar{\mathscr{U}} = \bigcup_{k=1}^{L}(\boldsymbol{x}_{i_k} + \gamma_{i_k})$, $L = 6$ in Example 5.2.

As an example of unimodular non-Pisot and the characteristic polynomial with non-irreducible, we propose the following.

Example 5.3 ([11]). *Let us consider the following matrix A:*

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

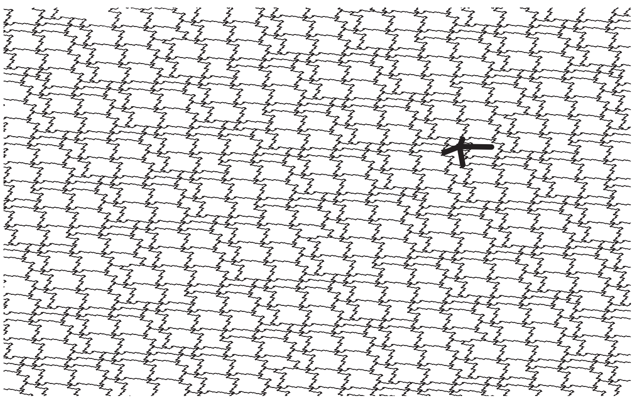
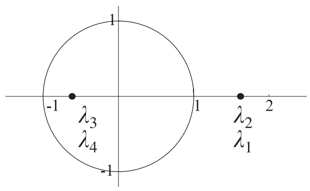


Fig. 25. Quasi-periodic GDSS tiling τ in Example 5.2.



Then the characteristic polynomial of A is $\Phi_A(x) = (x^2 - x - 1)^2$.

The set of symbolic parallelograms is chosen by

$$V_2 := \{1 \wedge 2, 3 \wedge 1, 4 \wedge 1, 2 \wedge 3, 2 \wedge 4, 4 \wedge 3\}.$$

Let us choose the automorphism σ (invertible substitution) and the mirror image θ of σ^{-1} is determined by

$$\sigma : \begin{cases} 1 \to 142 \\ 2 \to 321 \\ 3 \to 4^{-1}1^{-1} \\ 4 \to 2^{-1}3^{-1} \end{cases}, \qquad \theta : \begin{cases} 1 \to 24 \\ 2 \to 13 \\ 3 \to 3^{-1}1^{-1}4^{-1} \\ 4 \to 3^{-1}4^{-1}2^{-1} \end{cases}.$$

In this example, the covering substitution $E_2(\theta)$ sometimes produce the negative orientated prallelograms from the positive one (see Figure 26).

If $\mathcal{U}$ is chosen as

$$\begin{aligned} \mathcal{U} := &-(\boldsymbol{e}_1 - \boldsymbol{e}_3 + \boldsymbol{e}_4, 1 \wedge 2) + (\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3 + 2\boldsymbol{e}_4, 3 \wedge 1) \\ &+ (\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3 + \boldsymbol{e}_4, 4 \wedge 1) + (2\boldsymbol{e}_1 - \boldsymbol{e}_3 + \boldsymbol{e}_4, 2 \wedge 3) \\ &+ (\boldsymbol{e}_1 - \boldsymbol{e}_3, 2 \wedge 4) + (2\boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3 + \boldsymbol{e}_4, 4 \wedge 3) + (\boldsymbol{e}_2 - \boldsymbol{e}_3 + \boldsymbol{e}_4, 4 \wedge 1), \end{aligned}$$

then $\mathcal{U}$ satisfies the seed condition:

(1) $E_2(\theta)\mathcal{U} \succ \mathcal{U}$;

(2) $d(\partial E_2^n(\theta)\mathcal{U}, \mathbf{0}) \to 0 \ (n \to \infty)$

(see Figure 27).

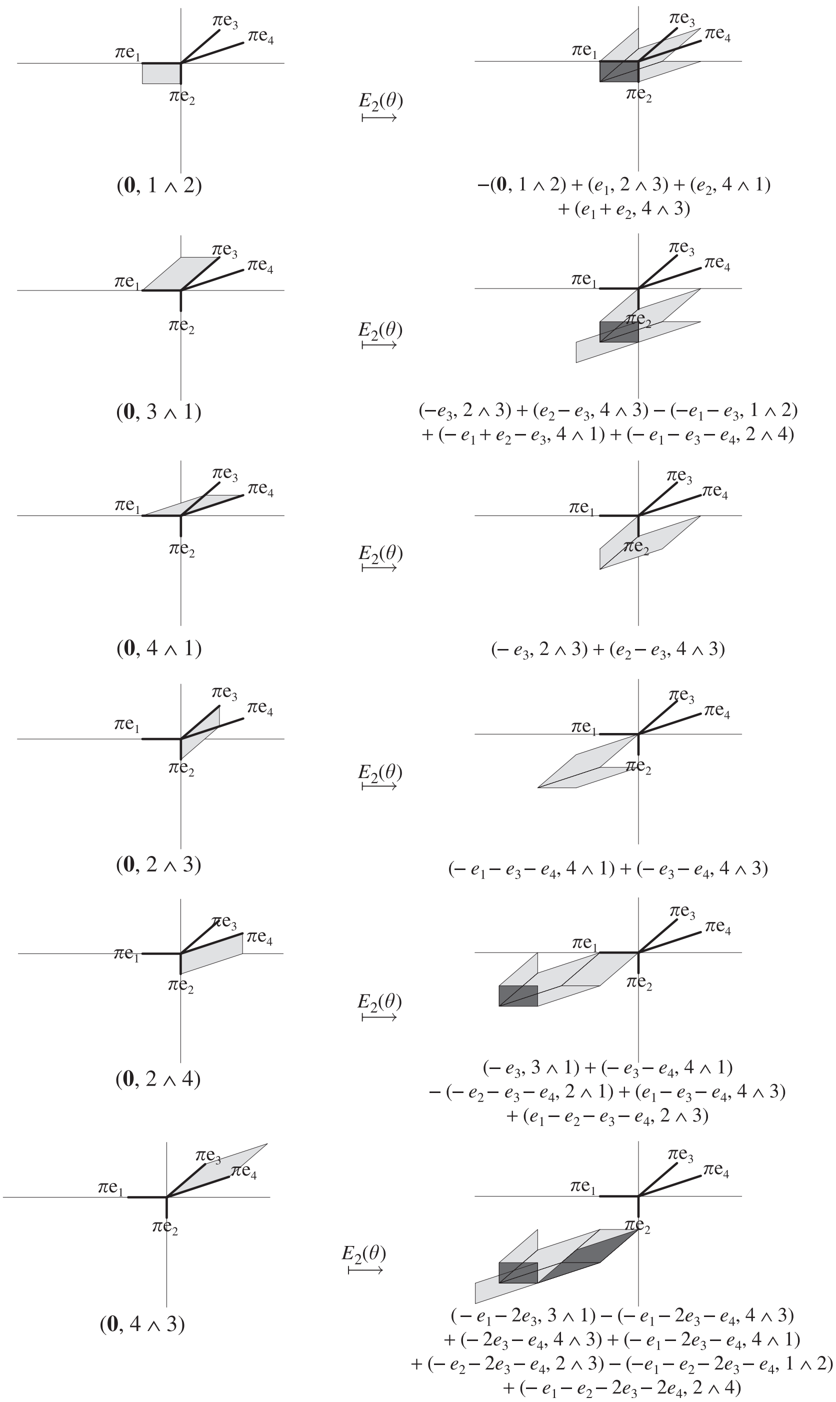


Fig. 26. $(\boldsymbol{0}, i \wedge j)$ and $E_2(\theta)(\boldsymbol{0}, i \wedge j)$, $i \wedge j \in V_2$ in Example 5.3.

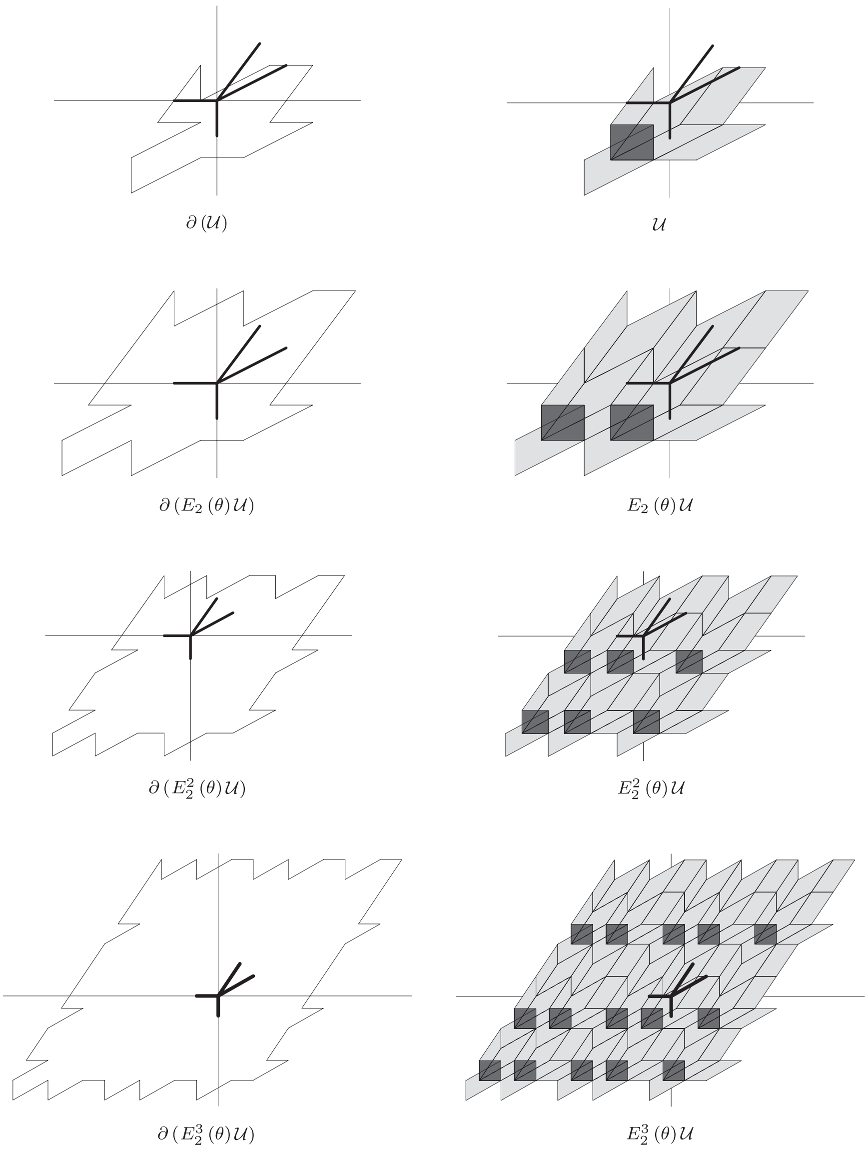


Fig. 27. $\partial(E_2^n(\theta)\mathscr{U})$ and $E_2^n(\theta)\mathscr{U}$, $n = 0, 1, 2, 3$ in Example 5.3.

Let us define the family of blocking patches B associated with σ by

$$\begin{cases} \gamma_1 := -(\mathbf{0}, 1 \wedge 2) + (\boldsymbol{e}_2, 4 \wedge 1) + (-\boldsymbol{e}_4, 2 \wedge 4) \\ \gamma_2 := (\mathbf{0}, 3 \wedge 1) \\ \gamma_3 := (\mathbf{0}, 2 \wedge 3) + (\boldsymbol{e}_2, 4 \wedge 3) \\ \gamma_4 := (\mathbf{0}, 4 \wedge 1) \end{cases}$$

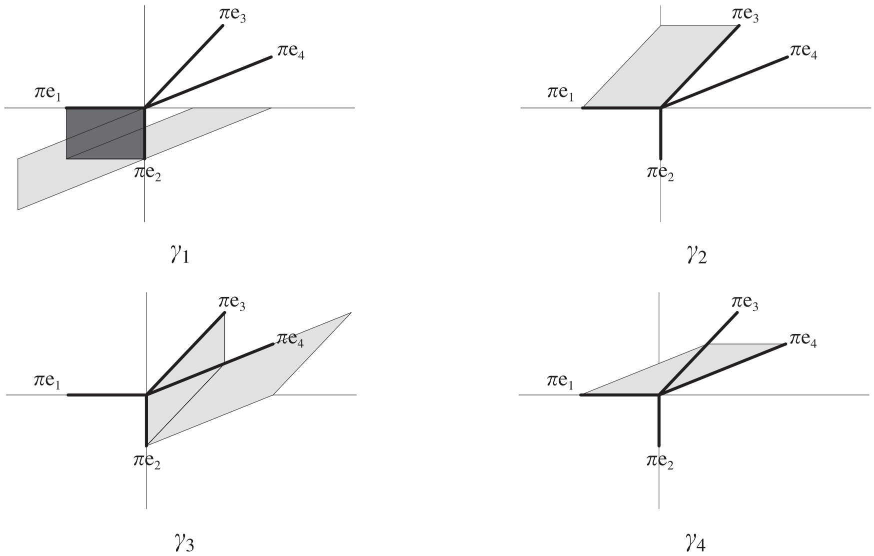


Fig. 28. Elements of the family of blocking patches B associated with σ in Example 5.3.

(see Figure 28*). Then, we see that the incidence matrix* $\hat{M}$ *of* $E_2(\theta)$ *is given by*

$$\hat{M} = \begin{array}{c} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{array} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and we know it's maximal eigenvalue $\lambda_{\hat{M}}$ *of* $\hat{M}$ *coincides with* $|\lambda_1| \cdot |\lambda_2|$ *where* λ_1, λ_2 *are eigenvalues of* A *where* $|\lambda_1| = |\lambda_2| > 1 > |\lambda_3| = |\lambda_4|$ *(c.f. The characteristic polynomial of* $\hat{M}$ *is given by* $(x^2 - 3x + 1)(x + 1)^2$ *and the first polynomial coincides with the characteristic polynomial of* A^**). Let us consider the family of patches* $\bar{\mathscr{U}}$ *instead of the seed* $\mathscr{U}$ *by*

$$\bar{\mathscr{U}} := \{\boldsymbol{e}_1 - \boldsymbol{e}_3 + \boldsymbol{e}_4 + \gamma_1, \boldsymbol{e}_1 + \boldsymbol{e}_2 - \boldsymbol{e}_3 + 2\boldsymbol{e}_4 + \gamma_2,$$
$$2\boldsymbol{e}_1 - \boldsymbol{e}_3 + \boldsymbol{e}_4 + \gamma_3, \boldsymbol{e}_2 - \boldsymbol{e}_3 + \boldsymbol{e}_4 + \gamma_4\}.$$

Then, we can see that B *satisfies Definition* 3.1 (1), (2), (3) (a)(b) *as* $N' = 1$.

Finally, we know the quasi-periodic GDSS tiling τ *can be found as* τ*-lattice in the crystal geometory (see* [8]*).*

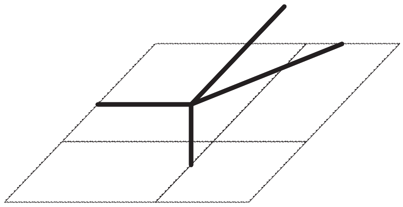


Fig. 29. $X = \bigcup_{k=1}^{L}(\boldsymbol{x}_{i_k} + X_{i_k})$ where $\bar{\mathscr{U}} = \bigcup_{k=1}^{L}(\boldsymbol{x}_{i_k} + X_{i_k})$, $L = 4$ in Example 5.3.

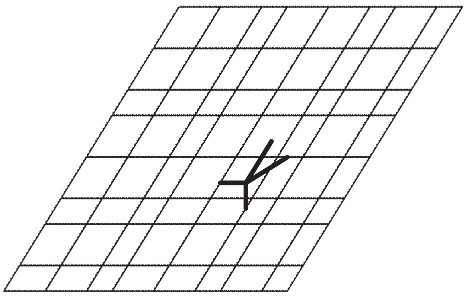


Fig. 30. Quasi-periodic GDSS tiling τ in Example 5.3.

Acknowledgement

I would like to thank my supervisor, Prof. Shunji ITO for his invaluable comments, guidance and encouragement during my work on this paper. I wish also to thank the referee for many helpful suggestions and comments.

References

[1] S. Akiyama and N. Gjini, Connectedness of number theoretic tilings, Discrete Math. Theor. Comput. Sci. **7** (2005), no. 1, 269–312.

[2] P. Arnoux, V. Berthé, H. Ei and S. Ito, Tilings, quasicrystals, discrete planes, generalized substitutions, and multidimensional continued fractions, Discrete Math. Theor. Comput. Sci. Proc. **AA(DM-CCG)**, 2001, 059–078.

[3] P. Arnoux, M. Furukado, E. Harriss and S. Ito, Tilings and Dynamical Systems from non-Pisot Unimodular Integer Matrix—Research REPORT—, unpublished.

[4] P. Arnoux and S. Ito, Pisot substitutions and Rauzy fractals, Bull. Belg. Math. Soc. Simon Stevin **8** (2001), no. 2, 181–207.

[5] M. F. Barnsley, Fractals Everywhere, Academic Press, New York, NY, 1988.

[6] G. A. Edgar, Measure, Topology, and Fractal Geometry, Springer-Verlag, New York, 1990.

[7] H. Ei, Some properties of invertible substitutions of rank d, and higher dimensional substitutions, Osaka J. Math. **40** (2003), no. 2, 543–562.

[8] A. Elkharrat, C. Frougny, J. P. Gazeau and J. L. Verger-Gaugry, Symmetry groups for beta-lattices, Theoret. Comput. Sci. **319** (2004), no. 1–3, 281–305.

[9] H. Ei, S. Ito, Tilings from some non-irreducible, Pisot substitutions, Discrete Math. Theor. Comput. Sci. **7** (2005), no. 1, 81–121.

[10] D. J. Feng, M. Furukado, S. Ito and J. Wu, Pisot substitutions and the Hausdorff dimension of boundaries of atomic surfaces, to appear in Tsukuba Journal of Mathematics.

[11] M. Furukado, S. Ito and E. A. Robinson, Tilings associated with non-Pisot matrices, to appear in Annales de l' Institut Fourier.

[12] M. Hall, Jr., The theory of groups, The Macmillan Company, New York (1959).

[13] S. Ito, J. Fujii, H. Higashino, S. Yasutomi, On simultaneous approximation to (α, α^2) with $\alpha^3 + k\alpha - 1 = 0$, J. Number Theory **99** (2003), no. 2, 255–283.

[14] S. Ito and H. Rao, Atomic surfaces, tilings and coincidence I. Irreducible case, Israel J. Math. **153** (2006), 129–156.

[15] S. Ito, Y. Sano, On periodic β-expansions of Pisot numbers and Rauzy fractals, Osaka J. Math. **38** (2001), no. 2, 349–368.

[16] J. M. Luck, C. Godrehe, A. Janner and T. Janssen, The nature of the atomic surfaces of quasiperiodic self-similar structures, J. Phys. A: Math. Gen. **26** (1993), 1951–1999.

[17] J. Lagarias and Y. Wang, Self-affine tiles in $\boldsymbol{R}^n$, Adv. Math. **121** (1996), no. 1, 21–49.

[18] B. Praggastis, Numeration systems and Markov partitions from self-similar tilings, Trans. Amer. Math. Soc. **351** (1999), no. 8, 3315–3349.

[19] M. Queffélec, Substitution dynamical systems—spectral analysis, Lecture Notes in Mathematics, **1294** (1987), Springer-Verlag.

[20] G. Rauzy, Nombres algébriques et substitutions, Bull. Soc. Math. France **110** (1982), no. 2, 147–178.

[21] V. F. Sirvent and Y. Wang, Self-affine tiling via substitution dynamical systems and Rauzy fractals, Pacific J. Math. **206** (2002), no. 2, 465–485.

[22] J. Thuswaldner, Unimodular Pisot substitutions and their associated tiles, J. Théor. Nombres Bordearux, to appear, 2006.

Maki Furukado
Faculty of Business Administration
Yokohama National University
79-4, *Tokiwadai, Hodogaya-ku*
Yokohama 240-8501, *Japan*
E-mail address: furukado@ynu.ac.jp