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On the asymptotics of solutions for some Schrödinger equations with dissipative perturbations of rank one

Dedicated to Professor S. T. Kuroda for his 70's birthday

Mitsuteru KADOWAKI, Hideo NAKAZAWA and Kazuo WATANABE*

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ABSTRACT. The classification of solutions for some dissipative systems by the information of the spectrum is established. Its generator is non self-adjoint Schrödinger operator with rank one singular perturbation. For the proof, a generalized Parseval formula is constructed.

1. Introduction

Let $\mathscr{H}$ be a Hilbert space. We shall consider the relation between the asymptotics of solutions for the equation

$$(1.1) \qquad i\partial_t u = Hu, \qquad u|_{t=0} = f, \quad f \in \mathscr{H},$$

where H is some maximal dissipative operators in $\mathscr{H}$, and the spectral structure of the operator H. By the analogy of the general theory of ordinary differential equations with constant coefficients (cf. Coddington and Levinson [4]), we especially expect that $\sigma(H) \cap \mathbf{R}$ brings non-decay, i.e. $\lim_{t\to\infty} e^{-itH}f \neq 0$ and $\sigma(H) \cap \mathbf{C}_-$ brings decay, i.e. $\lim_{t\to\infty} e^{-itH}f = 0$, where $\sigma(H)$ and $\mathbf{C}_-$ denote the spectrum for operators H and the complex lower half-plain, respectively. We also know some examples which suggest the above situation. These are stated in Appendix C below.

In order to define an operator with singular perturbation, we prepare some notations. Let $H_0 = -d^2/dx^2$ in $\mathscr{H} = L^2(\mathbf{R})$. Then H_0 is a self-adjoint operator with the domain $\mathfrak{D}(H_0) = \mathscr{H}^2$, where

$$\mathscr{H}^s = \left\{ f \mid \|f\|_{\mathscr{H}^s}^2 = \int_{\mathbf{R}^1} (1+|k|^2)^s |(\mathscr{F}_0 f)(k)|^2 dk < \infty \right\} \qquad \text{for } s \in \mathbf{R}$$

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is the usual Sobolev space and $\mathscr{F}_0$ is the Fourier transform in the sense of tempered distribution. $\langle \cdot , \cdot \rangle$ is the inner product of $\mathscr{H}$ and we use the same symbol for the dual coupling of $\mathscr{H}^s$ and $\mathscr{H}^{-s}$, (in the case $s = 0$, i.e. $\mathscr{H}^0 = \mathscr{H}$, its norm is denoted by $\| \cdot \|$).

In this paper we shall deal with the Schrödinger equation (1.1) with

$$\mathscr{H} = L^2(\mathbf{R}^1), \qquad H = H_\alpha = -\frac{d^2}{dx^2} + \alpha \langle \cdot , \delta \rangle \delta,$$

where $\delta (\in \mathscr{H}^{-1})$ is the Dirac delta and $\alpha = \alpha_1 + i\alpha_2$ with $\alpha_1 \leqq 0$ and $\alpha_2 \leqq 0$.

We define the domain of H_α, $\mathfrak{D}(H_\alpha)$ as follows (see also section 2):

$$\mathfrak{D}(H_\alpha) = \{ U = u + aH_0(H_0^2 + 1)^{-1}\delta \mid u \in \mathscr{H}^2, a \in \mathbf{C},$$
$$\langle u, \delta \rangle = -a(\alpha^{-1} + \langle \delta, H_0(H_0^2+1)^{-1}\delta \rangle)\} \qquad (\alpha \neq 0).$$

Then it follows from Appendix A that H_α with $\alpha_2 < 0$ is maximal dissipative (the case $\alpha_2 = 0$ is self-adjoint), i.e., H_α with $\alpha_2 < 0$ generates a contraction semi-group $\{e^{-itH_\alpha}\}_{t \geqq 0}$ (the case $\alpha_2 = 0$ generates a unitary group $\{e^{-itH_{\alpha_1}}\}_{t \in \mathbf{R}}$). Keeping

$$H_0(H_0^2+1)^{-1}\delta = \frac{1}{2}((H_0 + i)^{-1}\delta + (H_0 - i)^{-1}\delta)$$

in mind, we can rewrite the domain of H_α as follows:

$$\mathfrak{D}(H_\alpha) = \{ U \in \mathscr{H}^1; U'(0+) - U'(0-) = \alpha U(0), \chi_{(0,\infty)}U'' + \chi_{(-\infty,0)}U'' \in \mathscr{H} \},$$

where χ_I is the characteristic function on I.

Our aim is to classify the asymtotics of the solutions of dissipative system (1.1) (see Corollary 1.5).

To state our results, we prepare several definitions.

$\sigma_p(A) = \{ z \in \sigma(A) \mid \text{there exists } f \neq 0 \text{ such that } Af = zf \}$: the set of point spectrum of A.

$\sigma_r(A) = \{ z \in \sigma(A) \mid z \notin \sigma_p(A), \text{the range space of } (A - z) \text{ is not dense in } X \}$: the set of residual spectrum of A.

$\sigma_c(A) = \{ z \in \sigma(A) \backslash (\sigma_p(A) \cup \sigma_r(A)) \}$: the set of continuous spectrum of A.

$\sigma_{ess}(A) = \{ z \in \sigma(A) \backslash \sigma_d(A) \}$: the set of essential spectrum of A,

where $\sigma_d(A) = \{ z \in \sigma(A) \mid z \text{ is an isolated eigenvalue with finite multiplicity} \}$ (the set of discrete spectrum).

The first result is the following theorem:

THEOREM 1.1 (Spectral structure of H_α). *Let* $\alpha = \alpha_1 + i\alpha_2$ *with* $\alpha_1 \leqq 0$, $\alpha_2 \leqq 0$. *Then the spectrum of* H_α *is given by*

$$\sigma(H_\alpha) = \begin{cases} [0, \infty) \cup \left\{-\frac{\alpha^2}{4}\right\} & (\alpha_1 < 0), \\ [0, \infty) \quad (\alpha_1 = 0). \end{cases}$$

Exact classification of the spectrum $\sigma(H_\alpha)$ *is*

$$\sigma_{ess}(H_\alpha) = \sigma_c(H_\alpha) = [0, \infty), \qquad \sigma_r(H_\alpha) = \emptyset$$

and

$$\sigma_p(H_\alpha) = \begin{cases} \sigma_d(H_\alpha) = \left\{-\frac{\alpha^2}{4}\right\} & (\alpha_1 < 0), \\ \emptyset \quad (\alpha_1 = 0). \end{cases}$$

Moreover the projection with respect to $-\frac{\alpha^2}{4}$ $(\alpha_1 \neq 0)$ *is given by*

$$P_{-\alpha^2/4} f = -\alpha/2 \langle f, e^{(\bar{\alpha}|\cdot|)/2} \rangle e^{(\alpha|x|)/2}.$$

REMARK 1.2. The condition $\alpha_1 < 0$ and $\alpha_2 < 0$ is necessary and sufficient for the existence of a point spectrum in the complex lower half-plane (cf. section 2).

To state main theorem (Theorem 1.3) we note that

$$\text{Ker } P_{-\alpha^2/4} + \text{Range } P_{-\alpha^2/4} = \mathscr{H}$$

and

$$\text{Ker } P_{-\alpha^2/4} \cap \text{Range } P_{-\alpha^2/4} = \{0\}$$

(cf. Reed-Simon [21], Theorem XII.5), where

$$\text{Ker } A = \{f \in \mathfrak{D}(A) \mid Af = 0\}, \qquad \text{Range } A = \{Af \mid f \in \mathfrak{D}(A)\}$$

for an operator A. Thus for each $f \in \mathscr{H}$, we obtain a unique decomposition:

$$f = f_s + f_d, \tag{1.2}$$

where

$$f_s \equiv f - P_{-\alpha^2/4} f \in \text{Ker } P_{-\alpha^2/4}$$

and

$$f_d \equiv P_{-\alpha^2/4} f \in \text{Range } P_{-\alpha^2/4}.$$

Note that $f \in \text{Ker } P_{-\alpha^2/4}$ if and only if

$$\langle f, e^{(\bar{\alpha}|\cdot|)/2} \rangle = 0. \tag{1.3}$$

As is explained later, we can define the wave operator $W(\alpha)$

$$W(\alpha) = \text{s-}\lim_{t\to+\infty} e^{itH_0}e^{-itH_\alpha}$$

as a non-trivial operator from $\mathscr{H}$ to $\mathscr{H}$ (see Proposition 3.1), where $\alpha_1 \leqq 0$ and $\alpha_2 < 0$.

The existence of $W(\alpha)$ implies that the asymptotics of solutions for (1.1) with $\alpha_1 \leqq 0$ and $\alpha_2 < 0$ is scattering (asymptotic free, non-decay) or decay.

We have the following main theorem:

Theorem 1.3.

(i) *Assume that* $\alpha_1 < 0$ *and* $\alpha_2 < 0$. *Then*

$$\text{Ker } W(\alpha) = \text{Range } P_{-\alpha^2/4}.$$

(ii) *Assume that* $\alpha_1 = 0$ *and* $\alpha_2 < 0$. *Then*

$$\text{Ker } W(i\alpha_2) = \{0\}.$$

Remark 1.4.

(1) For the case $\alpha_1 < 0$ and $\alpha_2 < 0$, it is easy to show that $f_s = 0$ is a sufficient condition for $\lim_{t\to\infty}\|e^{-itH_\alpha}f\| = 0$ (see Corollary 3.2). However it is not clear that $f_s = 0$ is a necessary condition for $\lim_{t\to\infty}\|e^{-itH_\alpha}f\| = 0$. In order to show the necessity, we require a generalized Parseval formula (see Lemma 4.1).

(2) For the case $\alpha_1 = 0$ and $\alpha_2 < 0$, the situation changes, i.e., the point $\alpha_2^2/4$ is not an eigenvalue. According to Reed-Simon [21], XII.6, we may call this point *resonance*. Therefore we must analyze this effect to construct the generalized Parseval formula (see Proposition 5.1).

(3) For the case $\alpha_1 > 0$ and $\alpha_2 < 0$, there are no eigenvalues and no resonance. So we can obtain $\text{Ker } W(\alpha) = \{0\}$. Since the proof is done similarly with Theorem 1.3 (i), we omit the proof.

Corollary 1.5 (The classification of asymptotics by the initial data).

(i) *Assume that* α *is the same as in Theorem* 1.3 (i). *Then for each* $f \in \mathscr{H}$ *decomposed as in* (1.2), *we have the following characterization:*

$$\text{(S)} \qquad f_s \neq 0 \qquad \textit{if and only if} \qquad \begin{cases} \lim_{t\to\infty}\|e^{-itH_\alpha}f - e^{-itH_0}W(\alpha)f\| = 0, \\ W(\alpha)f \neq 0 \end{cases}$$

and

$$\text{(D)} \quad f_s = 0 \qquad \textit{if and only if} \qquad \lim_{t\to\infty}\|e^{-itH_\alpha}f\| = 0 \quad (e^{-itH_\alpha}f = e^{i(\alpha^2/4)t}f_d).$$

(ii) *Assume that* α *is the same as in Theorem* 1.3 (ii). *Then we have*

$$f \in \mathscr{H} \text{ and } f \neq 0 \quad \text{if and only if} \quad \begin{cases} \lim_{t\to\infty} \|e^{-itH_{i\alpha_2}} f - e^{-itH_0} W(i\alpha_2) f\| = 0, \\ W(i\alpha_2) f \neq 0. \end{cases}$$

In the case $\alpha_2 = 0$, the asymptotics of the solutions of (1.1) is well-known since H_{α_1} is self-adjoint operators (cf. Enss [6], Kuroda [13] and Reed-Simon [21]). Indeed, let $E_{\alpha_1}(\lambda)$ be the spectral family of H_{α_1}. Then we have by Theorem 1.1 and spectral theory

$$\text{Range } E_{\alpha_1}((0,\infty)) \oplus \text{Range } E_{\alpha_1}\left(\left\{-\frac{\alpha_1^2}{4}\right\}\right) = \mathscr{H}.$$

Furthermore scattering theory implies that

$$f \in \text{Range } E_{\alpha_1}((0,\infty)) \qquad \text{if and only if}$$
$$\lim_{t\to\pm\infty} \|e^{-itH_{\alpha_1}} f - e^{-itH_0} \tilde{W}_{\pm}(\alpha_1) f\| = 0$$

and

$$f \in \text{Range } E_{\alpha_1}\left(\left\{-\frac{\alpha_1^2}{4}\right\}\right) \qquad \text{if and only if} \qquad e^{-itH_\alpha} f = e^{i(\alpha_1^2/4)t} f,$$

where

$$\tilde{W}_{\pm}(\alpha_1) = \text{s-}\lim_{t\to\pm\infty} e^{itH_0} e^{-itH_{\alpha_1}} E_{\alpha_1}((0,\infty)).$$

Remark 1.6. It is well-known that the existence of $\tilde{W}_{\pm}(\alpha_1)$ is equivalent to the asymptotic completeness for

$$\tilde{\Omega}_{\pm}(\alpha_1) = \text{s-}\lim_{t\to\pm\infty} e^{itH_{\alpha_1}} e^{-itH_0}.$$

Corollary 1.5 asserts that it is possible to construct a formulation for dissipative systems (1.1) which is similar to the self-adjoint case. There are many works studying the asymptotics of solutions for dissipative systems. However it seems that there are no works dealing with a classification like Corollary 1.5.

We mention related works. Schrödinger operators with rank one perturbation are known as point interaction (Albeverio, Gesztesy, Høegh-Krohn and Holden [2], Albeverio and Kurasov [3]). Self-adjoint realizations are considered by Watanabe [23], Kurasov and Watanabe [11], [12]. Kato [10] deals with scattering theory for a perturbation of rank one. Non self-adjoint scattering theory was investigated by Kato [9], in which he developed the smooth perturbation theory. Pavlov [18] studied the spectral properties for the one-dimensional non self-adjoint Schrödinger operator, in which he derived "*generalized*" Parseval formula. Using Kato's theory, scattering theory for wave equations with dissipative terms is considered by Mochizuki [14], [15], Nakazawa [17] and Kadowaki [7]. Kadowaki [8] used the Enss method to prove the existence of the wave operators for some dissipative systems. Adamyan and Neidhardt [1] treated the non self-adjoint Friedrichs model and studied the absolute continuity of the spectrum for it.

The present paper is organized as follows.

In section 2, we prove Theorem 1.1. The existence of the wave operator is shown in section 3. In section 4 and 5, we prove Theorem 1.3 (i) and Theorem 1.3 (ii), respectively. In their proof, we construct a generalized Parseval formula (Lemma 4.1 and Proposition 5.1, respectively). In Appendix A, we show that H_α with $\alpha_2 < 0$ is maximal dissipative (H_{α_1} is self-adjoint). In Appendix B, we mention Kadowaki's results [8] which supplement subject in section 3. In Appendix C, we state two examples by which our work is motivated.

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2. Proof of Theorem 1.1

We consider the operator

$$\tilde{H}_\alpha = H_0 + \alpha\langle\cdot\,,\varphi\rangle\varphi$$

with the domain

$$\mathfrak{D}(\tilde{H}_\alpha) = \begin{cases} \{U = u + aH_0(H_0^2+1)^{-1}\varphi \mid u \in \mathscr{H}^2,\ a \in \mathbf{C}, \\ \quad \langle u, \varphi\rangle = -a(\alpha^{-1} + \langle\varphi, H_0(H_0^2+1)^{-1}\varphi\rangle)\} \quad (\alpha \neq 0), \\ \mathscr{H}^2 \quad (\alpha = 0), \end{cases}$$

where $\alpha \in \mathbf{C}$, $\varphi \in \mathscr{H}^{-1}\backslash\mathscr{H}$.

For $\alpha \neq 0$, $U \in \mathfrak{D}(\tilde{H}_\alpha)$ means $\tilde{H}_\alpha U \in \mathscr{H}$ for any $U \in \mathfrak{D}(\tilde{H}_\alpha)$ since

$$\tilde{H}_\alpha U = H_0 u - a(H_0^2+1)^{-1}\varphi. \tag{2.1}$$

Put $\alpha = \alpha_1 + i\alpha_2$ with $\alpha_1 \leqq 0$ and $\alpha_2 \leqq 0$. Then $\tilde{H}_\alpha$ is dissipative, i.e., $\mathrm{Im}\langle\tilde{H}_\alpha U, U\rangle \leqq 0$ for $U \in \mathfrak{D}(\tilde{H}_\alpha)$, and $\tilde{H}_{\bar{\alpha}}$ is accretive, i.e., $\mathrm{Im}\langle\tilde{H}_{\bar{\alpha}} V, V\rangle \geqq 0$ for $V \in \mathfrak{D}(\tilde{H}_{\bar{\alpha}})$. Moreover we have the following properties:

(i) $\tilde{H}_\alpha$ is a maximal dissipative operator,
(ii) $\tilde{H}_{\bar{\alpha}}$ is a maximal accretive operator,
(iii) $\tilde{H}_\alpha^* = \tilde{H}_{\bar{\alpha}}$.

These will be proven in Appendix A.

Especially, H_α denotes the operator $\tilde{H}_\alpha$ defined by choosing $\varphi = \delta$ (Dirac delta) $\in \mathscr{H}^s$ $(s < -1/2)$. We also denote by $\tilde{R}_\alpha(z)$ (resp. $R_\alpha(z)$) the resolvent $(\tilde{H}_\alpha - z)^{-1}$ (resp. $(H_\alpha - z)^{-1}$) of $\tilde{H}_\alpha$ (resp. H_α) for $z \in \rho(\tilde{H}_\alpha)$ (resp. $z \in \rho(H_\alpha)$), where $\rho(A)$ is the resolvent set of a closed operator A in $\mathscr{H}$.

The proof of Theorem 1.1 can be divided into several steps. First of all, consider the representation of the resolvent of H_α.

Lemma 2.1. *Assume that* $\alpha = \alpha_1 + i\alpha_2$ *with* $\alpha_1 \leqq 0$ *and* $\alpha_2 \leqq 0$. *Then we have for any* $f \in \mathscr{H}$,

$$\tilde{R}_\alpha(z)f = R_0(z)f - \alpha\{1 + \alpha\langle R_0(z)\varphi, \varphi\rangle\}^{-1}\langle R_0(z)f, \varphi\rangle R_0(z)\varphi$$

for any $z \in \rho(H_0) \cap \{z \in \mathbf{C} \mid 1 + \alpha\langle R_0(z)\varphi, \varphi\rangle \neq 0\}$.

Proof. The above equality can be obtained by using the arguments similar to these used by S. Albeverio and P. Kurasov [3], Theorem 1.1.1. □

Lemma 2.2. *Suppose* $\varphi = \delta$ *in addition to the assumption of Lemma* 2.1. *Then we obtain for any* $f \in \mathscr{H}$,

$$(R_\alpha(z)f)(x) = (R_0(z)f)(x) + \int_{\mathbf{R}^1} K(x, y; z) f(y) dy, \tag{2.2}$$

$$\text{where } K(x, y; z) = -\frac{\alpha}{2i\sqrt{z}(2i\sqrt{z} - \alpha)} e^{i\sqrt{z}(|x|+|y|)} \in L^2(\mathbf{R}_x^1 \times \mathbf{R}_y^1)$$

with $\operatorname{Im}\sqrt{z} > 0$, *where*

$$z \in \rho(H_\alpha) = \begin{cases} \mathbf{C}\backslash([0, \infty) \cup \{-\alpha^2/4\}) & (\alpha_1 < 0), \\ \mathbf{C}\backslash[0, \infty) & (\alpha_1 = 0). \end{cases}$$

Proof. The equality (2.2) is easily obtained by using the explicit formula for the free resolvent $R_0(z)$. □

The equality (2.2) implies the following corollary.

Corollary 2.3. *Under the same assumptions as in Lemma* 2.2, *we have* $\sigma_{ess}(H_\alpha) = [0, \infty)$.

Next we shall show that $-\alpha^2/4$ $(\alpha_1 < 0)$ is the eigenvalue of H_α.

LEMMA 2.4. *Assume* $\alpha_1 < 0$ *and* $\alpha_2 \leqq 0$. *Then* $-\alpha^2/4$ *is the eigenvalue of* H_α. *Moreover the projection with respect to* $-\frac{\alpha^2}{4}$, $P_{-\alpha^2/4}$, *is given by*

$$(P_{-\alpha^2/4}f)(x) = -\alpha/2\langle f, e^{(\bar{\alpha}|\cdot|)/2}\rangle e^{(\alpha|x|)/2}. \tag{2.3}$$

PROOF. Since $-\alpha^2/4$ is an isolated point of $\sigma(H_\alpha)$, Reed-Simon [21], Theorem XII.5 and the equality (2.2) give for any f and $g \in \mathscr{H}$,

$$\begin{aligned}\langle P_{-\alpha^2/4}f, g\rangle = -(2\pi i)^{-1}\Bigg\{ &\int_C \langle R_0(z)f, g\rangle dz \\ &+ \int_C \frac{\alpha}{2i\sqrt{z}(2i\sqrt{z}-\alpha)}\left(\int_{\mathbf{R}_x^1} e^{i\sqrt{z}|x|}f(x)dx\right)\left(\int_{\mathbf{R}_y^1} e^{i\sqrt{z}|y|}\overline{g(y)}dy\right)dz\Bigg\},\end{aligned} \tag{2.4}$$

where C is a closed curve enclosed $-\alpha^2/4$ in $\mathbf{C}_-$ and $\operatorname{Im}\sqrt{z} > 0$. Firstly we find

$$\int_C \langle R_0(z)f, g\rangle dz = 0.$$

On the other hand, since the point $z = -\alpha^2/4$ is the simple pole, the residual theorem gives

$$\begin{aligned}&\int_C \frac{\alpha}{2i\sqrt{z}(2i\sqrt{z}-\alpha)}\left(\int_{\mathbf{R}_x^1} e^{i\sqrt{z}|x|}f(x)dx\right)\left(\int_{\mathbf{R}_y^1} e^{i\sqrt{z}|y|}\overline{g(y)}dy\right)dz \\ &= \frac{-\alpha}{2}\left(\int_{\mathbf{R}_x^1} e^{(\alpha|x|)/2}f(x)dx\right)\left(\int_{\mathbf{R}_y^1} e^{(\alpha|y|)/2}\overline{g(y)}dy\right) \\ &= \frac{-\alpha}{2}\langle f, e^{(\bar{\alpha}|\cdot|)/2}\rangle\langle e^{(\alpha|\cdot|)/2}, g\rangle.\end{aligned} \tag{2.5}$$

Therefore we find $-\alpha^2/4 \in \sigma_p(H_\alpha)$ and (2.3) from (2.4) and (2.5). □

LEMMA 2.5. *Under the same assumption as in Lemma* 2.1, *we have for* $U, V \in \mathfrak{D}(\tilde{H}_\alpha)$,

$$\langle \tilde{H}_\alpha U, V\rangle - \langle U, \tilde{H}_\alpha V\rangle = \frac{2i \operatorname{Im}\alpha}{|\alpha|^2}a\bar{b}, \tag{2.6}$$

where

$$a = -\frac{\langle u, \varphi\rangle}{\alpha^{-1} + \langle \varphi, H_0(H_0^2+1)^{-1}\varphi\rangle} \qquad \textit{and} \qquad b = -\frac{\langle v, \varphi\rangle}{\alpha^{-1} + \langle \varphi, H_0(H_0^2+1)^{-1}\varphi\rangle}$$

for some $u, v \in \mathscr{H}^2$.

Proof. Using self-adjointness of H_0 and (2.1), we find the left hand side of (2.3) becomes

$$\langle u, bH_0^2(H_0^2+1)^{-1}\varphi\rangle - \langle a(H_0^2+1)^{-1}\varphi, v\rangle + \langle u, b(H_0^2+1)^{-1}\varphi\rangle - \langle aH_0^2(H_0^2+1)^{-1}\varphi, v\rangle.$$

Combining the first and the third terms, and the second and the fourth terms, respectively, we have

$$\bar{b}\langle u, \varphi\rangle - a\langle \varphi, v\rangle.$$

Noting the relation on $\langle u, \varphi\rangle$ in $\mathfrak{D}(\tilde{H}_\alpha)$, we easily obtain the desired results. □

Lemma 2.6. *Assume* $\alpha_1 \leqq 0$ *and* $\alpha_2 < 0$. *Then we find*

$$\sigma_p(\tilde{H}_\alpha) \cap \mathbf{R} = \emptyset, \tag{2.7}$$

$$\sigma_p(\tilde{H}_{\bar{\alpha}}) \cap \mathbf{R} = \emptyset, \tag{2.8}$$

$$\sigma_r(\tilde{H}_\alpha) \cap \mathbf{R} = \emptyset. \tag{2.9}$$

Proof. Firstly we shall show (2.7). Assume that U_λ is the eigenfunction of the operator H_α with respect to the $\lambda \in \sigma_p(H_\alpha) \cap \mathbf{R}$. Taking $U = V = U_\lambda$ in (2.6) of Lemma 2.5, we have

$$0 = \lambda\|U\|^2 - \lambda\|U\|^2 = \langle \lambda U, U\rangle - \langle U, \lambda U\rangle = \frac{2i \operatorname{Im} \alpha}{|\alpha|^2}|a|^2.$$

Hence it follows $a = 0$. Therefore, we have $U = u$ (see the definition of $\mathfrak{D}(\tilde{H}_\alpha)$) and $\tilde{H}_\alpha U = H_0 u$ by (2.1). It then follows that

$$H_0 u = \tilde{H}_\alpha U = \lambda U = \lambda u.$$

This means $\lambda \in \sigma_p(H_0) \cap \mathbf{R}$, which is the contradiction.

The similar argument is applicable to show (2.8).

Finally, we shall show (2.9). Assume $\lambda \in \sigma_r(\tilde{H}_\alpha) \cap \mathbf{R}$. Then $\lambda = \bar{\lambda} \in \sigma_p(\tilde{H}_\alpha^*) = \sigma_p(\tilde{H}_{\bar{\alpha}})$ and this contradicts with (2.8). □

Since the spectral theory for the self-adjoint operator implies $\sigma_r(H_{\alpha_1}) = \emptyset$, the proof of Theorem 1.1 is complete.

In the rest of this section, we give the principle of limiting absorption which follows from (2.2). Let

$$L^{2,s} = \left\{ f \mid \|f\|_{L^{2,s}}^2 = \int_{\mathbf{R}^1} (1+|x|^2)^s |f(x)|^2 dx < \infty \right\} \qquad \text{for } s \in \mathbf{R}$$

be the weighted L^2 space.

PROPOSITION 2.7 (The principle of limiting absorption for H_α). *Let $s > 1/2$. Then there exist the limits*

$$\lim_{\varepsilon \downarrow 0} R_\alpha(\lambda \pm i\varepsilon)$$

in the uniform operator topology of $B(L^{2,s}, L^{2,-s})$, i.e., we have

$$R_\alpha(\lambda \pm i0) = \lim_{\varepsilon \downarrow 0} R_\alpha(\lambda \pm i\varepsilon) \qquad (\alpha_1 < 0, \alpha_2 \leqq 0),$$

$$R_{i\alpha_2}(\lambda + i0) = \lim_{\varepsilon \downarrow 0} R_{i\alpha_2}(\lambda + i\varepsilon) \qquad (\alpha_2 < 0)$$

for every $\lambda \in (0, \infty)$. In particular,

$$R_{i\alpha_2}(\lambda - i0) = \lim_{\varepsilon \downarrow 0} R_{i\alpha_2}(\lambda - i\varepsilon) \qquad (\alpha_2 < 0)$$

exists for every $\lambda \in (0, \infty) \Big\backslash \left\{\frac{\alpha_2^2}{4}\right\}$.

PROOF. It is well-known that for every $\lambda > 0$, the existence of the limits

$$\lim_{\varepsilon \downarrow 0} R_0(\lambda \pm i\varepsilon)(= R_0(\lambda \pm i0))$$

in the uniform operator topology of $B(L^{2,s}, L^{2,-s})$. Moreover it is easy to see that for every $\lambda > 0$,

$$\lim_{\pm\varepsilon \downarrow 0} \int_{-\infty}^{\infty} (1 + |x|^2)^{-s} |e^{i\sqrt{\lambda + i\varepsilon}|x|} - e^{\pm i\sqrt{\lambda}|x|}|^2 dx = 0.$$

Therefore we find that the following limits exist in the uniform operator topology of $B(L^{2,s}, L^{2,-s})$,

$$\lim_{\varepsilon \downarrow 0} R_\alpha(\lambda \pm i\varepsilon) = R_0(\lambda \pm i0) \mp \frac{\alpha e^{\pm i\sqrt{\lambda}|\cdot|}}{2i\sqrt{\lambda}(\pm 2i\sqrt{\lambda} - \alpha)} \langle e^{\pm i\sqrt{\lambda}|\cdot|}, \bar{\cdot} \rangle$$

for α and λ as in the conclusion of Proposition 2.7. □

3. Existence of wave operators

In this section we show the existence of wave operator and construct a generalized Fourier transform for H_α (Propositions 3.1 and 3.7).

PROPOSITION 3.1 (Existence of the wave operator). *Let $\alpha = \alpha_1 + i\alpha_2$ with $\alpha_1 \leqq 0$, $\alpha_2 < 0$. Then there exists*

$$W(\alpha) = \text{s-}\lim_{t \to +\infty} e^{itH_0} e^{-itH_\alpha}$$

as non-trivial operator from $\mathscr{H}$ to $\mathscr{H}$.

From Theorem 1.1 and Proposition 3.1, we find the following Corollary:

Corollary 3.2. *Let* $\alpha = \alpha_1 + i\alpha_2$ *with* $\alpha_1 < 0$ *and* $\alpha_2 < 0$. *Then we have*

$$\text{Range } P_{-\alpha^2/4} \subset \text{Ker } W(\alpha) = \left\{ f \mid \lim_{t\to+\infty} \|e^{-itH_\alpha}f\| = 0 \right\}.$$

On the other hand, we can show the existence of the following wave operators by Cook-Kuroda method;

$$\Omega_-(\alpha) = \text{s-}\lim_{t\to+\infty} e^{-itH_\alpha}e^{itH_0},$$

$$\Omega_+(\alpha) = \text{s-}\lim_{t\to+\infty} e^{itH_{\bar\alpha}}e^{-itH_0}$$

in $\mathscr{H}$. So, we can define the scattering operator $S(\alpha)$ by

$$S(\alpha) = W(\alpha)\Omega_-(\alpha).$$

We show Proposition 3.1 by the argument in Kadowaki [8] (see Appendix B) which is due to Enss method (c.f. Enss [6], Simon [22], Kuroda [13] and Perry [19], [20]).

Remark 3.3. To show Proposition 3.1 we may apply Simon [22], Theorem 9.3 with simple modifications (compare our condition on perturbation with that of [22], Theorem 9.3).

According to Theorem B in Appendix B, Proposition 3.1 follows from lemmas below (Lemmas 3.4–3.6) and $\sigma_p(H_\alpha) \cap \mathbf{R} = \emptyset$ (see Theorem 1.1).

The following lemma is well known.

Lemma 3.4. (*cf.* (*B*1) *in* Appendix B) $\sigma(H_0) = \sigma_{ac}(H_0) = [0, \infty)$.

Lemma 3.5. (*cf.* (*B*2) *in* Appendix B) $K = (H_\alpha - i)^{-1} - (H_0 - i)^{-1}$ *is a compact operator in* $\mathscr{H}$.

Proof. Lemma 2.1 implies that K is the Hilbert-Schmidt. Thus K is compact. □

Lemma 3.6. (*cf.* (*B*3) *in* Appendix B) *Let* P_+ *and* P_- *be the positive and negative spectral projections for the generator of dilation* $\frac{1}{2i}\left(x\frac{d}{dx} + \frac{d}{dx}x\right)$, *respectively. Then we have*

$$\int_0^\infty \|Ke^{-itH_0}\psi(H_0)P_+\|_{B(\mathscr{H},\mathscr{H})}dt < \infty, \tag{3.1}$$

$$\int_0^\infty \|K^*e^{-itH_0}\psi(H_0)P_+\|_{B(\mathscr{H},\mathscr{H})}dt < \infty, \tag{3.2}$$

$$\int_0^\infty \|K^* e^{itH_0}\psi(H_0)P_-\|_{B(\mathscr{H},\mathscr{H})}dt < \infty, \tag{3.3}$$

$$w\text{-}\lim_{t\to+\infty} e^{itH_0}\psi(H_0)P_- f_t = 0 \tag{3.4}$$

for each $\psi \in C_0^\infty((0,\infty))$ *and* $\{f_t\}_{t\in\mathbf{R}}$ *satisfying* $\sup_{t\in\mathbf{R}}\|f_t\| < \infty$, *where* $\|\cdot\|_{B(\mathscr{H},\mathscr{H})}$ *is the operator norm for bounded operators in* $\mathscr{H}$.

PROOF. In the proof we use the Mellin transforms estimates (Perry [19], Lemma 1). First we show (3.1)–(3.3).

For $f \in \mathscr{H}$, we find by Lemma 2.1

$$\|Ke^{-itH_0}\psi(H_0)P_+ f\| = |C_\alpha|\,|\langle e^{-itH_0}\psi(H_0)P_+ f, \overline{e^{i\sqrt{i}|\cdot|}}\rangle|\,\|e^{i\sqrt{i}|\cdot|}\|,$$

$$\|K^*e^{-itH_0}\psi(H_0)P_+ f\| = |\overline{C_\alpha}|\,|\langle e^{-itH_0}\psi(H_0)P_+ f, e^{i\sqrt{i}|\cdot|}\rangle|\,\|\overline{e^{i\sqrt{i}|\cdot|}}\|,$$

$$\|K^*e^{itH_0}\psi(H_0)P_- f\| = |\overline{C_\alpha}|\,|\langle e^{itH_0}\psi(H_0)P_- f, e^{i\sqrt{i}|\cdot|}\rangle|\,\|\overline{e^{i\sqrt{i}|\cdot|}}\|,$$

where $C_\alpha = \frac{-1}{2i\sqrt{i}}\,\frac{\alpha}{2i\sqrt{i}-\alpha}$.

Moreover noting that for some $\delta > 0$

$$|e^{i\sqrt{i}|x|}|, |\overline{e^{i\sqrt{i}|x|}}| = O(e^{-\delta|x|}) \qquad (|x| \to \infty)$$

and using Perry [19], Lemma 1, we have

$$\frac{1}{\|f\|}\int_0^\infty |\langle e^{itH_0}\psi(H_0)P_+ f, \overline{e^{i\sqrt{i}|\cdot|}}\rangle| dt < \infty,$$

$$\frac{1}{\|f\|}\int_0^\infty |\langle e^{-itH_0}\psi(H_0)P_+ f, e^{i\sqrt{i}|\cdot|}\rangle| dt < \infty,$$

$$\frac{1}{\|f\|}\int_0^\infty |\langle e^{itH_0}\psi(H_0)P_- f, e^{i\sqrt{i}|\cdot|}\rangle| dt < \infty.$$

Thus (3.1)–(3.3) hold.

Finally, (3.4) follows from Perry [19], Remark 2. □

PROPOSITION 3.7 (Generalized Fourier transform for H_α). *Assume that* α *satisfies the same condition as in Proposition* 3.1 *and define*

$$\mathscr{F}_\alpha = \mathscr{F}_0 W(\alpha).$$

Then the representation of $\mathscr{F}_\alpha$ *is given by*

$$(\mathscr{F}_\alpha f)(k) = \lim_{R\to+\infty}\int_{|x|<R} \overline{\psi_\alpha(x,k)} f(x)dx \qquad \text{in } \mathscr{H} \tag{3.5}$$

for any $f \in \mathscr{H}$, *where*

$$\overline{\psi_\alpha(x,k)} = (2\pi)^{-1/2}\left(e^{-ixk} + \frac{\alpha}{(2i|k|-\alpha)}e^{i|x|\,|k|}\right).$$

Furthermore, we obtain

$$(\mathscr{F}_\alpha H_\alpha f)(k) = |k|^2(\mathscr{F}_\alpha f)(k) \qquad \text{for } f \in \mathfrak{D}(H_\alpha). \tag{3.6}$$

Proof. The above result follows from Kuroda [13], Chapter 5. First of all, note that the standard argument in the stationary scattering theory implies

$$\langle W(\alpha)u, v\rangle = \lim_{\kappa\to 0}\frac{\kappa}{\pi}\int_{-\infty}^{\infty}\langle R_\alpha(\lambda+i\kappa)u, R_0(\lambda+i\kappa)v\rangle d\lambda$$

for $u, v \in \mathscr{H}$.

Moreover we find by Lemma 2.2

$$\begin{aligned}(W(\alpha)u, v) &= \lim_{\kappa\to 0}\frac{1}{2\pi i}\int_0^{\infty}\langle (R_0(\lambda+i\kappa) - R_0(\lambda-i\kappa))u, v\rangle d\lambda \\ &\quad + \lim_{\kappa\to 0}\frac{\kappa}{\pi}\int_0^{\infty}\frac{\alpha}{2i\sqrt{\lambda+i\kappa}-\alpha} \\ &\quad \times \int_{-\infty}^{\infty} e^{i\sqrt{\lambda+i\kappa}|y|}u(y)dy\left(\overline{R_0(\lambda+i\kappa)R_0(\lambda-i\kappa)v}\right)(0)d\lambda.\end{aligned} \tag{3.7}$$

Taking $u, v \in L^{2,s}$ $(s > 1/2)$ and assuming that supp $\mathscr{F}_0 v$ does not contain 0 and is compact, we obtain by the standard calculation that

$$(\text{the first term of RHS of } (3.7)) = \int_{-\infty}^{\infty}\mathscr{F}_0 u(k)\overline{\mathscr{F}_0 v(k)}dk$$

and

$$\begin{aligned}&(\text{the second term of RHS of } (3.7)) \\ &= \lim_{\kappa\to 0}\frac{\kappa}{\pi}\int_0^{\infty}\frac{\alpha}{2i\sqrt{\lambda+i\kappa}-\alpha}\int_{-\infty}^{\infty}e^{i\sqrt{\lambda+i\kappa}|y|}u(y)dy\int_{-\infty}^{\infty}\frac{(2\pi)^{-1/2}\overline{\mathscr{F}_0 v(k)}}{(\lambda-k^2)^2+\kappa^2}\,dkd\lambda \\ &= (2\pi)^{-1/2}\int_{-\infty}^{\infty}\frac{\alpha}{2i|k|-\alpha}\int_{-\infty}^{\infty}e^{i|k|\,|y|}u(y)dy\overline{\mathscr{F}_0 v(k)}dk.\end{aligned}$$

The last equality is due to the property of Poisson integrals. Thus we have (3.5).

Next we show (3.6). Note that

$$We^{-itH_\alpha} = e^{-itH_0}W.$$

This implies that for $f \in \mathfrak{D}(H_\alpha)$,

$$\mathscr{F}_\alpha \frac{e^{-itH_\alpha}f - f}{t} = \frac{e^{-itk^2} - 1}{t}\mathscr{F}_\alpha f, \tag{3.8}$$

where $t > 0$. Thus $t \to 0$, we obtain (3.6) from (3.8). □

REMARK 3.8. In sections 4 and 5, in order to show a generalized Parseval formula, we also deal with $\mathscr{F}_{\bar{\alpha}}(k)$ ($\alpha = \alpha_1 + i\alpha_2$ with $\alpha_1 \leqq 0$, $\alpha_2 < 0$). Especially, in section 5 we have to note that $\mathscr{F}_{-i\alpha_2}(k)$ has singular points $\left(k = \pm\frac{\alpha_2}{2}\right)$ from resonance (cf. Remark 1.4(2) and Proposition 2.7).

4. Proof of Theorem 1.3 (i)

In this section we assume $\alpha_1 < 0$. We prove the following lemma, which is the generalized Parseval formula (cf. Pavlov [18], Theorem 2.1).

LEMMA 4.1. *For any* $f, g \in \mathscr{H} \cap L^1(\mathbf{R}^1)$ *and for* $\alpha \in \{\alpha = \alpha_1 + i\alpha_2; \alpha_1 < 0\} \equiv D$ *we have*

$$\langle \mathscr{F}_\alpha f, \mathscr{F}_{\bar{\alpha}} g \rangle = \langle f, g \rangle + \frac{\alpha}{2}\langle f, e^{(\bar{\alpha}|\cdot|)/2}\rangle \langle e^{(\alpha|\cdot|)/2}, g\rangle. \tag{4.1}$$

PROOF. We know the Parseval (Plancherel) formula

$$\langle \mathscr{F}_\alpha f, \mathscr{F}_\alpha g \rangle = \langle f, g \rangle + \frac{\alpha}{2}\langle f, e^{(\alpha|\cdot|)/2}\rangle \langle e^{(\alpha|\cdot|)/2}, g\rangle$$

for $\alpha \in (-\infty, 0)$. We can see that the second term on the right hand side is analytic in D. In fact, putting

$$H_n(\alpha) = \langle \chi_{\{|k|<n\}} \mathscr{F}_\alpha f, \mathscr{F}_{\bar{\alpha}} g \rangle = \int_{|k|<n} \mathscr{F}_\alpha f(k) \overline{\mathscr{F}_{\bar{\alpha}} g(k)} dk,$$

where $\chi_{\{|k|<n\}}$ is the characteristic function on $\{k; |k| < n\}$, we see that H_n is analytic in D and that H_n converges to $\langle \mathscr{F}_\alpha f, \mathscr{F}_{\bar{\alpha}} g\rangle$ locally uniformly in D. As a consequence of the above facts (by the identity theorem) we obtain (4.1). □

PROOF OF THEOREM 1.3 (i). The conclusion follows from

$$W(\alpha)f = 0 \quad \text{if and only if} \quad f_s = 0. \tag{4.2}$$

So we show (4.2).

Noting Corollary 3.2 we have

$$W(\alpha)f = W(\alpha)f_s. \tag{4.3}$$

Using (1.3) and Lemma 4.1 with a density arguement we find

$$\langle W(\alpha)f_s, W(\bar{\alpha})f_s\rangle = \langle \mathscr{F}_\alpha f_s, \mathscr{F}_{\bar{\alpha}} f_s\rangle = \|f_s\|^2. \tag{4.4}$$

Thus (4.3) and (4.4) imply (4.2). □

Proof of Corollary 1.5. Note that Corollary 3.2 and $f = f_s + f_d$. Therefore Corollary 1.5 also follows from (4.2). □

5. Proof of Theorem 1.3 (ii)

In this section, we prepare the generalized Parseval formula (Proposition 5.1) and several lemmas for $\alpha_1 = 0$, $\alpha_2 < 0$ to prove Theorem 1.3 (ii).

Proposition 5.1. *For any $f, g \in \mathscr{H} \cap L^1(\mathbf{R}^1)$, we have*

$$\lim_{\varepsilon \to 0} \langle \mathscr{F}_{i\alpha_2} f, \chi_\varepsilon \mathscr{F}_{-i\alpha_2} g\rangle = \langle f, g\rangle + \frac{i\alpha_2}{4}\int_{\mathbf{R}^1} e^{(i\alpha_2/2)|x|} f(x)dx \int_{\mathbf{R}^1} e^{(i\alpha_2/2)|y|}\overline{g(y)}dy,$$

where χ_a is the characteristic function on $\{k \in \mathbf{R}; a \leqq ||k| + \alpha_2/2|\}$ for $a > 0$.

Proof. For $0 < a < b$, let $\chi_{a,b}$ be the characteristic function on $\{k \in \mathbf{R}; a \leqq ||k| + \alpha_2/2| \leqq b\}$.

Since

$$\begin{aligned}
&\langle \mathscr{F}_{i\alpha_2} f, \chi_\varepsilon \mathscr{F}_{-i\alpha_2} g\rangle \\
&\quad = \lim_{R\to\infty}\int_{\mathbf{R}^1} \chi_{\varepsilon,R}(k)\mathscr{F}_{i\alpha_2} f(k)\overline{\mathscr{F}_{-i\alpha_2} g(k)}dk \\
&\quad = \langle \mathscr{F}_0 f, \chi_\varepsilon \mathscr{F}_0 g\rangle \\
&\qquad + (2\pi)^{-1}\lim_{R\to\infty}\int_{\mathbf{R}^1}\iint_{\mathbf{R}^2}\chi_{\varepsilon,R}(k)\left\{\frac{\alpha_2}{2|k|-\alpha_2}e^{-i(xk+|y|\,|k|)}\right. \\
&\qquad \left. - \frac{\alpha_2}{2|k|+\alpha_2}e^{i(|x|\,|k|+yk)} - \frac{\alpha_2^2}{4k^2-\alpha_2^2}e^{i(|x|-|y|)|k|}\right\} f(x)\overline{g(y)}dxdydk
\end{aligned}$$

and

$$\frac{\alpha_2^2}{4k^2-\alpha_2^2} = \frac{\alpha_2}{2}\left(\frac{1}{2k-\alpha_2} - \frac{1}{2k+\alpha_2}\right),$$

we have by Fubini's theorem

$$(5.1)\quad \lim_{\varepsilon\to 0}\langle \mathscr{F}_{i\alpha_2}f, \chi_\varepsilon \mathscr{F}_{-i\alpha_2}g\rangle = \langle \mathscr{F}_0 f, \mathscr{F}_0 g\rangle$$
$$+ (2\pi)^{-1}\lim_{\varepsilon\to 0}\lim_{R\to\infty}\int\int_{\mathbf{R}^2}\int_{\mathbf{R}^1}\chi_{\varepsilon,R}(k)\left\{\frac{\alpha_2}{2|k|-\alpha_2}e^{-i(xk+|y|\,|k|)}\right.$$
$$\left.-\frac{\alpha_2}{2|k|+\alpha_2}e^{i(|x|\,|k|+yk)} - \frac{\alpha_2}{2}\left(\frac{1}{2k-\alpha_2}-\frac{1}{2k+\alpha_2}\right)e^{i(|x|-|y|)|k|}\right\}$$
$$\times f(x)\overline{g(y)}dkdxdy.$$

Putting

$$I_{\varepsilon,R}(x, y) = \int_{\mathbf{R}^1}\chi_{\varepsilon,R}(k)\left\{\frac{\alpha_2}{2|k|-\alpha_2}e^{-i(xk+|y|\,|k|)}\right.$$
$$\left.-\frac{\alpha_2}{2|k|+\alpha_2}e^{i(|x|\,|k|+yk)} - \frac{\alpha_2}{2}\left(\frac{1}{2k-\alpha_2}-\frac{1}{2k+\alpha_2}\right)e^{i(|x|-|y|)|k|}\right\}dk,$$

we then find that

$$I_{\varepsilon,R}(x, y) = \frac{\alpha_2}{2}e^{i\alpha_2(|x|+|y|)/2}\left(\int_{-R}^{-\varepsilon}+\int_{\varepsilon}^{R}+\int_{R}^{-\alpha_2+R}-\int_{-\alpha_2-\varepsilon}^{-\alpha_2+\varepsilon}\right)\frac{e^{i(|x|+|y|)k}}{k}\,dk.$$

Indeed, we have for $x, y > 0$,

$$I_{\varepsilon,R}(x, y) = \left(\int_0^{-\varepsilon-\alpha_2/2}+\int_{\varepsilon-\alpha_2/2}^{R-\alpha_2/2}\right)\left\{\frac{\alpha_2}{2k-\alpha_2}e^{-i(x+y)k}\right.$$
$$\left.-\frac{\alpha_2}{2k+\alpha_2}e^{i(x+y)k} - \frac{\alpha_2}{2}\left(\frac{1}{2k-\alpha_2}-\frac{1}{2k+\alpha_2}\right)e^{i(x-y)k}\right\}dk$$
$$+\frac{\alpha_2}{2}\left(\int_{-R+\alpha_2/2}^{-\varepsilon+\alpha_2/2}+\int_{\varepsilon+\alpha_2/2}^{0}\right)\left(\frac{1}{2k-\alpha_2}-\frac{1}{2k+\alpha_2}\right)e^{-i(x-y)k}\,dk.$$

Changing k to $-k$ we obtain

(The second term of RHS of the above equality)

$$= \left(\int_{\varepsilon+\alpha_2/2}^{0}+\int_{-R+\alpha_2/2}^{-\varepsilon+\alpha_2/2}\right)\frac{\alpha_2}{2k-\alpha_2}e^{-i(x+y)k}\,dk$$

and

(The fourth term of RHS of the above equality)

$$= \left(\int_0^{-\varepsilon-\alpha_2/2}+\int_{\varepsilon-\alpha_2/2}^{R-\alpha_2/2}\right)\frac{\alpha_2}{2}\left(\frac{1}{2k-\alpha_2}-\frac{1}{2k+\alpha_2}\right)e^{i(x-y)k}\,dk.$$

Thus it holds that

$$I_{\varepsilon,R}(x,y) = \left(\int_{-R+\alpha_2/2}^{-\varepsilon+\alpha_2/2} + \int_{\varepsilon+\alpha_2/2}^{-\varepsilon-\alpha_2/2} + \int_{\varepsilon-\alpha_2/2}^{R-\alpha_2/2}\right) \frac{\alpha_2}{2k-\alpha_2} e^{i(x+y)k}\, dk$$
$$= \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{-\varepsilon-\alpha_2} + \int_{\varepsilon-\alpha_2}^{R-\alpha_2}\right) \frac{\alpha_2}{2k} e^{i(x+y)(k+\alpha_2/2)}\, dk$$
$$= \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} + \int_{R}^{-\alpha_2+R} - \int_{-\alpha_2-\varepsilon}^{-\alpha_2+\varepsilon}\right) \frac{e^{i(x+y)k}}{k}\, dk \times \frac{\alpha_2}{2} e^{i\alpha_2(x+y)/2}.$$

Since we can deal with the other cases by a similar calculation, we omit the details.

Note that the following equality holds;

(5.2) $$\lim_{\varepsilon\to 0}\langle \mathscr{F}_{i\alpha_2} f, \chi_\varepsilon \mathscr{F}_{-i\alpha_2} g\rangle$$
$$= \langle f, g\rangle + \frac{i\alpha_2}{4}\int_{\mathbf{R}^1} e^{(i\alpha_2/2)|x|} f(x)dx \int_{\mathbf{R}^1} e^{(i\alpha_2/2)|y|}\overline{g(y)}dy$$
$$+ (2\pi)^{-1}\lim_{\varepsilon\to 0}\lim_{R\to\infty}\iint_{\mathbf{R}^2}\left\{I_{\varepsilon,R}(x,y) - i\pi\frac{\alpha_2}{2}e^{i\alpha_2(|x|+|y|)/2}\right\} f(x)\overline{g(y)}dxdy$$

by (5.1). Below we shall estimate $I_{\varepsilon,R}(x,y) - i\pi\frac{\alpha_2}{2}e^{i\alpha_2(|x|+|y|)/2}$.

By simple calculation we have

(5.3) $$\left|\int_{-\alpha_2-\varepsilon}^{-\alpha_2+\varepsilon}\frac{e^{i(|x|+|y|)k}}{k}\, dk\right| \leqq \log\left|\frac{\alpha_2-\varepsilon}{\alpha_2+\varepsilon}\right| \to 0 \qquad (\varepsilon\to 0)$$

and

(5.4) $$\left|\int_{R}^{-\alpha_2+R}\frac{e^{i(|x|+|y|)k}}{k}\, dk\right| \leqq \frac{1}{R} \to 0 \qquad (R\to\infty).$$

Moreover Cauchy's integral theorem implies

$$\left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right)\frac{e^{i(|x|+|y|)k}}{k}\, dk - i\pi = -\int_{C_\varepsilon^-}\frac{e^{i(|x|+|y|)z}}{z}\, dz - i\pi - \int_{C_R^+}\frac{e^{i(|x|+|y|)z}}{z}\, dz,$$

where $C_\varepsilon^- = \{z = \varepsilon e^{i\theta} : \theta = \pi \to \theta = 0\}$ and $C_R^+ = \{z = Re^{i\theta} : \theta = 0 \to \theta = \pi\}$.

Thus we obtain

(5.5) $$\left|\left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R}\right)\frac{e^{i(|x|+|y|)k}}{k}\, dk - i\pi\right|$$
$$\leqq \int_0^\pi |e^{i(|x|+|y|)\varepsilon(\cos\theta + i\sin\theta)} - 1|d\theta + \int_0^\pi |e^{-(|x|+|y|)R\sin\theta}|d\theta$$
$$\to 0 \qquad (\varepsilon\to 0, R\to\infty).$$

It follows from (5.3)–(5.5) that

$$I_{\varepsilon,R}(x,y) - i\pi\frac{\alpha_2}{2}e^{i\alpha_2(|x|+|y|)/2} \to 0 \qquad (\varepsilon \to 0, R \to \infty)$$

and

$$\left|\left\{I_{\varepsilon,R}(x,y) - i\pi\frac{\alpha_2}{2}e^{i\alpha_2(|x|+|y|)/2}\right\}f(x)\overline{g(y)}\right| \leqq C|f(x)|\,|\overline{g(y)}|,$$

where C is a positive constant which is independent of x, y, ε and R.

Therefore using Lebesgue's theorem we have the conclusion from (5.2). □

Define

$$\mathscr{E} = \left\{g \in \mathscr{H} \cap L^1(\mathbf{R}^1) : \int_{\mathbf{R}^1} |y|\,|g(y)|dy < \infty, \int_{\mathbf{R}^1} e^{-(i\alpha_2/2)|y|}g(y)dy = 0\right\}.$$

LEMMA 5.2. *Let $g \in \mathscr{E}$. Then $\mathscr{F}_{-i\alpha_2}g$ belongs to $\mathscr{H}$.*

PROOF. The equality

$$e^{i|y|\,|k|} - e^{-i\alpha_2|y|/2} = \int_{-(\alpha_2/2)|y|}^{|y|\,|k|} ie^{i\tau}\,d\tau$$

implies

$$|e^{i|y|\,|k|} - e^{-i\alpha_2|y|/2}| \leqq \left||k| + \frac{\alpha_2}{2}\right||y|.$$

Thus noting

$$\|\chi_\varepsilon F_{-i\alpha_2}g\| \leqq \|F_0 g\| + 2\int_{\mathbf{R}^1} |g(y)|dy\left(\int_{\mathbf{R}^1} \chi_R \frac{|\alpha_2|^2}{|2|k| + \alpha_2|^2}\,dk\right)^{1/2}$$
$$+ \int_{\mathbf{R}^1} |y|\,|g(y)|dy\left(\int_{\mathbf{R}^1} \chi_{\varepsilon,R}\frac{|\alpha_2|^2}{4}\,dk\right)^{1/2},$$

we have

$$\|\mathscr{F}_{-i\alpha_2}g\| = \lim_{\varepsilon\to 0}\|\chi_\varepsilon \mathscr{F}_{-i\alpha_2}g\| < \infty. \quad \square$$

LEMMA 5.3. *Let $f \in \mathscr{H}$ and $g \in \mathscr{E}$. Then it holds that*

$$\langle \mathscr{F}_{i\alpha_2}f, \mathscr{F}_{-i\alpha_2}g\rangle = \langle f, g\rangle.$$

PROOF. Let $h \in \mathscr{H} \cap L^1(\mathbf{R}^1)$. Then Proposition 5.1 and Lemma 5.2 imply

$$\langle \mathscr{F}_{i\alpha_2} h, \mathscr{F}_{-i\alpha_2} g \rangle = \langle h, g \rangle.$$

Since $\mathscr{H} \cap L^1(\mathbf{R}^1)$ is dense in $\mathscr{H}$, we have the conclusion. □

LEMMA 5.4. *$\mathscr{E}$ is dense in $\mathscr{H}$.*

PROOF. Define $\tilde{\mathscr{E}} = \{u \in \mathscr{H} \cap L^1(\mathbf{R}^1) : \int_{\mathbf{R}^1} |x|\,|u(x)|dx < \infty, 0 \notin \operatorname{supp} \mathscr{F}_0 u\}$. Since $\{v \in \mathscr{S}(\mathbf{R}^1) : \mathscr{F}_0 v \in C_0^\infty(\mathbf{R}^1 \backslash 0)\} (\subset \tilde{\mathscr{E}})$ is dense in $\mathscr{H}$, $\tilde{\mathscr{E}}$ is also dense. Therefore, for any $f \in \mathscr{H}$ and any $\varepsilon > 0$ there exists $u \in \tilde{\mathscr{E}}$ such that

$$\|u - e^{-i\alpha_2 |\cdot|/2} f\| < \varepsilon. \tag{5.6}$$

Put

$$g(x) = e^{i\alpha_2 |x|/2} u(x).$$

Then $g \in \mathscr{H} \cap L^1(\mathbf{R}^1)$ and

$$\int_{\mathbf{R}^1} |x|\,|g(x)|dx < \infty, \qquad \int_{\mathbf{R}^1} e^{-i\alpha_2 |x|/2} g(x)dx = \int_{\mathbf{R}^1} u(x)dx = 0$$

hold. Thus g belongs to $\mathscr{E}$. Moreover it follows from (5.6) that

$$\|g - f\| < \varepsilon.$$

The proof is complete. □

PROOF OF THEOREM 1.3 (ii). It suffices to show

$$W(i\alpha_2) f = 0 \Rightarrow f = 0.$$

Since $\mathscr{F}_\alpha = \mathscr{F}_0 W(\alpha)$, we assume $\mathscr{F}_{i\alpha_2} f = 0$. Then Lemma 5.3 implies $\langle f, g \rangle = 0$ for any $g \in \mathscr{E}$. Thus we obtain $f = 0$ by Lemma 5.4. □

Appendix A

In this appendix, we give a proof of the following properties for $\tilde{H}_\alpha$ defined in section 2.

PROPOSITION A.1. *Assume that $\alpha = \alpha_1 + i\alpha_2 \neq 0$ with $\alpha_1 \leqq 0$ and $\alpha_2 \leqq 0$. Then we have*

(i) *$\tilde{H}_\alpha$ is a maximal dissipative operator,*

(ii) *$\tilde{H}_{\bar{\alpha}}$ is a maximal accretive operator,*

(iii) *$\tilde{H}_\alpha^* = \tilde{H}_{\bar{\alpha}}$.*

First we show the following Lemma:

LEMMA A.2. *Let $\alpha \in \mathbf{C} \backslash \{0\}$. Then $\tilde{H}_\alpha$ is a closed operator.*

Proof. Suppose $U_n \in \mathfrak{D}(\tilde{H}_\alpha)$, $(n = 1, 2, 3, \ldots)$ and

$$\lim_{n\to\infty} \|U_n - U\| = \lim_{n\to\infty} \|\tilde{H}_\alpha U_n - W\| = 0.$$

Noting that

$$U_n = u_n + a_n H_0 (H_0^2 + 1)^{-1}\varphi \qquad \text{and} \qquad \tilde{H}_\alpha U_n = H_0 u_n - a_n (H_0^2 + 1)^{-1}\varphi$$

for some $u_n \in \mathscr{H}^2$ and $a_n \in \mathbf{C}$ satisfying

$$\langle u_n, \varphi\rangle = -a_n(\alpha^{-1} + \langle \varphi, H_0(H_0^2 + 1)^{-1}\varphi\rangle)$$

(see the definition of $\tilde{H}_\alpha$), we have

$$\begin{aligned}\|U_n - U_m\|^2 &+ \|\tilde{H}_\alpha U_n - \tilde{H}_\alpha U_m\|^2 \\ &= \|u_n - u_m\|^2 + \|H_0(u_n - u_m)\|^2 \\ &\quad + |a_n - a_m|^2 (\|(H_0^2 + 1)^{-1}\varphi\|^2 + \|H_0(H_0^2 + 1)^{-1}\varphi\|^2),\end{aligned}$$

where $m = 1, 2, 3, \ldots$.

Since the above equality means that $\{u_n\}_{n\in\mathbf{N}}, \{H_0 u_n\}_{n\in\mathbf{N}}$ and $\{a_n\}_{n\in\mathbf{N}}$ satisfy the Cauchy condition, there exist $u_0 \in \mathfrak{D}(H_0)$ and $a_0 \in \mathbf{C}$ such that

$$\lim_{n\to\infty} \|u_n - u_0\|_{\mathscr{H}^2} = 0, \qquad \lim_{n\to\infty} a_n = a_0$$

and

$$a_0 = -\frac{\langle u_0, \varphi\rangle}{\alpha^{-1} + \langle \varphi, H_0(H_0^2 + 1)^{-1}\varphi\rangle}.$$

Therefore we have $U = u_0 + a_0 H_0 (H_0^2 + 1)^{-1}\varphi \in \mathfrak{D}(\tilde{H}_\alpha)$ and $W = \tilde{H}_\alpha U$.

Now the proof is complete. □

Proof of Proposition A.1. First we consider the proof of (i) and (ii).

Note that $\tilde{H}_\alpha$ and $\tilde{H}_{\bar{\alpha}}$ are dissipative and accretive, respectively. Then it follows from Lemma A.2 that $\text{Range}(\tilde{H}_\alpha - i)$ and $\text{Range}(\tilde{H}_{\bar{\alpha}} + i)$ are closed. Thus

(A.1) $$\text{Range}(\tilde{H}_\alpha - i)^\perp = \{0\} \ (\Leftrightarrow \text{Range}(\tilde{H}_\alpha - i) = \mathscr{H})$$

and

$$\text{Range}(\tilde{H}_{\bar{\alpha}} + i)^\perp = \{0\} \ (\Leftrightarrow \text{Range}(\tilde{H}_{\bar{\alpha}} + i) = \mathscr{H})$$

in the cases of (i) and (ii), respectively (see e.g. Kato [10]). Here we prove (A.1) only. The other is proved in the similar way.

Suppose $W \in \text{Range}(\tilde{H}_\alpha - i)^\perp$, i.e.,

(A.2) $$\langle (\tilde{H}_\alpha - i)U, W\rangle = 0$$

for any $U \in \mathfrak{D}(H_\alpha)$.

Noting the definition of $\mathfrak{D}(H_\alpha)$, we find that the above U has the following form: for any $u \in \mathscr{H}^2$,

$$U = u + aH_0(H_0^2+1)^{-1}\varphi,$$

where

$$a = -\frac{\langle u, \varphi\rangle}{\alpha^{-1} + \langle \varphi, H_0(H_0^2+1)^{-1}\varphi\rangle}. \tag{A.3}$$

Then it follows from (2.1) and (A.2) that

$$\langle (H_0 - i)u, W\rangle - a\langle i(H_0+i)^{-1}\varphi, W\rangle = 0. \tag{A.4}$$

Especially, taking $U = \tilde{u}$ satisfying $\langle \tilde{u}, \varphi\rangle = 0$ ($\Leftrightarrow a = 0$) we find by (A.4)

$$\langle (H_0 - i)\tilde{u}, W\rangle = 0. \tag{A.5}$$

Consider the operator H_0^0 defined by

$$\begin{cases} \mathfrak{D}(H_0^0) = \{\tilde{u} \in \mathscr{H}^2 \mid \langle \tilde{u}, \varphi\rangle = 0\}, \\ H_0^0 = H_0|_{\mathfrak{D}(H_0^0)}. \end{cases}$$

Then (A.5) implies $W \in \mathfrak{D}((H_0^0)^*)$. Therefore it follows from Albeverio and Kurasov [3], section 1.2.4 that

$$W = w + bH_0(H_0^2+1)^{-1}\varphi \tag{A.6}$$

for some $w \in \mathscr{H}^2$ and $b \in \mathbf{C}$.

Putting (A.6) into (A.5) and noting that $H_0^0 \subset H_0$ and $\langle \tilde{u}, \varphi\rangle = 0$, we obtain

$$\langle \tilde{u}, (H_0+i)w\rangle + \langle \tilde{u}, bi(H_0 - i)^{-1}\varphi\rangle = 0. \tag{A.7}$$

Then note that the following fact: *Let $v \in \mathscr{H}^{-2}$. If v satisfies that*

$$\langle \tilde{u}, v\rangle = 0 \qquad \textit{for any } \tilde{u} \in \mathfrak{D}(H_0^0),$$

then $v = c\varphi$ holds, where c is a constant.

Thus it follows from (A.7) that

$$(H_0+i)w + bi(H_0-i)^{-1}\varphi = c\varphi.$$

Since the LHS of the above equality belongs to $\mathscr{H}$ and φ belongs to $\mathscr{H}^{-1}\backslash\mathscr{H}$, we find $c = 0$, i.e.,

$$w = -bi(H_0^2+1)^{-1}\varphi. \tag{A.8}$$

Thus (A.6) and (A.8) imply

$$W = b(H_0+i)^{-1}\varphi. \tag{A.9}$$

Finally we have $b = 0$, i.e., $W = 0$. Indeed, putting (A.9) into (A.4) and noting (A.3), we find

$$\text{(A.10)} \qquad \bar{b}\langle u, \varphi\rangle \frac{\alpha^{-1} + \|H_0^{1/2}(H_0+i)^{-1}\varphi\|^2 + i\|(H_0+i)^{-1}\varphi\|^2}{\alpha^{-1} + \|H_0^{1/2}(H_0+i)^{-1}\varphi\|^2} = 0.$$

Note that $\langle u, \varphi\rangle \not\equiv 0$ and

$$\alpha^{-1} + \|H_0^{1/2}(H_0+i)^{-1}\varphi\|^2 + i\|(H_0+i)^{-1}\varphi\|^2 \neq 0$$

since $\alpha_2 \leqq 0$. Thus (A.10) implies $b = 0$. Now the proof of (A.1) is complete.

Finally we prove (iii). Let take $U \in \mathfrak{D}(\tilde{H}_\alpha)$ and $V \in \mathfrak{D}(\tilde{H}_{\bar{\alpha}})$. Then we have

$$\langle \tilde{H}_\alpha U, V\rangle = \langle U, \tilde{H}_{\bar{\alpha}} V\rangle.$$

This means $\tilde{H}_{\bar{\alpha}} \subset \tilde{H}_\alpha^*$. Thus (ii) implies $\tilde{H}_{\bar{\alpha}} = \tilde{H}_\alpha^*$. □

REMARK A.3. It follows from the above argument that H_{α_1} is self-adjoint.

Appendix B

We state an abstract result in Kadowaki [8] without a proof (see [8] for the proof).

Let $\mathscr{H}$ be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathscr{H}}$. The norm is denoted by $\|\cdot\|_{\mathscr{H}}$. Let $\{V(t)\}_{t\geqq 0}$ and $\{U_0(t)\}_{t\in\mathbf{R}}$ be a contraction semi-group in $\mathscr{H}$ and a unitary group in $\mathscr{H}_0$, respectively. We denote by A and A_0 the generator of $V(t)$ and $U_0(t)$, respectively ($V(t) = e^{-itA}$ and $U_0(t) = e^{-itA_0}$). We assume the following conditions on A and A_0.

(B1) $\sigma(A_0) = \sigma_{ac}(A_0) = \mathbf{R}$ or $[0, \infty)$.

(B2) $(A-i)^{-1} - (A_0-i)^{-1}$ defined as a form is extended to a compact operator K in $\mathscr{H}$.

(B3) There exist non-zero projection operators in $\mathscr{H}, P_+$ and P_-, such that $P_+ + P_- = I_d$ and

$$\text{(B3.1)} \qquad \int_0^\infty \|KU_0(t)\psi(A_0)P_+\|_{B(\mathscr{H},\mathscr{H})}dt < \infty,$$

$$\text{(B3.2)} \qquad \int_0^\infty \|K^*U_0(t)\psi(A_0)P_+\|_{B(\mathscr{H},\mathscr{H})}dt < \infty,$$

$$\text{(B3.3)} \qquad \int_0^\infty \|K^*U_0(-t)\psi(A_0)P_-\|_{B(\mathscr{H},\mathscr{H})}dt < \infty,$$

$$\text{(B3.4)} \qquad \text{w-}\lim_{t\to+\infty} U_0(-t)\psi(A_0)P_- f_t = 0$$

for each $\psi \in C_0^\infty(\mathbf{R}\backslash\{0\})$ and $\{f_t\}_{t\in\mathbf{R}}$ satisfying $\sup_{t\in\mathbf{R}}\|f_t\|_{\mathscr{H}} < \infty$, where $\|\cdot\|_{B(\mathscr{H},\mathscr{H})}$ is the operator norm of bounded operators in $\mathscr{H}$.

Let $\mathscr{H}_b$ be the space generated by the eigenvectors of A with real eigenvalues. Then we have the following theorem:

THEOREM B. *Assume that* (B1)–(B3). *Then for any* $f \in \mathscr{H}_b^{\perp}$, *the wave operator*

$$Wf = \lim_{t \to \infty} U_0(-t)V(t)f$$

exists. Moreover W *is not zero as an operator from* $\mathscr{H}_b^{\perp}$ *to* $\mathscr{H}$.

Appendix C

In the last section, we state some examples by which the present work is motivated.

EXAMPLE C.1 (Wave or Klein-Gordon equations with constant dissipation). Let

$$\mathscr{H} = \mathscr{H}(m) \equiv \{f = (f_1, f_2) \mid \|f\|^2_{\mathscr{H}(m)} < \infty\}$$

with

$$\|f\|^2_{\mathscr{H}(m)} = \int_{\mathbf{R}^N} (m^2|f_1|^2 + |\nabla f_1|^2 + |f_2|^2)dx$$

and

$$H(m) = i\begin{pmatrix} 0 & 1 \\ \Delta - m & -1 \end{pmatrix}$$

with $m = 0$ (wave) or $m = 1$ (Klein-Gordon). Then we obtain

$$\sigma(H(1)) = \sigma_c(H(1)) = \sigma_{ess}(H(1)) = \{z = \sigma - i/2 \in \mathbf{C}_- \mid |\sigma| \geqq \sqrt{3}/2\},$$

$$\sigma(H(0)) = \sigma_c(H(0)) = \sigma_{ess}(H(0))$$

$$= \{z = \sigma + i\tau \in \mathbf{C}_- \cup \mathbf{R} \mid \sigma \in \mathbf{R} \text{ and } \tau = -1/2 \text{ or } \sigma = 0 \text{ and } -1 \leqq \tau \leqq 0\}$$

and as $t \to +\infty$,

$$\|e^{-itH(m)}f\|_{\mathscr{H}(m)} = o(1)$$

for any $f \in \mathscr{H}(m)$.

PROOF. Conclusions are well known. Here we give a brief sketch of the proof.

For $(H - z)f = g$ with $f = (f_1, f_2)$ and $g = (g_1, g_2)$, we obtain $-\Delta f_2 - (z^2 + iz - m)f_2 = (i\Delta - im)g_1 + zg_2$.

(the case $m = 1$) $z^2 + iz - 1 \in [0, \infty) \Leftrightarrow |\sigma| \geqq \sqrt{3}/2$ and $\tau = -1/2$, where we put $z = \sigma + i\tau$ $(\sigma, \tau \in \mathbf{R})$.

(the case $m = 0$) $z^2 + iz \in [0, \infty) \Leftrightarrow \sigma \in \mathbf{R}$ and $\tau = -1/2$ or $\sigma = 0$ and $-1 \leqq \tau \leqq 0$.

The last assertion follows from Engel-Nagel [5], 2.22 Corollary. □

EXAMPLE C.2 (Wave equation with variable dissipation). Let $\mathscr{H} = \mathscr{H}(0)$ and consider the following operators:

$$H_0 = i\begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \qquad H_b = i\begin{pmatrix} 0 & 1 \\ \Delta & -b(x) \end{pmatrix}$$

with $0 \leqq b(x) \leqq b_0(1 + |x|)^{-1-\delta}$ in $\mathbf{R}^N$ for some positive constants b_0 and δ. Assuming b_0 is sufficiently small and $N \neq 2$, we find $\sigma(H_b) = \sigma_c(H_b) = \sigma_{ess}(H_b) = \mathbf{R}$ (Nakazawa [16]). In the case of $N = 2$ the spectral structure is not clear even if b_0 is small. However, without the assumption $N \neq 2$ and with the smallness of b_0, Mochizuki [15] and Nakazawa [17] showed that for any $f(\neq 0) \in \mathscr{H}(0)$, there exists $f_\pm(\neq 0) \in \mathscr{H}(0)$ such that

$$\text{(C.1)} \qquad \lim_{t \to \pm\infty} \|e^{-itH_b} f - e^{-itH_0} f_\pm\|_{\mathscr{H}(0)} = 0$$

and that the wave operator and the scattering operator exist. These are proved by smooth perturbation theory developed by Kato [9].

REMARK C.3. Without the smallness on b_0, Mochizuki [14] and Nakazawa [17] showed that there exists $f(\neq 0), f_+(\neq 0) \in \mathscr{H}(0)$ such that (C.1), i.e., the existence of scattering states, holds.

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Mitsuteru Kadowaki
Department of Mechanical Engineering, Ehime University
3 *Bunkyo-cho*
Matsuyama-shi, Ehime 790-8577, *Japan*
e-mail: mkadowaki@eng.ehime-u.ac.jp

Hideo Nakazawa
Current address: Department of Mathematics, Chiba Institute of Technology
2-1-1 *Shibazono, Narashino, Chiba* 275-0023, *Japan*
e-mail: nakazawa@pf.it-chiba.ac.jp
Department of Mathematics, Tokyo Metropolitan University
1-1 *Minami Ohsawa*
Hachioji-shi, Tokyo 192-0397, *Japan*
e-mail: hideo-n@comp.metro-u.ac.jp

Kazuo Watanabe
Department of Mathematics, Gakushuin University
1-5-1 *Mejiro*
Toshima-ku, Tokyo 171-8588, *Japan*
e-mail: kazuo.watanabe@gakushuin.ac.jp