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Congruence formulae for stable maps of surfaces

Shin SATOH

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ABSTRACT. We discuss some relationships among congruence formulae modulo two for the Euler characteristic of a closed surface M^2 and stable maps $f : M^2 \to \mathbf{R}^n$ $(1 \le n \le 4)$.

1. Introduction

Let M^2 be a closed surface and $f : M^2 \to \mathbf{R}^n$ a stable map. We have known several congruence formulae modulo two for the Euler characteristic $\chi(M^2)$ and the singularity/self-intersection sets of f. We denote by $S(f)$ and $I(f)$ $(\subset M^2)$ the sets of singularities and self-intersections of f respectively. We consider some relationships among the congruence formulae (1)–(9) below in case of $1 \le n \le 4$. Note that any stable map for $n \ge 5$ is an embedding and hence $S(f) = I(f) = \varnothing$ (cf. [**20**]).

Throughout this paper we work in the smooth category. For a finite set S, we denote by $|S|$ the number of elements in S. For a closed surface M^2, a circle L immersed in M^2 is said to be *of $\mathscr{A}$-type* (or *of $\mathscr{M}$-type*) if the normal bundle of an immersion $\lambda : S^1 \to M^2$ with $L = \lambda(S^1)$ which does not factor through a non-trivial covering map of S^1 is trivial (resp. non-trivial). Given a union U of immersed circles in M^2, we denote by $\|U\|$ the number of circles of $\mathscr{M}$-type in U.

1-dimensional case. Any stable map $f : M^2 \to \mathbf{R}^1$ is a Morse function and $S(f)$ is the set of critical points of f. The following congruence is obtained as a corollary of the Morse equality (cf. [**15**]).

$$\chi(M^2) \equiv |S(f)| \qquad (\mathrm{mod}\ 2). \tag{1}$$

2-dimensional case. For a stable map $f : M^2 \to \mathbf{R}^2$, the singularity set $S(f)$ of f forms a disjoint union of embedded circles in M^2. Then we have the following congruence.

$$\chi(M^2) \equiv \|S(f)\| \qquad (\mathrm{mod}\ 2). \tag{2}$$

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The singularity set $S(f)$ consists of fold singularities and cusp singularities (cf. [**22**]). We denote by $C(f)$ the subset of $S(f)$ consisting of cusp singularities of f. Note that $C(f)$ is finite. Then we have the following congruence (cf. [**10**, **14**, **19**]).

$$\chi(M^2) \equiv |C(f)| \qquad (\mathrm{mod}\ 2). \tag{3}$$

The congruences (2) and (3) are immediate consequences of the following well-known facts:

(i) the Poincaré duals of the homology classes $[S(f)] \in H_1(M^2; \mathbf{Z}_2)$ and $[C(f)] \in H_0(M^2; \mathbf{Z}_2)$ represented by $S(f)$ and $C(f)$ coincide with the first and the second Stiefel-Whitney classes $w_1(M^2) \in H^1(M^2; \mathbf{Z}_2)$ and $w_2(M^2) \in H^2(M^2; \mathbf{Z}_2)$ respectively, and

(ii) $\langle w_1(M^2) \cup w_1(M^2), [M^2] \rangle \equiv \langle w_2(M^2), [M^2] \rangle \equiv \chi(M^2)$ (mod 2), where $\cup$ denotes the cup product and $[M^2] \in H_2(M^2; \mathbf{Z}_2)$ is the fundamental class.

See [**19**], for example. Note that $\|U\| \equiv u \cup u$ (mod 2) for any finite union U of immersed circles in M^2, where $u \in H^1(M^2; \mathbf{Z}_2)$ denotes the Poincaré dual of the homology class $[U] \in H_1(M^2; \mathbf{Z}_2)$ represented by U; for it is not difficult to see that $\|U\|$ has the same parity as the self-intersection number of $[U]$.

Combined case in 2- and 3-dimensions. We consider stable maps $f : M^2 \to \mathbf{R}^2$ and $g : M^2 \to \mathbf{R}^3$ such that $f = \pi \circ g$ for a projection $\pi : \mathbf{R}^3 \to \mathbf{R}^2$.

$$\begin{tikzcd} & \mathbf{R}^3 \arrow[d, "\pi"] \\ M^2 \arrow[ur, "g"] \arrow[r, "f"'] & \mathbf{R}^2 \end{tikzcd}$$

Throughout this paper, we assume that the projection π is *generic* with respect to g in the sense of [**6**, §1.5]. The singularity set $S(g)$ of g consists of Whitney umbrella points, and the self-intersection set $I(g)$ is a union of immersed circles and open arcs in M^2 such that the closure $\overline{I(g)}$ in M^2 is the union $I(g) \cup S(g)$ (cf. [**21**]). By our assumption, we have that $S(f)$ intersects $\overline{I(g)}$ transversely in M^2 and misses the self-intersections of $\overline{I(g)}$. It is well-known that

(iii) the Poincaré dual of the homology class $[\overline{I(g)}] \in H_1(M^2; \mathbf{Z}_2)$ represented by $\overline{I(g)}$ coincides with the first Stiefel-Whitney class $w_1(M^2) \in H^1(M^2; \mathbf{Z}_2)$.

See [**8**]. Hence, we have the following congruence by the algebraic facts (i)–(iii) immediately.

$$\chi(M^2) \equiv |S(f) \cap \overline{I(g)}| \qquad (\mathrm{mod}\ 2). \tag{4}$$

Note that the congruence (4) holds for any stable maps $f : M^2 \to \mathbf{R}^2$ and $g : M^2 \to \mathbf{R}^3$ without the restriction $f = \pi \circ g$ for a projection $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ as long as $S(f)$ and $\overline{I(g)}$ are in general position.

The singularity set $S(f)$ of f is mapped into $\mathbf{R}^2$ by f as a union of immersed circles with cusps. Also, by our assumption, $\overline{I(g)}$ is mapped into $\mathbf{R}^2$ by f as a union of immersed arcs and circles. We denote by $W(f,g)$ the set of points in $\mathbf{R}^2$ where $f(\overline{I(g)})$ is tangent to $f(S(f))$. Then we have the following congruence.

Theorem 1.1. *Let M^2 be a closed surface. For any stable maps $f : M^2 \to \mathbf{R}^2$ and $g : M^2 \to \mathbf{R}^3$ such that $f = \pi \circ g$ for a generic projection $\pi : \mathbf{R}^3 \to \mathbf{R}^2$, we have*

$$\chi(M^2) \equiv |W(f,g)| \pmod 2. \tag{5}$$

3-dimensional case. Let $f : M^2 \to \mathbf{R}^3$ be a stable map. The closure $\overline{I(f)}$ of the self-intersection set of f is regarded as a union of immersed circles in M^2. The following congruence is obtained from the facts (ii) and (iii) immediately.

$$\chi(M^2) \equiv \|\overline{I(f)}\| \pmod 2. \tag{6}$$

We denote by $D(f)$ the multiple point set of f; that is, $D(f) = f(I(f))$. Then the closure $\overline{D(f)}$ is regarded as a union of immersed arcs and circles in $\mathbf{R}^3$ which are called *double curves*. Note that the intersections of double curves are non-tangential triple points and that the two ends of each arc-component correspond to Whitney umbrella points. There are two types of arc-components in $\overline{D(f)}$ called *$\mathscr{A}$-type* and *$\mathscr{M}$-type*, and there are three types of circle-components in $\overline{D(f)}$ called *$\mathscr{A}$-type*, *$\mathscr{M}$-type*, and *$\mathscr{N}$-type*. Their definitions are given in Section 4 (see Figure 8). We denote by $|||\overline{D(f)}|||$ the total number of double curves (arc- and circle-components) of $\mathscr{M}$-type in $\overline{D(f)}$. Then we have the following congruence.

Theorem 1.2. *Let M^2 be a closed surface. For any stable map $f : M^2 \to \mathbf{R}^3$, we have*

$$\chi(M^2) \equiv |||\overline{D(f)}||| \pmod 2. \tag{7}$$

We denote by $T(f)$ the subset of $D(f)$ consisting of triple points of f. Note that $T(f)$ is finite. For each point $P \in f(S(f))$, we take a point P' close to P in $\mathbf{R}^3 \backslash f(M^2)$ outside the Whitney umbrella (see Figure 10). Let $P_1', \dots, P_k'$ be the points in $\mathbf{R}^3 \backslash f(M^2)$ constructed as above associated with all the points $P_1, \dots, P_k$ in $f(S(f))$. The *linking number* of f, denoted by $l(f)$, is defined by Szűcs [**17**] to be the mod 2 linking number of the set $\{P_1', \dots, P_k'\}$

and $f(M^2)$. Since the number k is even, $l(f)$ is well-defined. Szűcs generalizes Banchoff's congruence [**3**] for the case that f is an immersion (that is, $S(f) = \emptyset$) as follows.

$$\chi(M^2) \equiv |T(f)| + l(f) \qquad (\mathrm{mod}\ 2). \tag{8}$$

4-dimensional case. Any stable map $f : M^2 \to \mathbf{R}^4$ is an immersion with transverse double points (cf. [**20**]). Hence, $D(f) = f(I(f))$ is the set of such double points and $S(f) = \emptyset$. Recall that the normal Euler number of f, denoted by $e(f)$, is defined to be the intersection number of $f(M^2)$ and its transverse push-off, where we ignore the intersections corresponding to the self-intersection points of $f(M^2)$ (cf. [**20**]). Refer also to [**4**, **5**, **16**]. The following congruence is known as a generalization of Whitney's congruence for the case that f is an embedding, that is, $D(f) = \emptyset$ (cf. [**20**]).

$$\chi(M^2) \equiv |D(f)| + \frac{e(f)}{2} \qquad (\mathrm{mod}\ 2). \tag{9}$$

This is an immediate consequence of the congruence formula proved by Mahowald [**12**] and Lannes [**9**] as follows. Let $f : M^n \to \mathbf{R}^{2n}$ be a stable map, where M^n is a closed n-manifold and $n \geq 2$ is even. They proved that $\overline{w}_1(M^n) \cup \overline{w}_{n-1}(M^n) \in H^n(M^n; \mathbf{Z}_2) = \mathbf{Z}_2$ has the same parity as $|D(f)| + e(f)/2$, where $\overline{w}_i(M^n) \in H^i(M^n; \mathbf{Z}_2)$ denotes the i-th dual Stiefel-Whitney class of M^n. Saeki and Sakuma gave a geometric proof of this congruence in [**16**]. The congruence (9) is also obtained from Yamada's congruence [**23**] as follows. Let $f : M^2 \to N^4$ be a stable map such that N^4 is an oriented 4-manifold. He defined a $\mathbf{Z}_4$-quadratic map $q : H_2(N^4; \mathbf{Z}_2) \to \mathbf{Z}_4$ and proved that $q([M^2])$ is congruent to $e(f) + 2\chi(M^2) + 2|D(f)|$ modulo 4, where $[M^2] \in H_2(N^4; \mathbf{Z}_2)$ denotes the homology class represented by $f(M^2)$ in N^4. See also [**1**, **2**, **11**].

The first aim of this paper is to give geometric proofs of the congruences (2) and (6) without using the facts (i)–(iii) (Lemmas 2.1 and 2.2). The second aim is to give geometric relationships among the congruences (1)–(9) as the following scheme shows, where the numbers attached to the arrows indicate the propositions which connect the congruences.

$$\begin{tikzcd}
& (2) \arrow[r, leftrightarrow, "2.3"] \arrow[d, leftrightarrow, "4.1"'] & (4) \arrow[r, leftrightarrow, "2.3"] \arrow[d, leftrightarrow, "4.3"'] & (6) \arrow[d, leftrightarrow, "5.1"] & \\
(1) \arrow[r, leftrightarrow, "3.1"'] & (3) \arrow[r, leftrightarrow, "4.2"'] & (5) \arrow[r, leftrightarrow, "4.4"'] & (7) \arrow[r, leftrightarrow, "6.1"'] \arrow[d, leftrightarrow, "5.2"] & (9) \\
& & & (8) &
\end{tikzcd}$$

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2. Immersed circles in a closed surface

Let M^2 be a closed surface and $U = \bigcup_{i=1}^m L_i$ a union of immersed circles $L_1, \dots, L_m$ in M^2 with transverse double points. We say that L_i is *of $\mathscr{A}$-type* (or *of $\mathscr{M}$-type*) if the normal bundle of an immersion $\lambda_i : S^1 \to M^2$ with $L_i = \lambda_i(S^1)$ which does not factor through a non-trivial covering map of S^1 is trivial (resp. non-trivial). Equivalently, L_i is of $\mathscr{A}$-type (or of $\mathscr{M}$-type) if and only if a regular neighborhood of L_i in M^2 is the image of a non-factorizable immersion of an annulus (or a Möbius band). We denote by $\|U\|$ the number of L_i's of $\mathscr{M}$-type in $\{L_1, \dots, L_m\}$.

The union U is regarded as a 4-valent graph with hoops embedded in M^2, where a hoop means a circle-edge with no vertex. Consider the condition ($\sharp$) for U as follows.

($\sharp$) The connected components of $M^2 \setminus U$ can be oriented so that adjacent components along U have incompatible orientations.

More precisely, let $R_1, \dots, R_n$ be the connected components of $M^2 \setminus U$. The condition ($\sharp$) says that each R_i is orientable and can be oriented so that if the closures $\overline{R_i}$ and $\overline{R_j}$ have a common edge e of U, then the orientation of e induced from $\overline{R_i}$ coincides with that induced from $\overline{R_j}$.

We give geometric proofs of the congruences (2) and (6). For the purpose, it is sufficient to prove the following two lemmas.

Lemma 2.1. *Let M^2 be a closed surface and $U = \bigcup_{i=1}^m L_i$ a union of immersed circles $L_1, \dots, L_m$ in M^2 with transverse double points. Suppose that U satisfies the condition* ($\sharp$). *Then we have*

$$\chi(M^2) \equiv \|U\| \qquad (\mathrm{mod}\ 2).$$

Lemma 2.2. *For both cases* (i) *and* (ii) *below, U satisfies the condition* ($\sharp$).

(i) $U = S(f)$ *for a stable map* $f : M^2 \to \mathbf{R}^2$.

(ii) $U = \overline{I(f)}$ *for a stable map* $f : M^2 \to \mathbf{R}^3$.

Note that since f is stable, $U = S(f)$ in Lemma 2.2(i) is a disjoint union of embedded circles in M^2, and $U = \overline{I(f)}$ in (ii) is a union of immersed circles in M^2 with transverse double points. Hence, the congruences (2) and (6) follow from Lemmas 2.1 and 2.2 immediately.

Proof of Lemma 2.1. We denote by $c(U)$ the number of double points of U. Let N_U be a regular neighborhood of U in M^2. Since U is a deformation retract of N_U, the Euler characteristic $\chi(N_U)$ is equal to

$c(U) - 2c(U) = -c(U)$. Furthermore, since $\overline{M^2 \backslash N_U}$ is a union of orientable compact surfaces by the condition ($\sharp$), its Euler characteristic has the same parity as the number of boundary circles of N_U. We denote this number by $d(U)$. Then we have

$$\chi(M^2) = \chi(N_U) + \chi(\overline{M^2 \backslash N_U}) - \chi(N_U \cap \overline{M^2 \backslash N_U}) \equiv c(U) + d(U) \pmod 2,$$

and it is sufficient to prove

$$\|U\| \equiv c(U) + d(U) \pmod 2.$$

We prove this congruence by induction on the number $c(U)$. If $c(U) = 0$, then $L_1, \dots, L_m$ are disjointly embedded in M^2, and hence, we have $\|U\| \equiv d(U) \pmod 2$; for L_i is of $\mathscr{A}$-type (or of $\mathscr{M}$-type) if and only if its regular neighborhood is an annulus (resp. a Möbius band) embedded in M^2. Consider the case $c(U) \geq 1$. We take a double point of U and splice the intersecting two arcs at the point. Then we obtain a new union U' of immersed circles such that $c(U') = c(U) - 1$, see Figure 1. It is not difficult to see that $\|U'\| = \|U\|$ or $\|U\| \pm 2$. Since the boundary circles of N_U can be oriented from the orientation of $\overline{M^2 \backslash N(U)}$ given by the condition ($\sharp$), it holds that $d(U') = d(U) \pm 1$. Since U' also satisfies ($\sharp$), we have

$$\|U\| \equiv \|U'\| \equiv c(U') + d(U') \equiv c(U) + d(U) \pmod 2$$

by applying our induction hypothesis to U'. This completes the proof. $\square$

Proof of Lemma 2.2. (i) Since $f|_{M^2 \backslash U} : M^2 \backslash U \to \mathbf{R}^2$ is an immersion, $M^2 \backslash U$ can be given the orientation induced from a fixed orientation of $\mathbf{R}^2$. This orientation of $M^2 \backslash U$ satisfies the condition ($\sharp$).

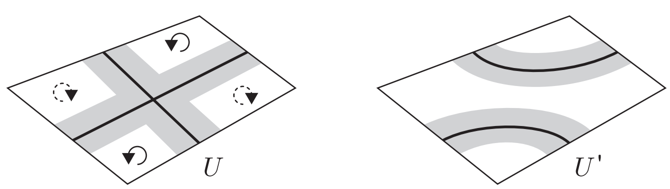

Fig. 1

(ii) Since $H_2(\mathbf{R}^3; \mathbf{Z}_2) = 0$, the connected components of $\mathbf{R}^3 \backslash f(M^2)$ admit a checkerboard coloring by black and white. We fix such a checkerboard coloring for $\mathbf{R}^3 \backslash f(M^2)$ and an orientation of $\mathbf{R}^3$. Then we orient $M^2 \backslash U$ so that the normal vector to each component of $f(M^2 \backslash U)$, which presents the orientation, points into the adjacent black region. This orientation of $M^2 \backslash U$ satisfies the condition ($\sharp$). $\square$

By Lemma 2.2, the following proposition relates the congruences (2) and (4), and (4) and (6).

Proposition 2.3. *For a closed surface M^2, let U and $\tilde{U}$ be unions of immersed circles in M^2. Suppose that $\tilde{U}$ satisfies the condition* ($\sharp$) *and that $U \cap \tilde{U}$ consists of transverse double points. Then we have*

$$\|U\| \equiv |U \cap \tilde{U}| \qquad (\mathrm{mod}\ 2).$$

Proof. We give an orientation to $M^2 \backslash \tilde{U}$ as in the condition ($\sharp$) such that the regions of $M^2 \backslash \tilde{U}$ have incompatible orientations on both sides of $\tilde{U}$. Let $L_1, \dots, L_m$ be the immersed circles of U. We take a regular neighborhood of L_i which is an annulus (or a Möbius band) immersed in M^2 provided that L_i is of $\mathscr{A}$-type (resp. of $\mathscr{M}$-type). See Figure 2, where the dotted curve represents L_i and the thick lines represent $\tilde{U}$ transverse to L_i. In the figure, we also indicate the fixed orientation of $M^2 \backslash \tilde{U}$ by shading the regions. Observing the orientation, we see that $|L_i \cap \tilde{U}| \equiv 1 \pmod 2$ if and only if L_i is of $\mathscr{M}$-type. Hence, we have

$$|U \cap \tilde{U}| = \sum_{i=1}^{m} |L_i \cap \tilde{U}| \equiv \|U\| \qquad (\mathrm{mod}\ 2). \qquad \square$$

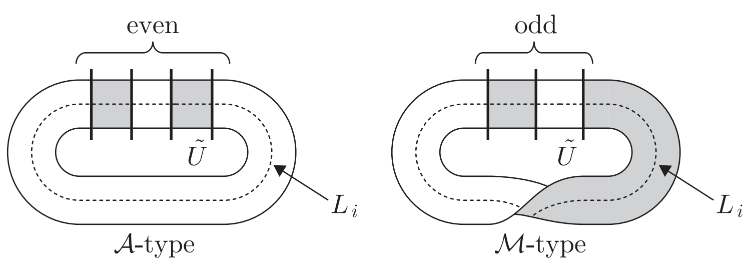

Fig. 2

By Lemmas 2.1, 2.2 and Proposition 2.3, we see that the congruence (4) holds for any stable maps $f : M^2 \to \mathbf{R}^2$ and $g : M^2 \to \mathbf{R}^3$ *without* the assumption that f and g are related by a projection $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ as $f = \pi \circ g$ as long as $S(f)$ and $\overline{I(g)}$ are in general position.

3. Critical points on a circle

In this section, we consider stable maps $f : M^2 \to \mathbf{R}^1$ and $g : M^2 \to \mathbf{R}^2$ such that $f = \pi \circ g$ for a generic projection $\pi : \mathbf{R}^2 \to \mathbf{R}^1$. In this case, f is a Morse function with the critical point set $S(f)$, and the singularity set $S(g)$ of g consists of fold singularities and cusp singularities as shown in Figure 3. Then we have $S(f) \subset S(g)$. The subset of $S(g)$ consisting of cusp singularities of g is denoted by $C(g)$. Note that $S(f) \subset S(g) \backslash C(g)$ by our genericity hypothesis.

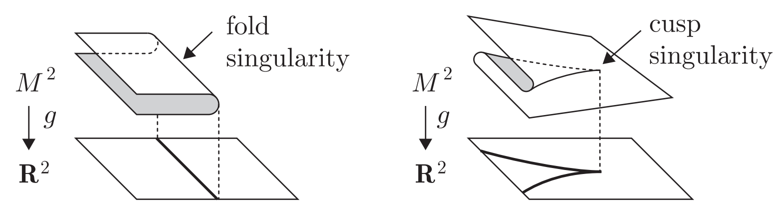

Fig. 3

For any stable map $f : M^2 \to \mathbf{R}^1$, there are a stable map $g : M^2 \to \mathbf{R}^2$ and a generic projection $\pi : \mathbf{R}^2 \to \mathbf{R}^1$ such that $f = \pi \circ g$. Conversely, for any stable map $g : M^2 \to \mathbf{R}^2$, there is a generic projection $\pi : \mathbf{R}^2 \to \mathbf{R}^1$ such that $f = \pi \circ g : M^2 \to \mathbf{R}^1$ is a stable map (cf. [**13**]). Hence, the following proposition connects the congruences (1) and (3).

PROPOSITION 3.1. *For a closed surface M^2, let $f : M^2 \to \mathbf{R}^1$ and $g : M^2 \to \mathbf{R}^2$ be stable maps such that $f = \pi \circ g$ for a generic projection $\pi : \mathbf{R}^2 \to \mathbf{R}^1$. Let $S(g) = \bigcup_{i=1}^{m} L_i$ be the union of embedded circles $L_1, \dots, L_m$ in M^2. For each L_i, we have*

$$|L_i \cap S(f)| \equiv |L_i \cap C(g)| \qquad (\mathrm{mod}\ 2).$$

Hence, we have $|S(f)| \equiv |C(g)|$ (mod 2).

PROOF. It is easy to see that the critical points of the map $\pi|_{g(S(g))} \circ g|_{S(g)} : S(g) \to \mathbf{R}^1$ form the union $S(f) \cup C(g)$. Figure 4 shows such an example, where $g(S(f))$ is marked by $\bullet$ and $g(C(g))$ is marked by $\circ$. Since the number of critical points of $\pi|_{g(S(g))} \circ g|_{S(g)}$ on each circle L_i is even, we have the first congruence. By taking the sum of these congruences for $L_1, \dots, L_m$, we have the second congruence. $\square$

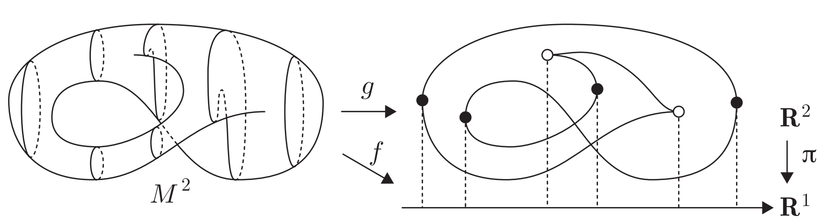

Fig. 4

4. Neighborhoods and colors of fold curves

In this section, we consider stable maps $f : M^2 \to \mathbf{R}^2$ and $g : M^2 \to \mathbf{R}^3$ such that $f = \pi \circ g$ for a projection $\pi : \mathbf{R}^3 \to \mathbf{R}^2$. We assume that the projection π is generic with respect to g in the sense of [**6**]. The singularity set $S(g)$ of g consists of Whitney umbrella points whose number is always even. Then we have $C(f) \subset S(f)$ and $S(g) \subset S(f)$. By our assumption, we have that $S(g) \subset S(f) \backslash C(f)$ (cf. [**6**]).

For any stable map $f : M^2 \to \mathbf{R}^2$, there are a stable map $g : M^2 \to \mathbf{R}^3$ and a generic projection $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ such that $f = \pi \circ g$. Conversely, for any stable map $g : M^2 \to \mathbf{R}^3$, there is a generic projection $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ such that $f = \pi \circ g : M^2 \to \mathbf{R}^2$ is a stable map (cf. [**13**]). For the lifting problem of f to an immersion g, refer to [**7**, **14**].

By dividing the circles of $S(f)$ into four classes with respect to the parity of the numbers of cusp singularities and Whitney umbrella points on the circles, we have the following proposition which connects the congruences (2) and (3).

Proposition 4.1. *For a closed surface M^2, let $f : M^2 \to \mathbf{R}^2$ and $g : M^2 \to \mathbf{R}^3$ be stable maps such that $f = \pi \circ g$ for a generic projection $\pi : \mathbf{R}^3 \to \mathbf{R}^2$. Let a, b, and c denote the numbers of L's such that L is an embedded circle of $S(f)$ satisfying the condition as shown in Table* 1. *Then we have*

$$\begin{cases} \|S(f)\| = a + b, \\ |C(f)| \equiv a + c \qquad \pmod 2, \\ |S(g)| \equiv b + c \equiv 0 \pmod 2. \end{cases}$$

Hence, we have $\|S(f)\| \equiv |C(f)| \pmod 2$ for any stable map $f : M^2 \to \mathbf{R}^2$.

We remark that the symbol "$*$" in Table 1 means there may exist such L's satisfying the condition but the number is not used in the proposition.

Table 1

<table>
<tr><td colspan="2" rowspan="2">$L \subset S(f)$</td><td colspan="2">$|L \cap S(g)|$</td></tr>
<tr><td>even</td><td>odd</td></tr>
<tr><td rowspan="2">$|L \cap C(f)|$</td><td>even</td><td>$*$</td><td>b</td></tr>
<tr><td>odd</td><td>a</td><td>c</td></tr>
</table>

Proof of Proposition 4.1. Recall that the singularity set $S(f)$ of f is a disjoint union of embedded circles in M^2. Let L be a connected component of $S(f)$. By observing a regular neighborhood of $g(L)$ in $g(M^2) \subset \mathbf{R}^3$, it is easy to see that L is of $\mathscr{M}$-type (that is, L has a Möbius band neighborhood in M^2) if and only if the sum of the numbers of cusp singularities and Whitney umbrella points on L is odd. See Figure 5. Hence, we have the first equality. The second and the third congruences follow from the definition immediately. Since $b \equiv c \pmod 2$, we have $\|S(f)\| \equiv a + b \equiv a + c \equiv |C(f)| \pmod 2$. □

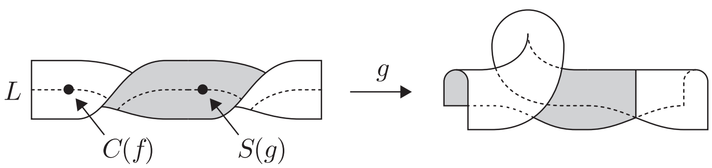

Fig. 5

Let $I(g) \subset M^2$ and $D(g) \subset \mathbf{R}^3$ denote the self-intersection set and the multiple point set of $g : M^2 \to \mathbf{R}^3$ respectively; that is, $I(g) = \{x \in M^2 \mid g^{-1}(g(x)) \neq \{x\}\}$ and $D(g) = g(I(g))$. Then $\pi(\overline{D(g)})$ is regarded as a union of immersed arcs and circles in $\mathbf{R}^2$ such that the ends of arc-components belong to $f(S(g))$. By our genericity hypothesis, we have that the intersections of $f(S(f))$ and $\pi(\overline{D(g)})$ are (i) transverse double points, (ii) points at which an arc-component of $\pi(\overline{D(g)})$ ends on $f(S(f))$ transversely, or (iii) points at which $\pi(\overline{D(g)})$ is tangent to $f(S(f))$ (cf. [**6**, **13**]). See Figure 6, where the thin and thick curves in $\mathbf{R}^2$ represent $\pi(\overline{D(g)})$ and $f(S(f))$ respectively.

We denote by $W(f,g)$ the set of points of type (iii) where $\pi(\overline{D(g)})$ is tangent to $f(S(f))$. The following proposition connects the congruences (3) and (5).

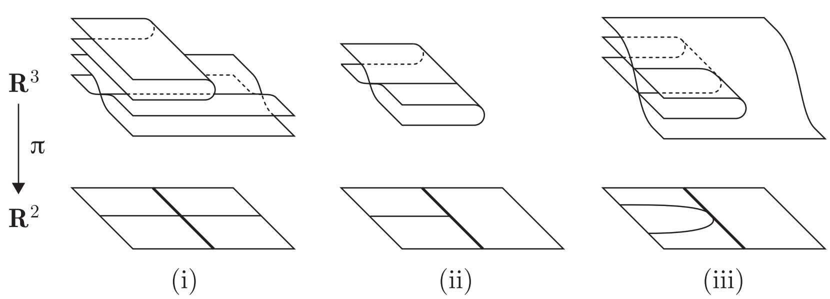

Fig. 6

PROPOSITION 4.2. *For a closed surface M^2, let $f : M^2 \to \mathbf{R}^2$ and $g : M^2 \to \mathbf{R}^3$ be stable maps such that $f = \pi \circ g$ for a generic projection*

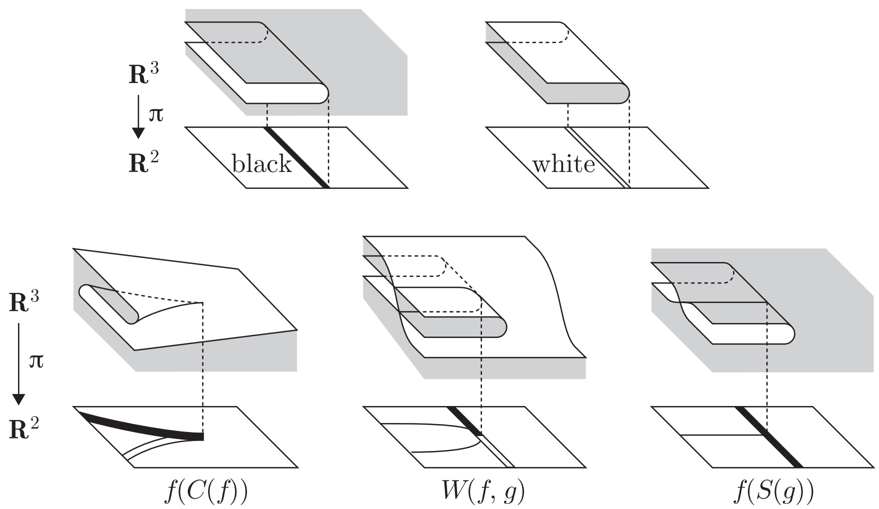

Fig. 7

$\pi : \mathbf{R}^3 \to \mathbf{R}^2$. *Let* $S(f) = \bigcup_{i=1}^m L_i$ *be the union of embedded circles* $L_1, \dots, L_m$ *in* M^2. *For each* L_i, *we have*

$$|L_i \cap C(f)| \equiv |f(L_i) \cap W(f,g)| \qquad (\text{mod } 2).$$

Hence, we have $|C(f)| \equiv |W(f,g)|$ (mod 2).

Proof. We fix a checkerboard coloring for the components of $\mathbf{R}^3 \backslash g(M^2)$ (see the proof of Proposition 2.2(ii)). Then each curve $f(L_i)$ outside the points in $f(C(f)) \cup W(f,g) \cup f(S(g))$ can be colored black and white as shown in the upper half of Figure 7. Such colors along $f(L_i)$ change alternately on both sides of the points in $f(C(f)) \cup W(f,g)$ and do not change at the points in $f(S(g))$, see the bottom of the figure. Hence the number of points in $f(C(f)) \cup W(f,g)$ on $f(L_i)$ is even, and we have the first congruence. By taking the sum of these congruences for $L_1, \dots, L_m$, we have $|C(f)| \equiv |W(f,g)|$ (mod 2). □

The closure $\overline{I(g)} \subset M^2$ is regarded as a union of immersed circles in M^2. By our genericity assumption, we have that $S(f)$ and $\overline{I(g)}$ intersect transversely on M^2 (cf. [**6**]). The following proposition connects the congruences (4) and (5).

Proposition 4.3. *For a closed surface* M^2, *let* $f : M^2 \to \mathbf{R}^2$ *and* $g : M^2 \to \mathbf{R}^3$ *be stable maps such that* $f = \pi \circ g$ *for a generic projection*

$\pi: \mathbf{R}^3 \to \mathbf{R}^2$. *Let* $S(f) = \bigcup_{i=1}^m L_i$ *be the union of embedded circles* $L_1, \dots, L_m$ *in* M^2. *For each* L_i, *we have*

$$|L_i \cap \overline{I(g)}| - |L_i \cap S(g)| = |f(L_i) \cap W(f,g)|.$$

Hence, we have $|S(f) \cap \overline{I(g)}| \equiv |W(f,g)| \pmod 2$.

PROOF. For each point $x \in L_i \cap \overline{I(g)}$, we have either that x belongs to $S(g)$ or that $f(x)$ belongs to $W(f,g)$. See Figure 6 again. Hence, we have the first equality. By taking the sum of these equalities for $L_1, \dots, L_m$, we have $|S(f) \cap \overline{I(g)}| - |S(g)| = |W(f,g)|$. Since $|S(g)|$ is always even, we have the second congruence. $\square$

The closure $\overline{D(g)}$ of the multiple point set $D(g) = g(I(g))$ of a stable map $g: M^2 \to \mathbf{R}^3$ is regarded as a union of immersed arcs and circles in $\mathbf{R}^3$ which are called *double curves*. An arc-component K of $\overline{D(g)}$ is said to be *of* $\mathscr{A}$-/$\mathscr{M}$-*type* if when we walk along K from one end outside the Whitney umbrella, we reach the other end outside/inside the umbrella respectively. A circle-component K of $\overline{D(g)}$ is said to be *of* $\mathscr{A}$-/$\mathscr{M}$-/$\mathscr{N}$-*type* if a neighborhood of K presents zero-/quarter-/half-twist modulo full-twist respectively. Equivalently, K is of $\mathscr{A}$-/$\mathscr{M}$-/$\mathscr{N}$-type if when we walk along K while keeping one of the four quadrants of $\mathbf{R}^3 \backslash g(M^2)$ around K, we return to the same/adjacent/diagonal quadrant compared with the starting quadrant respectively. See Figure 8. We denote by $|||\overline{D(g)}|||$ the total number of double curves (arc- and circle-components) of $\mathscr{M}$-type in $\overline{D(g)}$.

The following proposition connects the congruences (5) and (7).

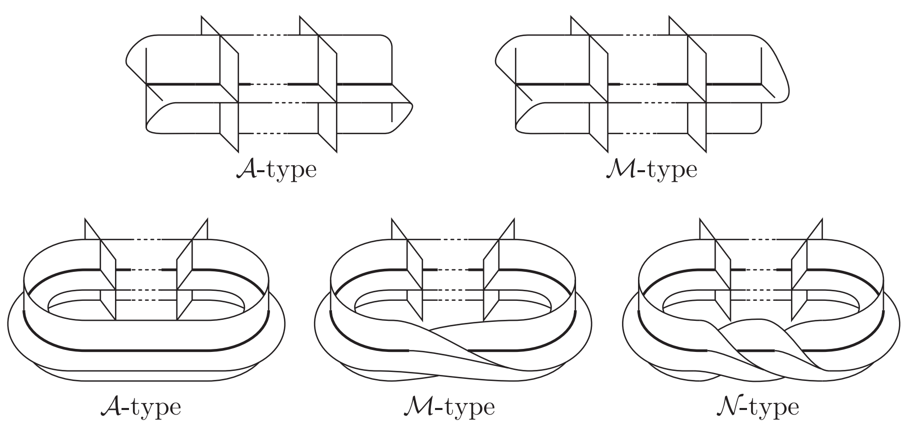

Fig. 8

Proposition 4.4. *For a closed surface M^2, let $f : M^2 \to \mathbf{R}^2$ and $g : M^2 \to \mathbf{R}^3$ be stable maps such that $f = \pi \circ g$ for a generic projection $\pi : \mathbf{R}^3 \to \mathbf{R}^2$. Let $\overline{D(g)} = \bigcup_{j=1}^n K_j$ be the union of double curves $K_1, \dots, K_n$ of $\overline{D(g)}$. For each K_j, we have $|\pi(K_j) \cap W(f,g)| \equiv 1 \pmod 2$ if and only if K_j is of $\mathscr{M}$-type. Hence, we have $|W(f,g)| \equiv |||\overline{D(g)}||| \pmod 2$.*

Proof. There are four quadrants of $\mathbf{R}^3 \backslash g(M^2)$ around a double curve K_j. We divide them into two types (type I and II) with respect to the projection direction of π as shown in the left half of Figure 9; that is, each quadrant of type I is mapped by π on both sides of the projected curve $\pi(K_j)$ in $\mathbf{R}^2$, and each quadrant of type II is mapped on one side of $\pi(K_j)$.

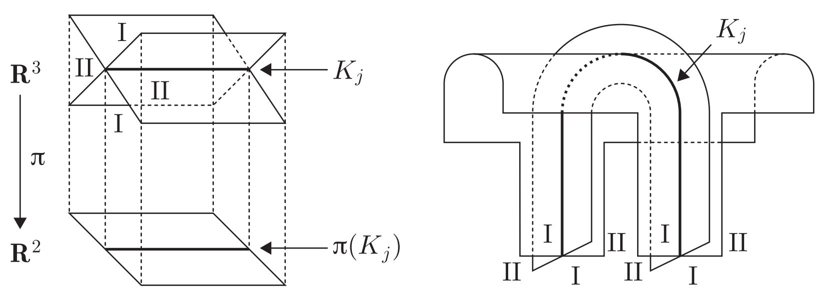

Fig. 9

We first consider the case that K_j is an arc-component of $\overline{D(g)}$. We start from one end of K_j outside the Whitney umbrella which is a quadrant of type I, see Figure 6(ii). On the way to the other end of K_j, when we pass through a point corresponding to $W(f,g)$, the type of the quadrant where we walk must change. See the right half of Figure 9, where the projection direction of π is taken to be perpendicular to the paper. Hence, the following three are equivalent to each other.

(i) K_j is of $\mathscr{A}$-type (or of $\mathscr{M}$-type).

(ii) We reach the other end of K_j in a quadrant of type I (resp. II).

(iii) There are even (resp. odd) number of points corresponding to $W(f,g)$ on K_j.

Similarly, in case that K_j is a circle-component of $\overline{D(g)}$, we see that K_j is of $\mathscr{A}$-/$\mathscr{N}$-type (or $\mathscr{M}$-type) if and only if there are an even (resp. odd) number of points corresponding to $\pi(K_j) \cap W(f,g)$. In both cases, we have $|\pi(K_j) \cap W(f,g)| \equiv 1 \pmod 2$ if and only if K_j is of $\mathscr{M}$-type. Hence, we have

$$|W(f,g)| = \sum_{j=1}^n |\pi(K_j) \cap W(f,g)| \equiv |||\overline{D(g)}||| \pmod 2. \qquad \square$$

We remark that for a circle-component K of $\overline{D(g)}$ we cannot distinguish between $\mathscr{A}$-type and $\mathscr{N}$-type only by the even number $|\pi(K) \cap W(f, g)|$ in general.

5. Multiple point set and triple points

In this section, we consider a stable map $f : M^2 \to \mathbf{R}^3$. The following proposition connects the congruences (6) and (7).

Proposition 5.1. *For a closed surface M^2, let $f : M^2 \to \mathbf{R}^3$ be a stable map. Let p, q, and r denote the numbers of K's such that K is a double curve (arc- or circle-component) of $\overline{D(f)}$ satisfying the condition as shown in Table 2. Then we have*

$$\begin{cases} \|\overline{I(f)}\| = p + q + 2r, \\ |||\overline{D(f)}||| = p + q. \end{cases}$$

Hence, we have $\|\overline{I(f)}\| \equiv |||\overline{D(f)}||| \pmod 2$.

Table 2

$K \subset \overline{D(f)}$	arc-component		circle-component		
	$\mathscr{A}$-type	$\mathscr{M}$-type	$\mathscr{A}$-type	$\mathscr{M}$-type	$\mathscr{N}$-type
number	$*$	p	$*$	q	r

We remark that the symbols "$*$" in Table 2 mean there may exist such K's satisfying the conditions but the numbers are not used in the proposition.

Proof of Proposition 5.1. An immersed circle of $\mathscr{M}$-type in $\overline{I(f)} \subset M^2$ is mapped into an arc-component of $\mathscr{M}$-type, a circle-component of $\mathscr{M}$-type, or a circle-component of $\mathscr{N}$-type in $\overline{D(f)} \subset \mathbf{R}^3$. Conversely, there is a unique circle of $\mathscr{M}$-type in $\overline{I(f)}$ which is mapped into a given arc- or circle-component of $\mathscr{M}$-type in $\overline{D(f)}$, and there is a pair of immersed circles of $\mathscr{M}$-type in $\overline{I(f)}$ which are mapped into the same circle-component of $\mathscr{N}$-type in $\overline{D(f)}$. Hence, we have the first equality. The second equality is exactly the definition of $|||\overline{D(f)}|||$. It follows from these equalities that $\|\overline{I(f)}\| \equiv |||\overline{D(f)}||| \pmod 2$ immediately. □

We denote by $T(f)$ the subset of $\overline{D(f)}$ consisting of triple points of f. Note that $T(f)$ is finite.

Recall that the singularity set $S(f)$ of a stable map $f : M^2 \to \mathbf{R}^3$ consists of Whitney umbrella points. Put $f(S(f)) = \{P_1, \dots, P_k\}$. We take a point

P_i' close to P_i missing $f(M^2)$ which is located outside the Whitney umbrella around P_i. See Figure 10. Szűcs [**17**] defines the *linking number* $l(f)$ of f to be the linking number of $\{P_1', \dots, P_k'\}$ and $f(M^2)$ modulo 2. Since k is even, $l(f)$ is well-defined. Equivalently, for a fixed checkerboard coloring for the regions of $\mathbf{R}^3 \backslash f(M^2)$, let β (or ω) denote the number of P_i''s such that P_i' is located in a black (resp. white) region. Then we have $l(f) \equiv \beta \equiv \omega \pmod 2$.

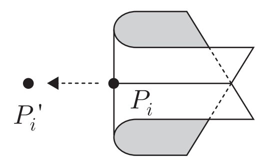

Fig. 10

The following proposition connects the congruences (7) and (8).

Proposition 5.2. *For a closed surface M^2, let $f : M^2 \to \mathbf{R}^3$ be a stable map. Let s, t, u and q denote the numbers of K's such that K is a double curve (arc- or circle-component) of $\overline{D(f)}$ satisfying the condition as shown in Table 3. Then we have*

$$\begin{cases} |T(f)| \equiv s + u + q & \pmod 2, \\ l(f) \equiv s + t & \pmod 2. \end{cases}$$

Hence, we have $\| \overline{D(f)} \| \equiv |T(f)| + l(f) \pmod 2$.

We remark that the symbols "$*$" in Table 3 mean there may exist such K's satisfying the conditions and that the symbols "—" mean there never exist such K's.

Proof of Proposition 5.2. We fix a checkerboard coloring for the regions of $\mathbf{R}^3 \backslash f(M^2)$. The colors around a double curve change alternately when we pass through a triple point. Hence, the number of triple points on

Table 3

$K \subset \overline{D(f)}$		arc-component		circle-component	
		$\mathscr{A}$-type	$\mathscr{M}$-type	$\mathscr{A}$-/$\mathscr{N}$-type	$\mathscr{M}$-type
$\lvert K \cap T(f) \rvert$	even	$*$	t	$*$	—
	odd	s	u	—	q

a circle-component K of $\overline{D(f)}$ is even (or odd) if and only if K is of $\mathscr{A}$-/$\mathscr{N}$-type (resp. of $\mathscr{M}$-type). See the upper half of Figure 11. Thus we have the first congruence.

Let P_i and P_j be the two boundary points of an arc-component K of $\overline{D(f)}$. Then the linking number of $\{P_i', P_j'\}$ and $f(M^2)$ is congruent to 1 modulo 2 if and only if K is of $\mathscr{A}$-type and $|K \cap T(f)|$ is odd, or K is of $\mathscr{M}$-type and $|K \cap T(f)|$ is even. See the lower half of Figure 11. Hence, we have the second congruence.

By taking the sum of these congruences, we have $|T(f)| + l(f) \equiv t + u + q = \|\|\overline{D(f)}\|\|$ (mod 2). $\square$

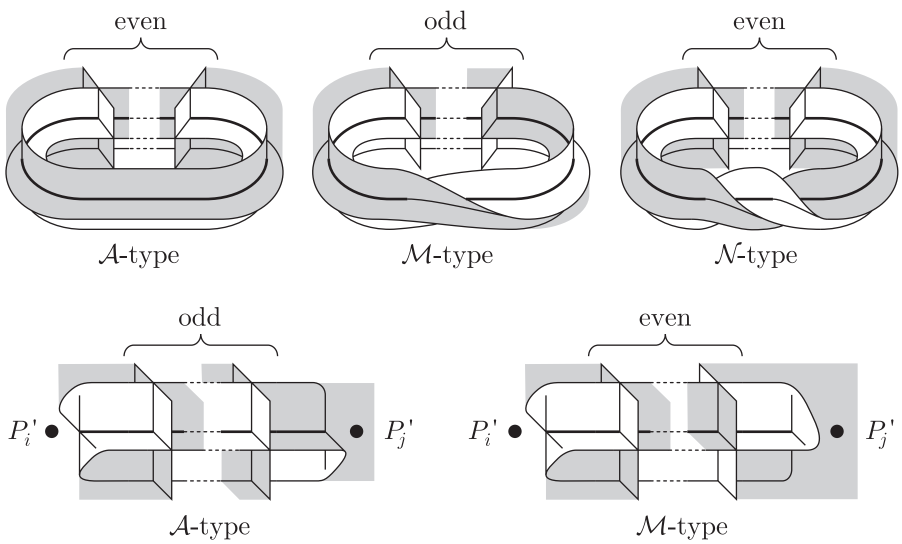

Fig. 11

6. Diagrams of immersed surfaces in 4-space

In this section, we consider stable maps $f : M^2 \to \mathbf{R}^3$ and $g : M^2 \to \mathbf{R}^4$ such that $f = \pi \circ g$ for a generic projection $\pi : \mathbf{R}^4 \to \mathbf{R}^3$. Then g is an immersion with transverse double points, and hence, $D(g) = g(I(g))$ consists of such double points. We denote by $e(g)$ the normal Euler number of g.

We use a (*broken surface*) *diagram* to describe an immersed surface $g(M^2) \subset \mathbf{R}^4$ (cf. [**5**]). Along each double curve of $\overline{D(f)} \subset \mathbf{R}^3$ we indicate the over-under information with respect to the projection direction of $\pi : \mathbf{R}^4 \to \mathbf{R}^3$ by breaking the 'lower' of the two intersecting sheets. See the first from the left in Figure 12. Any double point in $D(g)$ is mapped by π into $D(f) \backslash T(f)$

by our genericity assumption. The over-under relations along a double curve are opposite to each other on the two sides of a projected double point, and are the same on both sides of a triple point on the curve. See the second and the third figures from the left in Figure 12, where the dot • represents a projected double point in $\pi(D(g))$. We define the sign for each Whitney umbrella point in $f(S(f))$ as shown in the first and the second figures from the right (cf. [**5**, **16**]).

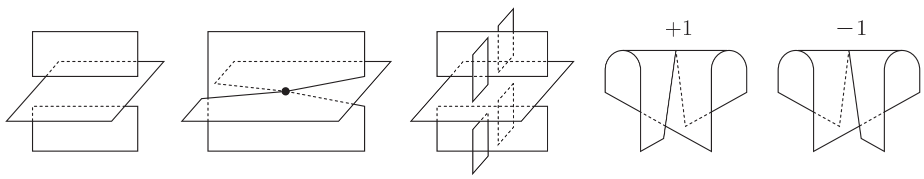

Fig. 12

For any stable map $f: M^2 \to \mathbf{R}^3$, there are a stable map $g: M^2 \to \mathbf{R}^4$ and a generic projection $\pi: \mathbf{R}^4 \to \mathbf{R}^3$ such that $f = \pi \circ g$. Conversely, for any stable map $g: M^2 \to \mathbf{R}^4$, there is a generic projection $\pi: \mathbf{R}^4 \to \mathbf{R}^3$ such that $f = \pi \circ g: M^2 \to \mathbf{R}^3$ is a stable map (cf. [**13**]). Hence, the following proposition connects the congruences (7) and (9). Refer to [**16**, **18**].

Proposition 6.1. *For a closed surface M^2, let $f: M^2 \to \mathbf{R}^3$ and $g: M^2 \to \mathbf{R}^4$ be stable maps such that $f = \pi \circ g$ for a generic projection $\pi: \mathbf{R}^4 \to \mathbf{R}^3$. Let x, y, z and q denote the numbers of K's such that K is a double curve (arc- or circle-component) of $\overline{D(f)}$ satisfying the condition as shown in Table* 4. *Then we have*

$$\begin{cases} |D(g)| \equiv x + z + q & \pmod 2, \\ \dfrac{e(g)}{2} \equiv x + y & \pmod 2. \end{cases}$$

Hence, we have $|||\overline{D(f)}||| \equiv |D(g)| + e(g)/2 \pmod 2$.

Table 4

$K \subset \overline{D(f)}$		arc-component		circle-component	
		$\mathscr{A}$-type	$\mathscr{M}$-type	$\mathscr{A}$-/$\mathscr{N}$-type	$\mathscr{M}$-type
$\lvert K \cap \pi(D(g))\rvert$	even	$*$	y	$*$	—
	odd	x	z	—	q

We remark that the symbols "$*$" in Table 4 mean there may exist such K's satisfying the conditions and that the symbols "—" mean there never exist such K's.

PROOF OF PROPOSITION 6.1. Recall that the over-under information along a double curve interchanges (or is invariant) when we pass through a point in $\pi(D(g))$ (resp. $T(f)$). Hence, a circle-component K of $\overline{D(f)}$ has an odd number of points in $\pi(D(g))$ if and only if K is of $\mathscr{M}$-type. See the top of Figure 13. Thus we have the first congruence. Similarly, the two ends of an arc-component K of $\overline{D(f)}$ have the same sign if and only if K is of $\mathscr{A}$-type and $|K \cap \pi(D(g))| \equiv 1 \pmod 2$, or K is of $\mathscr{M}$-type and $|K \cap \pi(D(g))| \equiv 0 \pmod 2$. See the bottoms of Figure 13, where we only show the case that the ends have positive signs. Since the normal Euler number $e(g)$ is equal to the sum of signs of all the points of $f(S(f))$ (cf. [**4**, **5**, **16**]), we have the second congruence. By taking the sum of these congruences, we have $|D(g)| + e(g)/2 \equiv y + z + q = \|\overline{D(f)}\| \pmod 2$. $\square$

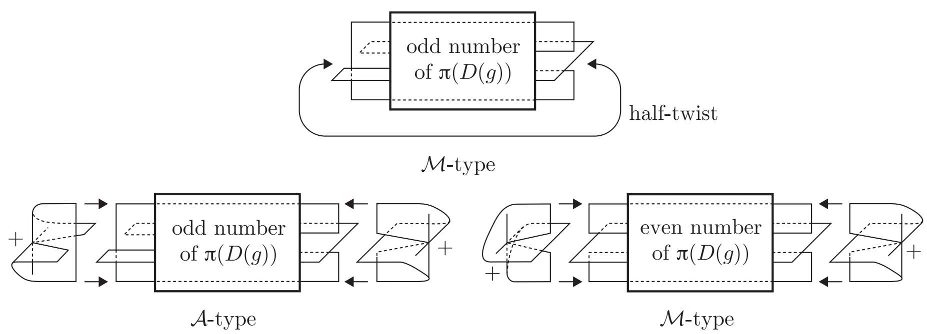

Fig. 13

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Research Institute for
Mathematical Sciences
Kyoto University
Kyoto, 606-8502, *Japan*
satoh@kurims.kyoto-u.ac.jp