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Extendibility and stable extendibility of vector bundles over real projective spaces

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Abstract. The purpose of this paper is to study the extendibility and the stable extendibility of vector bundles of real projective spaces and those of their complexifications. We determine the dimension m for which the complexification of the tangent bundle of the n-dimensional real projective space RP^n is extendible to RP^m for $n = 6$ or $n > 7$, and determine the dimension n for which the square of the tangent bundle of RP^n or its complexification is extendible to RP^m for every $m > n$.

1. Introduction

Let X be a space and A be its subspace. A t-dimensional F-vector bundle ζ over A is called extendible (respectively stably extendible) to X, if there is a t-dimensional F-vector bundle over X whose restriction to A is equivalent (respectively stably equivalent) to ζ as F-vector bundles, where F is the real number field R, the complex number field C or the quaternion number field H (cf. [8] and [3]). Let R^n be the n-dimensional Euclidean space, RP^n be the n-dimensional real projective space and $\tau(RP^n)$ be the tangent bundle of RP^n.

First, we study the question: Determine the dimension m with $m > n$ for which a vector bundle over RP^n is extendible to RP^m. We have obtained the complete answer for the tangent bundle $\tau(RP^n)$ in [4, Theorem 6.6].

For an R-vector bundle and a C-vector bundle over RP^n we have

Theorem 1. *Let ζ be a t-dimensional R-vector bundle over RP^n. If $n < t$, ζ is extendible to RP^m for every m with $n < m \le t$.*

Theorem 2. *Let ζ be a t-dimensional C-vector bundle over RP^n. If $n < 2t+1$, ζ is extendible to RP^m for every m with $n < m \le 2t+1$.*

For the complexification of the tangent bundle $\tau(RP^n)$, we have

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Theorem 3. *Let $c\tau$ be the complexification $c\tau(RP^n)$ of the tangent bundle $\tau(RP^n)$. Then $c\tau$ is extendible to RP^{2n+1}, but is not stably extendible to RP^{2n+2} if $n = 6$ or $n > 7$. If $n = 1, 3$ or 7, $c\tau$ is extendible to RP^m for every m with $m > n$.*

Second, we study the question: Determine the dimension n for which a vector bundle over RP^n is extendible to RP^m for every m with $m > n$. We have obtained the complete answer for the normal bundle ν associated to an immersion of RP^n in R^{2n+1} in [7, Theorem A], and its complexification $c\nu$ in [7, Theorem 4.4]. For the square $\tau^2 = \tau(RP^n) \otimes \tau(RP^n)$ of the tangent bundle $\tau(RP^n)$ and its complexification $c\tau^2$, we have

Theorem 4. *Let τ^2 be the square $\tau(RP^n) \otimes \tau(RP^n)$ of the tangent bundle of RP^n. Then the following three conditions are equivalent:*
(1) *τ^2 is extendible to RP^m for every m with $m > n$,*
(2) *τ^2 is stably extendible to RP^m for every m with $m > n$,*
(3) $1 \leq n \leq 16$.

Theorem 5. *Let $c\tau^2$ be the complexification $c(\tau(RP^n) \otimes \tau(RP^n))$ of the square of the tangent bundle of RP^n. Then the following three conditions are equivalent:*
(1) *$c\tau^2$ is extendible to RP^m for every m with $m > n$,*
(2) *$c\tau^2$ is stably extendible to RP^m for every m with $m > n$,*
(3) $1 \leq n \leq 17$.

This note is arranged as follows. We prove Theorem 1 in §2. In §3 we study the Whitney sum decomposition of the square of the tangent bundle of the real projective space and prove Theorem 4. In §4 we prove Theorem 2 and the part of the extendibility of Theorem 3. In §5 we prove the part of the non-extendibility of Theorem 3, study the Whitney sum decomposition of the complexification of the square of the tangent bundle, and prove Theorem 5.

2. Extendibility of an R-vector bundle over the real projective space

Let ξ_n be the canonical R-line bundle over RP^n. The ring structure of $KO(RP^n)$ is determined in [1]. We recall the results which are necessary for our proofs. We use the same letter for a vector bundle and its isomorphism class.

(2.1) [1, Theorem 7.4]. (1) *The reduced KO-group $\widetilde{KO}(RP^n)$ is isomorphic to the cyclic group $Z/2^{\phi(n)}$, generated by $\xi_n - 1$, where $\phi(n)$ is the number of integers s such that $0 < s \leq n$ and $s \equiv 0, 1, 2$ or $4 \bmod 8$.* (2) $(\xi_n)^2 = 1$.

Let d denote $\dim_R F$, where $F = R, C$ or H. For a real number x, let $\lceil x \rceil$ denote the smallest integer n with $x \le n$. The following facts are known (cf. [2, Theorems 1.2 and 1.5, p. 99–p. 100]).

(2.2). *Let $m = \lceil (n+1)/d - 1 \rceil$. Then every t-dimensional F-vector bundle over an n-dimensional CW-complex X is isomorphic to $\alpha \oplus (t-m)$ for some m-dimensional F-vector bundle α over X if $m \le t$, where $\oplus$ denotes the Whitney sum and $t-m$ denotes the $(t-m)$-dimensional trivial F-vector bundle over X.*

(2.3). *Let $\ell = \lceil (n+2)/d - 1 \rceil$. If α and β are two t-dimensional F-vector bundles over an n-dimensional CW-complex X such that $\ell \le t$ and $\alpha \oplus k = \beta \oplus k$ for some k-dimensional trivial F-bundle k over X, then $\alpha = \beta$.*

Using (2.1), (2.2) and (2.3), we prove Theorem 1.

Proof of Theorem 1. It follows from (2.1)(1) that ζ is stably equivalent to $k\xi_n$ for some non-negative integer k. So there are trivial R-bundles u and v over RP^n of dimensions u and v respectively such that $\zeta \oplus u = k\xi_n \oplus v$. Let $n < m$. Then

$$\zeta \oplus u = i^*(k\xi_m \oplus v),$$

where $i : RP^n \to RP^m$ is the inclusion. Note that $k + v - m \ge k + v - m - u = t + u - m - u = t - m \ge 0$ if $m \le t$. By (2.2) there is an R-vector bundle α over RP^m of dimension m $(= \lceil (m+1)/1 - 1 \rceil)$ such that

$$k\xi_m \oplus v = \alpha \oplus (k + v - m),$$

since $m \le t$. Hence we have

$$\zeta \oplus u = i^*(\alpha \oplus (k + v - m)) = i^*(\alpha \oplus (k + v - m - u)) \oplus u.$$

Therefore, by (2.3),

$$\zeta = i^*(\alpha \oplus (k + v - m - u)),$$

since $\lceil (n+2)/1 - 1 \rceil = n + 1 \le t$. q.e.d.

Theorem 2.4. *Let w be an integer > 1 and let τ^w be the w-fold tensor product $\tau(RP^n)^w$ of the tangent bundle $\tau = \tau(RP^n)$. Then τ^w is extendible to RP^m for every m with $n < m \le n^w$.*

Proof. Since $\dim \tau^w = n^w$, the result follows from Theorem 1. q.e.d.

3. Extendibility and stable extendibility of the square of the tangent bundle of the real projective space

Lemma 3.1. *Let $\tau^2 = \tau(RP^n) \otimes \tau(RP^n)$ be the square of the tangent bundle $\tau = \tau(RP^n)$ of RP^n. Then the equality*

$$\tau^2 = (a2^{\phi(n)} - 2n - 2)\xi_n + (n+1)^2 + 1 - a2^{\phi(n)}$$

holds in $KO(RP^n)$, where a is any integer.

Proof. Since $\tau \oplus 1 = (n+1)\xi_n$, we have, by (2.1), $\tau^2 = (n+1)^2(\xi_n)^2 - 2(n+1)\xi_n + 1 = (a2^{\phi(n)} - 2n - 2)\xi_n + (n+1)^2 + 1 - a2^{\phi(n)}$ in $KO(RP^n)$ for any integer a. q.e.d.

Theorem 3.2. *Let $\tau(RP^n)^2 = \tau(RP^n) \otimes \tau(RP^n)$ be the square of the tangent bundle of RP^n. Then we have the Whitney sum decompositions:*

$\tau(RP^1)^2 = 1$, $\tau(RP^2)^2 = 2\xi_2 \oplus 2$, $\tau(RP^3)^2 = 9$,
$\tau(RP^4)^2 = 6\xi_4 \oplus 10$, $\tau(RP^5)^2 = 4\xi_5 \oplus 21$, $\tau(RP^6)^2 = 2\xi_6 \oplus 34$,
$\tau(RP^7)^2 = 49$, $\tau(RP^8)^2 = 14\xi_8 \oplus 50$, $\tau(RP^9)^2 = 12\xi_9 \oplus 69$,
$\tau(RP^{10})^2 = 42\xi_{10} \oplus 58$, $\tau(RP^{11})^2 = 40\xi_{11} \oplus 81$, $\tau(RP^{12})^2 = 102\xi_{12} \oplus 42$,
$\tau(RP^{13})^2 = 100\xi_{13} \oplus 69$, $\tau(RP^{14})^2 = 98\xi_{14} \oplus 98$, $\tau(RP^{15})^2 = 96\xi_{15} \oplus 129$
and $\tau(RP^{16})^2 = 222\xi_{16} \oplus 34$.

Proof. For $2 \le n \le 8$, put $a = 2$ and for $9 \le n \le 16$, put $a = 1$. Then the equalities above follow from Lemma 3.1 and (2.3). The case $n = 1$ is clear. q.e.d.

Remark 3.3. $\tau(RP^{17})^2 = 476\xi_{17} - 187$.

The following result is Theorem 4.1 in [6] which is the *stably extendible* version of Theorem 6.2 in [4].

(3.4). *Let ζ be a t-dimensional R-vector bundle over RP^n. Assume that there is a positive integer ℓ such that ζ is stably equivalent to $(t+\ell)\xi_n$ and $t + \ell < 2^{\phi(n)}$. Then $n < t + \ell$ and ζ is not stably extendible to RP^m for every m with $m \ge t + \ell$.*

Lemma 3.5. *Define $\ell(n) = 2^{\phi(n)} - n^2 - 2n - 2$. Then $\ell(n) > 0$ if $n \ge 17$.*

Proof. As is seen in the table below, the inequality $\ell(n) > 0$ holds for integers n with $17 \le n \le 24$ clearly.

n	17	18	19	20	21	22	23	24
$8s+i$	$i=1$ $s=2$	$i=2$ $s=2$	$i=3$ $s=2$	$i=4$ $s=2$	$i=5$ $s=2$	$i=6$ $s=2$	$i=7$ $s=2$	$i=0$ $s=3$
$\ell(n)$	187	662	623	1606	1563	1518	1471	3470

For larger values of n, we have $\ell(n) > 0$ by induction on s. q.e.d.

Theorem 3.6. *$\tau^2 = \tau(RP^n) \otimes \tau(RP^n)$ is not stably extendible to RP^m for every m with $m \geq 2^{\phi(n)} - 2n - 2$ if $n \geq 17$.*

Proof. According to Lemma 3.1, τ^2 is stably equivalent to $(2^{\phi(n)} - 2n - 2)\xi_n$. Setting $\zeta = \tau^2$, $t = n^2$ and $\ell = \ell(n) = 2^{\phi(n)} - n^2 - 2n - 2$ in (3.4), we see that τ^2 is not stably extendible to RP^m for every m with $m \geq 2^{\phi(n)} - 2n - 2$, since $2^{\phi(n)} - n^2 - 2n - 2 > 0$ for $n \geq 17$ by Lemma 3.5 and $2^{\phi(n)} - 2n - 2 < 2^{\phi(n)}$. q.e.d.

Example 3.7. *$\tau^2 = \tau(RP^{17}) \otimes \tau(RP^{17})$ is not stably extendible to RP^m for every m with $m \geq 476$.*

Proof of Theorem 4. It is clear that (1) implies (2). The fact that (2) implies (3) is a consequence of Theorem 3.6. Since ξ_n and the trivial bundles over RP^n are extendible to RP^m for every m with $m > n$, Theorem 3.2 shows that (3) implies (1). q.e.d.

4. Extendibility of a C-vector bundle over the real projective space

The ring structure of $K(RP^n)$ is determined in [1]. We recall the result which is necessary for our proofs.

(4.1) [1, Theorem 7.3]. *The reduced K-group $\tilde{K}(RP^n)$ is isomorphic to the cyclic group $Z/2^{\lfloor n/2 \rfloor}$, generated by $c(\xi_n - 1)$, where $\lfloor n/2 \rfloor$ denotes the integral part of $n/2$.*

Using (4.1), (2.2) and (2.3), we prove Theorem 2.

Proof of Theorem 2. It follows from (4.1) that ζ is stably equivalent to $kc\xi_n$ for some non-negative integer k. So there are trivial C-bundles u and v over RP^n of dimensions u and v respectively such that $\zeta \oplus u = kc\xi_n \oplus v$. Let $n < m$. Then

$$\zeta \oplus u = i^*(kc\xi_m \oplus v),$$

where $i : RP^n \to RP^m$ is the inclusion. Note that $k + v - \ell \geq k + v - \ell - u = t + u - \ell - u = t - \ell \geq 0$ if $\ell \leq t$. By (2.2) there is a C-vector bundle α over RP^m of dimension ℓ $(= \lceil (m+1)/2 - 1 \rceil)$ such that

$$kc\xi_m \oplus v = \alpha \oplus (k + v - \ell),$$

since $\ell \leq t$. Hence we have

$$\zeta \oplus u = i^*(\alpha \oplus (k + v - \ell)) = i^*(\alpha \oplus (k + v - \ell - u)) \oplus u.$$

The inequalities $n < m \leq 2t + 1$ show that $\lceil (n+2)/2 - 1 \rceil \leq \lceil (m+1)/2 - 1 \rceil = \ell \leq t$. Therefore, by (2.3), we obtain

$$\zeta = i^*(\alpha \oplus (k + v - \ell - u)). \qquad \text{q.e.d.}$$

THEOREM 4.2. *Let w be a positive integer and let $c\tau^w$ be the complexification of the w-fold tensor product $\tau^w = \tau(RP^n)^w$. Then $c\tau^w$ is extendible to RP^m for every m with $n < m \leq 2n^w + 1$*

PROOF. Since $\dim c\tau^w = n^w$, the result follows from Theorem 2. q.e.d.

COROLLARY 4.3. *$c\tau = c\tau(RP^n)$ is extendible to RP^{2n+1}.*

5. Extendibility and stable extendibility of the complexification of the square of the tangent bundle of the real projective space

LEMMA 5.1. *Let $c\tau^2 = c(\tau(RP^n) \otimes \tau(RP^n))$ be the complexification of the square of the tangent bundle of RP^n. Then the equality*

$$c\tau^2 = (b2^{\lfloor n/2 \rfloor} - 2n - 2)c\xi_n + (n+1)^2 + 1 - b2^{\lfloor n/2 \rfloor}$$

holds in $K(RP^n)$, where b is any integer.

PROOF. Complexifying the equality $\tau^2 = -2(n+1)\xi_n + (n+1)^2 + 1$ (cf. Lemma 3.1) and using (4.1), we have the equality above. q.e.d.

THEOREM 5.2. *Let $c\tau(RP^n)^2 = c(\tau(RP^n) \otimes \tau(RP^n))$ be the complexification of the square $\tau(RP^n) \otimes \tau(RP^n)$. Then we have the Whitney sum decompositions:*

$$\begin{array}{lll}
c\tau(RP^1)^2 = 1, & c\tau(RP^2)^2 = 4, & c\tau(RP^3)^2 = 9, \\
c\tau(RP^4)^2 = 6c\xi_4 \oplus 10, & & c\tau(RP^5)^2 = 25, \\
c\tau(RP^6)^2 = 2c\xi_6 \oplus 34, & & c\tau(RP^7)^2 = 49, \\
c\tau(RP^8)^2 = 14c\xi_8 \oplus 50, & & c\tau(RP^9)^2 = 12c\xi_9 \oplus 69, \\
c\tau(RP^{10})^2 = 42c\xi_{10} \oplus 58, & & c\tau(RP^{11})^2 = 40c\xi_{11} \oplus 81, \\
c\tau(RP^{12})^2 = 102c\xi_{12} \oplus 42, & & c\tau(RP^{13})^2 = 100c\xi_{13} \oplus 69, \\
c\tau(RP^{14})^2 = 98c\xi_{14} \oplus 98, & & c\tau(RP^{15})^2 = 96c\xi_{15} \oplus 129, \\
c\tau(RP^{16})^2 = 222c\xi_{16} \oplus 34 \ \textit{and} & & c\tau(RP^{17})^2 = 220c\xi_{17} \oplus 69.
\end{array}$$

PROOF. Since $\lfloor n/2 \rfloor \leq \phi(n)$, complexifying the equalities in Theorem 3.2, we have the equalities above except for $n = 2, 5$ and 17. Considering the relations $2c\xi_2 - 2 = 0$ for $n = 2$, $4c\xi_5 - 4 = 0$ for $n = 5$ and $256c\xi_{17} - 256 = 0$ for $n = 17$ (cf. (4.1)) and using (2.3), we have the equalities above from those in Theorem 3.2 and Remark 3.3. q.e.d.

The following result is Theorem 2.1 in [6] which is the *stably extendible* version of Theorem 4.2 for $d = 1$ in [5].

(5.3). *Let ζ be a t-dimensional C-vector bundle over RP^n. Assume that there is a positive integer ℓ such that ζ is stably equivalent to $(t+\ell)c\xi_n$ and $t+\ell < 2^{\lfloor n/2 \rfloor}$. Then $\lfloor n/2 \rfloor < t+\ell$ and ζ is not stably extendible to RP^m for every m with $m \geq 2t+2\ell$.*

Theorem 5.4. *Let $c\tau = c\tau(RP^n)$ be the complexification of the tangent bundle $\tau(RP^n)$. If $n=6$ or $n>7$, $c\tau$ is not stably extendible to RP^m for every m with $m \geq 2n+2$.*

Proof. Putting $\zeta = c\tau(RP^n)$, $t=n$ and $\ell=1$, we have the result from (5.3), since $c\tau(RP^n)\oplus 1 = (n+1)c\xi_n$ and $n+1 < 2^{\lfloor n/2 \rfloor}$ if $n=6$ or $n>7$. q.e.d.

Proof of Theorem 3. The first part follows from Corollary 4.3 and Theorem 5.4. The second part is a consequence of the fact that $\tau(RP^n)$ is trivial for $n=1,3$ and 7. q.e.d.

Lemma 5.5. *Define $\ell(n) = 2^{\lfloor n/2 \rfloor} - n^2 - 2n - 2$. Then $n^2 + \ell(n) < 2^{\lfloor n/2 \rfloor}$, and if $n \geq 18$, $\ell(n) > 0$.*

Proof. Since $2^{\lfloor n/2 \rfloor} - n^2 - \ell(n) = 2n+2 > 0$, we have $n^2 + \ell(n) < 2^{\lfloor n/2 \rfloor}$. The inequality $\ell(n) > 0$ holds for $n = 18$ and 19 clearly. For larger values of n, we have $\ell(n) > 0$ by induction. q.e.d.

Theorem 5.6. *$c\tau^2 = c(\tau(RP^n)\otimes\tau(RP^n))$ is not stably extendible to RP^m for every m with $m \geq 2^{\lfloor n/2 \rfloor+1} - 4n - 4$ if $n \geq 18$.*

Proof. According to Lemma 5.1, $c\tau^2$ is stably equivalent to $(2^{\lfloor n/2 \rfloor} - 2n - 2)c\xi_n$. Setting $\zeta = c\tau^2$, $t = n^2$ and $\ell = \ell(n) = 2^{\lfloor n/2 \rfloor} - n^2 - 2n - 2$ in (5.3), we see that $c\tau^2$ is not stably extendible to RP^m for every m with $m \geq 2^{\lfloor n/2 \rfloor+1} - 4n - 4$, since $\ell(n) > 0$ and $t + \ell(n) < 2^{\lfloor n/2 \rfloor}$ then by Lemma 5.5. q.e.d.

Example 5.7. *$c\tau^2 = c(\tau(RP^{18})\otimes\tau(RP^{18}))$ is not stably extendible to RP^m for every m with $m \geq 948$.*

Proof of Theorem 5. It is clear that (1) implies (2). The fact that (2) implies (3) is a consequence of Theorem 5.6. Since $c\xi_n$ and the trivial bundles over RP^n are extendible to RP^m for every m with $m > n$, Theorem 5.2 shows that (3) implies (1). q.e.d.

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