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Kernels of derivations in positive characteristic

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ABSTRACT. Let R be an integral domain that is finitely generated over a field k. Let $D: R \to R$ be a derivation over k. Our aim is to compute $\operatorname{Ker} D$.

Under the assumptions that the characteristic of k is zero, D is locally nilpotent and $\operatorname{Ker} D$ is finitely generated over k, Essen gave an explicit algorithm based on the exponential of the derivation. In this paper we give an analogous algorithm in the positive characteristic case using a truncated version of the exponential. It does not require the nilpotence of D. We give several computational examples of application of our algorithm.

Also using higher derivations, we obtain a word-by-word translation of Essen's formula to positive characteristics.

1. Introduction

Let R be an integral domain that is finitely generated over a field k. Let $D: R \to R$ be a derivation over k. The subject of this paper is to compute the kernel $\operatorname{Ker} D$ in positive characteristic.

When the characteristic of k is zero, several techniques to compute the derivation kernel are known. Essen gave one of such techniques in the case that D is locally nilpotent and that $\operatorname{Ker} D$ is finitely generated over k ([6], [7, Corollary 1.3.23]). The technique is based on the exponential of the derivation. Note that in characteristic 0, some derivation kernels are not finitely generated (counter-examples to Hilbert's Fourteenth Problem), even if the derivations are locally nilpotent (see [4], [8], [9], [14], [15]). Essen's algorithm does not work for these derivations.

In this paper we give an algorithm to compute the derivation kernel in the positive characteristic case, inspired by Essen's algorithm. And using our algorithm, we can compute the kernel for any derivation, without assuming the nilpotency of the derivations. The key points of our algorithm are that we regard the derivation kernel as a $k[R^p]$-module, where $k[R^p]$ is a sub k-domain of R generated by $\{x^p \mid x \in R\}$ and that we compute the generators of the kernel as the $k[R^p]$-module using the truncated version of the exponential of the der-

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ivation (Definition 2.2). In section 2, we calculate generators for the kernel of D in case there exists an element $s \in R$ satisfying $D(s) = 1$ (we call such an s as a slice of D) and $D^p = 0$ (Theorem 2.3 (1)). And we describe our algorithm in Theorem 2.3 (2) and section 3. We exhibit our calculation techniques by concrete examples in section 4 and give a complete list of the kernels of monomial derivations in two variables in characteristic 2 and 3 (Appendix A).

Another idea to compute the derivation kernel is to consider a derivation $D : R \to R$ as a $k[R^p]$-module homomorphism, and to compute the kernel by "linear algebra" as a $k[R^p]$-module homomorphism (actually this is pointed out by the referee). Experiments show that the linear algebra method is faster than our method, but our method gives a filtration for Ker $D \subset R$, which should reflect more informations of the derivation.

In fact, our algorithm is related with $\mathbf{G}_a$-actions. One of the aims of Essen's algorithm is to compute the invariant ring of $\mathbf{G}_a$-action on an affine variety in case the ring is finitely generated ([6]). In zero characteristic case, $\mathbf{G}_a$-actions correspond to locally nilpotent derivations, and the invariant rings correspond to the derivation kernels. In positive characteristic case, as we will see in section 5, $\mathbf{G}_a$-action on an affine variety is related to a locally finite iterative higher derivation, and we can also calculate the invariant ring mimicking Essen's algorithm (Proposition 5.7).

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Notation and Convention.

Throughout this paper, k denotes a field, p its characteristic. Unless otherwise stated, we assume that $p > 0$. When R is a commutative k-domain, $Q(R)$ is the quotient field of R and $k[R^p]$ is a sub k-domain of R generated by $\{x^p \in R \mid x \in R\}$.

A map $D : R \to R$ (resp. $R \to Q(R)$) is said to be a k-derivation of R (resp. k-derivation of R with values in $Q(R)$) if D satisfies $D(f+g) = D(f) + D(g)$ and $D(fg) = D(f)g + fD(g)$ for $f, g \in R$, and $D(a) = 0$ for $a \in k$. The set of all k-derivations of R (resp. with values $Q(R)$) is denoted by $\mathrm{Der}_k R$ (resp. $\mathrm{Der}_k(R, Q(R))$).

For $D \in \mathrm{Der}_k(R, Q(R))$, we denote the unique extension of D to $\mathrm{Der}_k Q(R)$ by $\tilde{D}$. By abuse of notation, the map $D^l : R \to Q(R)$ is the restriction of $\tilde{D}^l : Q(R) \to Q(R)$.

A k-derivation D is called locally nilpotent if for every $f \in R$ there exists a positive integer N such that $D^N(f) = 0$.

An element $s \in R$ is called a slice of a k-derivation D if $D(s) = 1$.

2. Kernels of derivations in positive characteristic

Throughout this section, R is a finitely generated commutative k-domain. Let D be a k-derivation of R. We denote by $\mathrm{Ker}\, D$ the kernel of D, which is the set $\{f \in R \mid D(f) = 0\}$. When the characteristic $p > 0$, note that we have $x^p \in \mathrm{Ker}\, D$ for $x \in R$, hence $\mathrm{Ker}\, D$ contains $k[R^p]$.

In this section, we claim that we have an algorithm to calculate the kernel of a k-derivation D of a finitely generated k-domain when k is of positive characteristic p, and calculate generators for the kernel of D in case that D has a slice and $D^p = 0$. Our algorithm is inspired by Essen's algorithm in the zero characteristic case.

Proposition 2.1 (Essen [6], [7, Corollary 1.3.23]). *Let k be a field of characteristic zero, $R := k[x_1, \ldots, x_n]$ a finitely generated k-domain and $D \in \mathrm{Der}_k R$ a locally nilpotent derivation.*

(1) *When D has a slice $s \in R$, we define a map $\varphi_{-s} : R \to R$ by*

$$\varphi_{-s}(f) := \exp(TD)(f)|_{T=-s} = \sum_{l=0}^{\infty} \frac{(-s)^l}{l!} D^l(f).$$

Then φ_{-s} is a k-algebra homomorphism and the image of φ_{-s} is equal to $\mathrm{Ker}\, D$. In particular $\mathrm{Ker}\, D = k[\varphi_{-s}(x_1), \ldots, \varphi_{-s}(x_n)]$.

(2) *When $\mathrm{Ker}\, D$ is a finitely generated k-algebra, without assuming the existence of a slice, taking any $t \in R$ with $D(t) = u \neq 0$, the derivation $\frac{1}{u} D \in \mathrm{Der}_k R\left[\frac{1}{u}\right]$ has a slice t, hence (1) calculates $\mathrm{Ker}\left(\frac{1}{u} D\right)$. One can compute $\mathrm{Ker}\, D = \mathrm{Ker}\left(\frac{1}{u} D\right) \cap R$ by Groebner basis theory.*

When one tries to use this algorithm in the positive characteristic, we have three difficulties.

The first difficulty is that in the definition of φ_{-s} we cannot define $\frac{1}{p!}$ when the characteristic $p > 0$. So we use the truncated version of φ_{-s}.

Definition 2.2. *When $D \in \mathrm{Der}_k(R, Q(R))$ and $a \in R$, we define a $\mathrm{Ker}\, D$-module homomorphism $E_a D : R \to Q(R)$ by*

$$E_a D(f) = \sum_{l=0}^{p-1} \frac{a^l}{l!} D^l(f).$$

Remark 2.2.1. In the following, we will use only $E_{-s}D$ with a slice s as in Proposition 2.1.

Remark 2.2.2. It is a merit of this definition not to assume that D is locally nilpotent, compared to the case of characteristic zero.

The second difficulty is that $E_{-s}D$ is not necessarily a ring homomorphism. But the morphism $E_{-s}D$ is a Ker D-module homomorphism, so we will get the generators, not as a ring but as a Ker D-module.

The last difficulty is that the image of $E_{-s}D$ is not necessarily equal to Ker D, unless $D^p = 0$. Hence by restricting D to Ker D^p (where D^p is also a k-derivation), we can calculate Ker D. Now it is enough to calculate Ker D^p, and we can proceed by induction.

Theorem 2.3. *Let k be a field of characteristic $p > 0$, $R := k[x_1, \dots, x_n]$ a finitely generated k-domain and $D \in \mathrm{Der}_k R$.*

(1) *We assume that D has a slice s and $D^p = 0$. Then* Ker D *is equal to the image of $E_{-s}D$. In particular, as a $k[R^p]$-module,*

$$\mathrm{Ker}\, D = \sum_{0 \le i_1, \dots, i_n < p} k[R^p] E_{-s} D(x_1^{i_1} \dots x_n^{i_n}),$$

where $k[R^p] = k[x_1^p, \dots, x_n^p] \subset \mathrm{Ker}\, D$.

When $D' \in \mathrm{Der}_k(R, Q(R))$ with a slice s and $D'^p = 0$, we have $\mathrm{Ker}\, D' = \mathrm{Im}\, E_{-s}D' \cap R$.

(2) *Let D be a k-derivation, without assuming that D has a slice, D is locally nilpotent nor $D^p = 0$, still we have an algorithm to calculate* Ker D *by repeatedly localizing R, using* (1) *and using the Groebner basis theory.*

Remark 2.3.1. Unlike the zero characteristic case, the derivation kernel is always a finitely generated k-algebra, because it is a sub $k[R^p]$-module of R. This fact is essentially due to Nowicki-Nagata [13, Proposition 4.1].

In the rest of this section we prove (1) of Theorem 2.3. And we give an algorithm (2) in the next section.

Proposition 2.4. *Let $D \in \mathrm{Der}_k(R, Q(R))$ be a k-derivation having a slice s. Then we have*

$$\tilde{D}(E_{-s}D(f)) = -s^{p-1} D^p(f).$$

Hence, we have $\tilde{D}(E_{-s}D(f)) = 0$ if and only if $D^p(f) = 0$.

Proof. Easy calculation. □

Proof of Theorem 2.3 (1). Let $f \in R$. When $f \in \mathrm{Im}\, E_{-s}D$, then $f \in \mathrm{Ker}\, D$ by Proposition 2.4. If $f \in \mathrm{Ker}\, D$, we have $f = E_{-s}D(f) \in \mathrm{Im}\, E_{-s}D$. We also have the equality $\mathrm{Ker}\, D = \sum_{0 \le i_1, \dots, i_n < p} k[R^p]E_{-s}D(x_1^{i_1} \dots x_n^{i_n})$, since R is a finitely generated $k[R^p]$-module generated by the elements $x_1^{i_1} \dots x_n^{i_n}$ $(0 \le i_1, \dots, i_n < p)$.

We can show the last statement using the same proof. □

3. Algorithm

The goal of this section is to give the algorithm explicitly, claimed in Theorem 2.3 (2).

Let $D \neq 0$ be a k-derivation of $R = k[x_1, \dots, x_n]$, and we assume that the characteristic of k is $p > 0$. We do not assume that D has a slice or that $D^p = 0$. In this section, we give an algorithm to calculate the kernel of D using Theorem 2.3 (1).

There exists $s_1 \in R$ such that $D(s_1) \neq 0$. Define $D_1 := \frac{1}{D(s_1)} D \in \mathrm{Der}_k(R, Q(R))$, then D_1 has a slice s_1. We note that $D_1^p(s_1) = 0$, because $D_1(D_1(s_1)) = D_1(1) = 0$. For every integer $i > 0$, we define $D_i \in \mathrm{Der}_k(R, Q(R))$ inductively as follows. Since k is of characteristic $p > 0$, D_{i-1}^p is also k-derivation of R with values in $Q(R)$. If $D_{i-1}^p \neq 0$, then there exists $s_i \in R$ such that $D_{i-1}^p(s_i) \neq 0$. Define $D_i := \frac{1}{D_{i-1}^p(s_i)} D_{i-1}^p \in \mathrm{Der}_k(R, Q(R))$. Then s_i is a slice of D_i. By definition, we have $\mathrm{Ker}\, D_i = \mathrm{Ker}\, D_{i-1}^p$.

Lemma 3.1. *There exists $\ell \le n$ such that $D_{\ell+1} = 0$.*

Proof. Since $\mathrm{Ker}\, D$ contains $k[R^p]$, the field extension degree $[Q(R) : Q(\mathrm{Ker}\, D)]$ is p^v with some $v \le n$. On the other hand, we have

$$\mathrm{Ker}\, D = \mathrm{Ker}\, D_1 \subset \mathrm{Ker}\, D_2 \subset \cdots \subset \mathrm{Ker}\, D_i \subset \mathrm{Ker}\, D_{i+1} \subset \cdots \subset R$$

and $[Q(\mathrm{Ker}\, D_{i+1}) : Q(\mathrm{Ker}\, D_i)] \ge p$ since D_i has a slice $s_i \in \mathrm{Ker}(D_{i+1}) \backslash \mathrm{Ker}(D_i)$. □

We take the integer ℓ such that $D_\ell \neq 0$ and $D_{\ell+1} = 0$. We calculate the kernels $\mathrm{Ker}\, D_\ell, \mathrm{Ker}\, D_{\ell-1}, \dots,$ and $\mathrm{Ker}\, D$, inductively.

When $i = \ell$, since $D_\ell^p = 0$, we have $\mathrm{Ker}\, D_\ell^p = R$, which has trivial generators $\{x_1^{i_1} \dots x_n^{i_n}\}$ as a $k[R^p]$-module. When $i < \ell$, $\mathrm{Ker}\, D_i^p = \mathrm{Ker}\, D_{i+1}$ whose generators are known by the induction hypothesis.

Proposition 3.2. *We have $\mathrm{Ker}\, D_i = E_{-s_i}D_i(\mathrm{Ker}\, D_i^p) \cap R$. In particular, when $\mathrm{Ker}\, D_i^p = k[R^p]a_1 + \cdots + k[R^p]a_t$, then we have*

$$\mathrm{Ker}\, D_i = (k[R^p]E_{-s_i}D_i(a_1) + \cdots + k[R^p]E_{-s_i}D_i(a_t)) \cap R.$$

Proof. In order to calculate $\mathrm{Ker}\, D_i$, we consider D_i as a k-derivation of Ker D_i^p with values in $Q(\text{Ker } D_i^p)$. Then we have $D_i^p = 0$ and s_i is a slice of D_i. So we apply Theorem 2.3 (1) to D, the kernel of D_i is equal to the intersection of $E_{-s_i} D_i(\text{Ker } D_i^p)$ and Ker D_i^p. Hence we have Ker $D_i = E_{-s_i} D_i(\text{Ker } D_i^p) \cap R$. For the last statement immediately follows. □

Lemma 3.3. *We can explicitely find generators of*

$$(k[R^p]E_{-s_i}D_i(a_1) + \cdots + k[R^p]E_{-s_i}D_i(a_t)) \cap R,$$

as a $k[R^p]$*-module.*

Proof. One can find $b \in k[R^p]$ and $c_j \in R$ such that $E_{-s_i}D_i(a_j) = \frac{c_j}{b}$ for every j (find the common divisor b' and multiply the numerator and the denominator by b'^{p-1}). Generators of the $k[R^p]$-modules $k[R^p]c_1 + \cdots + k[R^p]c_t$ and bR are known, so we need to calculate generators of the intersection of two modules. For that purpose, one can apply the standard technique of calculating the ideal intersection (see [1, §6.2], [2, Ch. 4 §3 Theorem 11]) also for the module intersections, using a suitable monomial order (for example, the TOP order) (see [3, §5.2] for the Groebner basis for modules). If we write the generators of $(k[R^p]c_1 + \cdots + k[R^p]c_t) \cap (bR)$ as $bd_1, \ldots, bd_{t'}$ with $d_1, \ldots, d_{t'} \in R$, then generators of $(k[R^p]E_{-s_i}D_i(a_1) + \cdots + k[R^p]E_{-s_i}D_i(a_t)) \cap R$ are $d_1, \ldots, d_{t'}$. □

Finally we get generators of Ker D_1 = Ker D as a $k[R^p]$-module, which we write as $f_1, \ldots, f_m$. Hence we deduce that Ker $D = k[x_1^p, \ldots, x_n^p, f_1, \ldots, f_m]$. This is the algorithm which we claim in Theorem 2.3 (2). (End of the construction of the algorithm, hence the proof of Theorem 2.3 (2).)

Remark 3.4. This algorithm can be implemented by the software "Singular" ([11]).

Remark 3.5. "Linear algebraic" approach expresses the derivation homomorphism as a $k[R^p]$-matrix, and computes its kernel by the Groebner basis for modules. When $M = k[R^p]m_1 + \cdots + k[R^p]m_r \subset R$ is a finitely generated $k[R^p]$-module and $\phi : M \to N$ an $k[R^p]$-homomorphism, then in $M \oplus N$, Ker $\phi = M \cap (k[R^p](m_1 - \phi(m_1)) + \cdots + k[R^p](m_r - \phi(m_r)))$, which can be computed by the module intersection algorithm, and can be implemented by "Singular". Experiment shows that this approach is faster, at least for small examples as in the next section.

4. Examples

We calculate the kernels of several derivations using the techniques in the previous section.

In Examples 4.1, 4.2, 4.3, 4.4 and 4.5, we treat monomial k-derivations D of the polynomial ring $k[X_1, \dots, X_n]$. A k-derivation $D \in \mathrm{Der}_k k[X_1, \dots, X_n]$ is called a monomial k-derivation when $D = a_1 \frac{\partial}{\partial X_1} + \cdots + a_n \frac{\partial}{\partial X_n}$ for some monomials $a_1, \dots, a_n \in k[X_1, \dots, X_n]$ with coefficient 1.

Example 4.6 is an example of a k-derivation of a finitely generated k-domain which is not a polynomial ring.

Example 4.1. In this example, we assume that $p = 3$ and write $\alpha \equiv \beta$ for $\alpha \equiv \beta \pmod 3$. We take a k-derivation D of the polynomial ring $k[x, y]$ as

$$D := \frac{\partial}{\partial x} + x^a y^b \frac{\partial}{\partial y}, \qquad (a, b \in \mathbf{Z}_{\geq 0}).$$

Let us calculate $\mathrm{Ker}\, D$. Note that D has a slice x. We know

$$D^3 = (a(a-1)x^{a-2}y^b + b(2b-1)x^{3a}y^{3b-2})\frac{\partial}{\partial y}.$$

In the right hand side, we have $a(a-1)x^{a-2}y^b + b(2b-1)x^{3a}y^{3b-2} = 0$ if and only if $a \not\equiv 2$ and $b \not\equiv 1$.

(i) In case of $a \not\equiv 2$ and $b \not\equiv 1$, we see $D^3 = 0$. We have the following calculations:

$$\begin{aligned}
E_{-x}D(1) &= 1, \\
E_{-x}D(y) &= y - (a+1)x^{a+1}y^b - bx^{2a+2}y^{2b-1}, \\
E_{-x}D(y^2) &= y^2 + (a+1)x^{a+1}y^{b+1} + (b+1)x^{2a+2}y^{2b}, \\
E_{-x}D(x) &= 0, \\
E_{-x}D(xy) &= -ax^{a+2}y^b - bx^{2a+3}y^{2b-1}, \\
E_{-x}D(xy^2) &= ax^{a+2}y^{b+1} + (b+1)x^{2a+3}y^{2b}, \\
E_{-x}D(x^2) &= 0, \\
E_{-x}D(x^2y) &= -(a+2)x^{a+3}y^b - bx^{2a+4}y^{2b-1}, \\
E_{-x}D(x^2y^2) &= (a+2)x^{a+3}y^{b+1} + (b+1)x^{2a+4}y^{2b}.
\end{aligned}$$

When $a \equiv b \equiv 0$, by the calculations above, we find that $\mathrm{Ker}\, D$ is generated by the following elements as a $k[x^3, y^3]$-module:

$$\begin{gathered}
1, y - x^{a+1}y^b, y^2 + x^{a+1}y^{b+1} + x^{2a+2}y^{2b}, x^{2a+3}y^{2b}, \\
-2x^{a+3}y^b, 2x^{a+3}y^{b+1} + x^{2a+4}y^{2b}.
\end{gathered}$$

Notice the following relations:

$$y^2 + x^{a+1}y^{b+1} + x^{2a+2}y^{2b} = (y - x^{a+1}y^b)^2,$$
$$2x^{a+3}y^{b+1} + x^{2a+4}y^{2b} = -x^{a+3}y^b(y - x^{a+1}y^b).$$

Therefore we deduce that $\operatorname{Ker} D$ is a $k[x^3, y^3]$-module generated by 1, $y - x^{a+1}y^b$ and $(y - x^{a+1}y^b)^2$, and hence

$$\operatorname{Ker} D = k[x^3, y^3, y - x^{a+1}y^b].$$

In the other cases, we can calculate similarly, and the results are as follows:

- when $a \equiv 0$ and $b \equiv 2$;
  $\operatorname{Ker} D = k[x^3, y^3, y - x^{a+1}y^b + x^{2a+2}y^{2b-1}, y^2 + x^{a+1}y^{b+1}]$,
- when $a \equiv 1$ and $b \equiv 0$;
  $\operatorname{Ker} D = k[x^3, y^3, y + x^{a+1}y^b]$,
- when $a \equiv 1$ and $b \equiv 2$;
  $\operatorname{Ker} D = k[x^3, y^3, y + x^{a+1}y^b + x^{2a+2}y^{2b-1}, y^2 - x^{a+1}y^{b+1}]$.

We remark that when $a \equiv 0$ and $b \equiv 2$ or when $a \equiv 1$ and $b \equiv 2$, we see

$$(y^2 \pm x^{a+1}y^{b+1})^2 = y^3(y \mp x^{a+1}y^b + x^{2a+2}y^{2b-1}).$$

So, as a ring, we need two generators over $k[x^3, y^3]$, but, as a field, we need one generator over $k(x^3, y^3)$.

(ii) In the other cases, namely $a \equiv 2$ or $b \equiv 1$, we see $\operatorname{Ker} D^3 = k[x, y^3]$, which is a $k[x^3, y^3]$-module generated by 1, x and x^2. We have $E_{-x}D(1) = 1$ and $E_{-x}D(x) = E_{-x}D(x^2) = 0$. Hence we deduce that $\operatorname{Ker} D = k[x^3, y^3]$.

EXAMPLE 4.2. In this example, we assume that $p = 2$ and write $\alpha \equiv \beta$ for $\alpha \equiv \beta \pmod 2$. We take a k-derivation D of the polynomial ring $k[x, y]$ as

$$D := x^a \frac{\partial}{\partial x} + y^b \frac{\partial}{\partial y}, \qquad (a, b \in \mathbf{Z}_{\geq 0}).$$

Let us calculate $\operatorname{Ker} D$. Put $D_1 := \frac{1}{x^a} D = \frac{\partial}{\partial x} + \frac{y^b}{x^a}\frac{\partial}{\partial y}$, then D_1 has a slice x. We know

$$D_1^2 = \frac{by^{2b-1} - ax^{a-1}y^b}{x^{2a}} \frac{\partial}{\partial y}.$$

The numerator $by^{2b-1} - ax^{a-1}y^b$ is 0 if and only if $a \equiv b \equiv 0$ or $a = b = 1$.

(i) When $a \equiv b \equiv 0$, then $D_1^2 = 0$. We have $E_{-x}D_1(1) = 1$, $E_{-x}D_1(x) = 0$, $E_{-x}D_1(y) = \frac{x^a y - xy^b}{x^a}$ and $E_{-x}D_1(xy) = -\frac{y^b}{x^{a-2}}$. So we see that $\operatorname{Ker} D$ is the intersection of $k[x, y]$ and the $k[x^2, y^2]$-module generated by 1, $\frac{x^a y - xy^b}{x^a}$ and $-\frac{y^b}{x^{a-2}}$. Therefore, for $f \in \operatorname{Ker} D$, we can write

$$f = \alpha + \beta\frac{x^a y - xy^b}{x^a} + \gamma\frac{-y^b}{x^{a-2}}$$

for some $\alpha, \beta, \gamma \in k[x^2, y^2]$, and

$$f = c_0 + c_1 x + c_2 y + c_3 xy$$

for some $c_0, c_1, c_2, c_3 \in k[x^2, y^2]$. We get

$$x^a(c_0 + c_1 x + c_2 y + c_3 xy) = x^a f = x^a\alpha + \beta(x^a y - xy^b) - \gamma x^2 y^b \in k[x, y],$$

and we obtain that $x^a c_0 = x^a\alpha - \gamma x^2 y^b$ and $x^a c_1 = -\beta y^b$ since $x^a, y^b \in k[x^2, y^2]$. So β is divisible by x^a. Put $\tilde{\beta} \in k[x^2, y^2]$ such that $\beta = \tilde{\beta} x^a$. We have

$$f = \frac{\alpha x^{a-2} - \gamma y^b}{x^{a-2}} + \frac{\beta}{x^a}(x^a y - xy^b) = c_0 + \tilde{\beta}(x^a y - xy^b).$$

Therefore we deduce that $\operatorname{Ker} D$ is a $k[x^2, y^2]$-module generated by 1 and $x^a y - xy^b$, hence $\operatorname{Ker} D = k[x^2, y^2, x^a y - xy^b]$.

(ii) When $a = b = 1$, then we see $D_1^2 = 0$. We have $E_{-x}D_1(1) = 1$, $E_{-x}D_1(x) = E_{-x}D_1(y) = 0$ and $E_{-x}D_1(xy) = -xy$. Hence, we deduce that $\operatorname{Ker} D = k[x^2, y^2, xy]$, similarly to (i).

(iii) In the other cases, we see $\operatorname{Ker} D_1^2 = k[x, y^2]$. Since $E_{-x}D_1(1) = 1$ and $E_{-x}D_1(x) = 0$, we deduce that $\operatorname{Ker} D = k[x^2, y^2]$.

Example 4.3. We treat the k-derivation

$$D := \frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}$$

of a polynomial ring $k[x_1, x_2, x_3, x_4]$.

Before considering the positive characteristic case, we calculate the kernel of D when $\operatorname{char} k = 0$, using [6], [7, Corollary 1.3.23] or Proposition 2.1.

The derivation D has a slice x_1. For the map φ_{-x_1} in Proposition 2.1 (1), we have

$$\begin{aligned}
\varphi_{-x_1}(x_1) &= 0,\\
\varphi_{-x_1}(x_2) &= x_2 - \frac{1}{2}x_1^2,\\
\varphi_{-x_1}(x_3) &= x_3 - x_1x_2 + \frac{1}{3}x_1^3,\\
\varphi_{-x_1}(x_4) &= x_4 - x_1x_3 + \frac{1}{2}x_1^2x_2 - \frac{1}{8}x_1^4.
\end{aligned}$$

Hence we obtain

$$\mathrm{Ker}\, D = k[2x_2 - x_1^2, 3x_3 - 3x_1x_2 + x_1^3, 8x_4 - 8x_1x_3 + 4x_1^2x_2 - x_1^4].$$

Now, we calculate the Ker D for $p = 2, 3$.

(i) We assume that $p = 2$.

We see $D^2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_4}$ and $D^4 = \frac{\partial}{\partial x_4}$, and we have

$$\mathrm{Ker}\, D^4 = k[x_1, x_2, x_3, x_4^2] = \sum_{i_1, i_2, i_3 = 0, 1} k[x_1^2, x_2^2, x_3^2, x_4^2] x_1^{i_1} x_2^{i_2} x_3^{i_3}.$$

Using Ker D^4, we see that Ker D^2 is a $k[x_1^2, x_2^2, x_3^2, x_4^2]$-module generated by 1, x_1, $x_3 - x_1x_2$, $x_1(x_3 - x_1x_2)$, since we have $E_{-x_2}D^2(x_3) = x_3 - x_1x_2$, $E_{-x_2}D^2(x_1x_3) = x_1(x_3 - x_1x_2)$ and so on.

We calculate $E_{-x_1}D(1) = 1$, $E_{-x_1}D(x_1) = 0$, $E_{-x_1}D(x_3 - x_1x_2) = x_3 - x_1x_2 + x_1^3$ and $E_{-x_1}D(x_1(x_3 - x_1x_2)) = -x_1^4$. Therefore we deduce that

$$\mathrm{Ker}\, D = k[x_1^2, x_2^2, x_3^2, x_4^2, x_3 - x_1x_2 + x_1^3].$$

To compare with the characteristic zero case, we have the following calculations;

$$2x_2 - x_1^2 \equiv x_1^2,$$

$$3x_3 - 3x_1x_2 + x_1^3 \equiv x_3 - x_1x_2 + x_1^3,$$

$$8x_4 - 8x_1x_3 + 4x_1^2x_2 - x_1^4 \equiv x_1^4.$$

(ii) We assume that $p = 3$.

We have $D^3 = \frac{\partial}{\partial x_3} + x_1 \frac{\partial}{\partial x_4}$ and $D^9 = 0$.

The k-derivation D^3 has a slice x_3. So we see that Ker D^3 is a $k[x_1^3, x_2^3, x_3^3, x_4^3]$-module generated by $\{x_1^{i_1} x_2^{i_2} (x_4 - x_1x_3)^{i_3} \mid 0 \le i_1, i_2, i_3 \le 2\}$, since $E_{-x_3}D^3(x_4) = x_4 - x_1x_3$ and so on.

Going on to Ker D, we see

$$E_{-x_1}D(x_2) = x_2 + x_1^2,$$

$$E_{-x_1}D(x_4 - x_1x_3) = x_4 + 2x_1x_3 + 2x_1^2x_2 + x_1^4,$$

and so on. So we have that Ker D is a $k[x_1^3, x_2^3, x_3^3, x_4^3]$-module generated by

$$\{(x_2 + x_1^2)^{i_1} (x_4 + 2x_1x_3 + 2x_1^2x_2 + x_1^4)^{i_2} \mid 0 \le i_1, i_2 \le 2\},$$

hence we deduce that

$$\mathrm{Ker}\, D = k[x_1^3, x_2^3, x_3^3, x_4^3, x_2 + x_1^2, x_4 + 2x_1x_3 + 2x_1^2x_2 + x_1^4].$$

To compare with the characteristic zero case, we have the following calculations;

$$2x_2 - x_1^2 \equiv 2(x_2 + x_1^2),$$

$$3x_3 - 3x_1x_2 + x_1^3 \equiv x_1^3,$$

$$8x_4 - 8x_1x_3 + 4x_1^2x_2 - x_1^4 \equiv 2(x_4 + 2x_1x_3 + 2x_1^2x_2 + x_1^4).$$

Example 4.4. We treat $D := x_1 \frac{\partial}{\partial x_1} + x_2^2 \frac{\partial}{\partial x_2} + x_3^3 \frac{\partial}{\partial x_3}$, a k-derivation of the polynomial ring $k[x_1, x_2, x_3]$.

We calculate the $\operatorname{Ker} D$ for $p = 2, 3$.

(i) We assume that $p = 2$.

We set

$$D_1 := \frac{1}{x_3^3} D = \frac{x_1}{x_3^3} \frac{\partial}{\partial x_1} + \frac{x_2^2}{x_3^3} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}.$$

Then we have

$$D_1^2 = \frac{x_1(1 + x_3^2)}{x_3^6} \frac{\partial}{\partial x_1} + \frac{x_2^2}{x_3^4} \frac{\partial}{\partial x_2}.$$

In addition, we set

$$D_2 := \frac{x_3^4}{x_2^2} D_1^2 = \frac{x_1(1 + x_3^2)}{x_2^2 x_3^2} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2},$$

a k-derivation of $k[x_1, x_2, x_3]$. Then we get

$$D_2^2 = \frac{x_1(1 + x_3^2)^2}{x_2^4 x_3^4} \frac{\partial}{\partial x_1},$$

so we see that $\operatorname{Ker} D_2^2$ is $k[x_1^2, x_2, x_3]$, which is a $k[x_1^2, x_2^2, x_3^2]$-module generated by 1, x_2, x_3 and x_2x_3.

We have $E_{-x_2}D_2(1) = 1$, $E_{-x_2}D_2(x_3) = x_3$ and $E_{-x_2}D_2(x_2) = E_{-x_2}D_2(x_2x_3) = 0$. So we obtain that $\operatorname{Ker} D_1^2$ is a $k[x_1^2, x_2^2, x_3^2]$-module generated by 1 and x_3.

We see that $E_{-x_3}D_1(1) = 1$ and $E_{-x_3}D_1(x_3) = 0$, hence we deduce that $\operatorname{Ker} D = k[x_1^2, x_2^2, x_3^2]$.

(ii) We assume that $p = 3$.

We consider, as in the case $p = 2$,

$$D_1 := \frac{1}{x_3^3} D = \frac{x_1}{x_3^3} \frac{\partial}{\partial x_1} + \frac{x_2^2}{x_3^3} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}.$$

Then we have $D_1^3 = \frac{x_1}{x_3^9} \frac{\partial}{\partial x_1}$ and we see that $\operatorname{Ker} D_1^3$ is $k[x_1^3, x_2, x_3]$, which is a $k[x_1^3, x_2^3, x_3^3]$-module generated by $\{x_2^{i_2} x_3^{i_3} \mid 0 \leq i_2, i_3 \leq 2\}$.

We have

$$E_{-x_3}D_1(x_2) = \frac{x_2x_3^6 + 2x_2^2x_3^4 + x_2^3x_3^2}{x_3^6}, \qquad E_{-x_3}D_1(x_2^2) = \frac{x_2^2x_3^3 + x_2^3x_3}{x_3^3}$$

and so on. So we obtain that Ker D is a $k[x_1^2, x_2^2, x_3^2]$-module generated by 1, $x_2^2x_3^3 + x_2^3x_3$ and $x_2x_3^6 + 2x_2^2x_3^4 + x_2^3x_3^2$, hence we deduce that

$$\text{Ker } D = k[x_1^3, x_2^3, x_3^3, x_2^2x_3^3 + x_2^3x_3, x_2x_3^6 + 2x_2^2x_3^4 + x_2^3x_3^2].$$

Example 4.5. In this example, we assume that $p = 2$.
We consider D, the k-derivation of the polynomial ring $k[x, s, t, u, v]$,

$$D := x^3\frac{\partial}{\partial s} + s\frac{\partial}{\partial t} + t\frac{\partial}{\partial u} + x^2\frac{\partial}{\partial v}.$$

When char $k = 0$, this D is an example of a k-derivation whose kernel is not a finitely generated k-algebra (Daigle and Freudenburg [4, Example 3.2]).
Set

$$D_1 := \frac{1}{x^2}D = x\frac{\partial}{\partial s} + \frac{s}{x^2}\frac{\partial}{\partial t} + \frac{t}{x^2}\frac{\partial}{\partial u} + \frac{\partial}{\partial v}.$$

Then we have $D_1^2 = \frac{1}{x}\frac{\partial}{\partial t} + \frac{s}{x^4}\frac{\partial}{\partial u}$.
In addition, we set

$$D_2 := xD_1^2 = \frac{\partial}{\partial t} + \frac{s}{x^3}\frac{\partial}{\partial u}.$$

We see that $D_2^2 = 0$ and that D_2 has a slice t. We have the following calculations; $E_{-t}D_2(1) = 1$, $E_{-t}D_2(t) = 0$, $E_{-t}D_2(u) = u - \frac{st}{x^3}$ and $E_{-t}D_2(tu) = \frac{st^2}{x^3}$, and so on. So we see that Ker D_1^2 is a $k[x^2, s^2, t^2, u^2, v^2]$-module generated by

$$\{x^{i_1}s^{i_2}v^{i_3}(x^3u - ts)^{i_4} \mid i_1, i_2, i_3, i_4 = 0, 1\}.$$

Furthermore, we have

$$E_{-v}D_1(s) = s + xv,$$

$$E_{-v}D_1(x^3u - st) = \frac{x^5u + x^2st + s^2v}{x^2},$$

$$E_{-v}D_1(s(x^3u - ts)) = \frac{(s + xv)(x^2st + s^2v + x^5u) - xs^2v^2}{x^2},$$

and so on. We obtain that Ker D_1 is a $k[x^2, s^2, t^2, u^2, v^2]$-module generated by

$$\{x^{i_1}(s + xv)^{i_2}(x^2st + s^2v + x^5u)^{i_3} \mid i_1, i_2, i_3 = 0, 1\},$$

hence Ker $D = k[x, s^2, t^2, u^2, v^2, s + xv, x^2st + s^2v + x^5u]$.

When char $k = 0$, Ker D is contained in $k + (x, s, t, u) \subset k[x, s, t, u, v]$, and contains elements of the form $xv^m +$ (lower-degree v-terms) for each $m \geq 1$, which implies that Ker D is not a finitely generated k-algebra ([4, Example 3.2], [8, §3], [10]). When $p = \text{char}\, k > 0$, then v^p is in Ker D, and the above argument fails.

Example 4.6. In this example, we assume that $p = 2$. We consider a finitely generated k-domain $R := k[X, Y]/(X^3 + Y^3 + 1) = k[x, y]$, where $k[X, Y]$ is the polynomial ring and x and y are images of X and Y in R respectively.

We can define the k-derivation $D \in \text{Der}_k R$ such that $D(x) = y^2$ and $D(y) = x^2$. We calculate the kernel of D.

We consider $\frac{1}{y^2} D$, the derivation of R, which has a slice x and which satisfies $\left(\frac{1}{y^2} D\right)^2 (x) = \left(\frac{1}{y^2} D\right)^2 (y) = 0$. So we see $\text{Ker}\left(\frac{1}{y^2} D\right)^2 = R$. We calculate as follows: $E_{-x}\left(\frac{1}{y^2} D\right)(1) = 1$, $E_{-x}\left(\frac{1}{y^2} D\right)(x) = 0$, $E_{-x}\left(\frac{1}{y^2} D\right)(y) = \frac{1}{y^2}$ and $E_{-x}\left(\frac{1}{y^2} D\right)(xy) = \frac{x^4}{y^2}$. Therefore,

$$\text{Ker}\left(\frac{1}{y^2} D\right) = \left(k[x^2, y^2]1 + k[x^2, y^2]\frac{1}{y^2} + k[x^2, y^2]\frac{x^4}{y^2}\right) \cap R = k[x^2, y^2].$$

Hence we deduce that Ker $D = k[x^2, y^2]$.

5. On higher derivations

The spirit of the Essen formula (Proposition 2.1) can be translated into positive characteristic, more straightforward, if we use the higher derivations. This section is a fruit of the referee's comment for the first draft of this paper.

Throughout this section, R is a finitely generated commutative k-domain.

Definition 5.1 ([12, 1.1]). A locally finite higher derivation *on R is a set of k-linear endomorphisms $\mathcal{D} = \{D_0, D_1, D_2, \dots\}$ of the k-vector space R satisfying the following conditions:*

(1) *The morphism D_0 is an identity, and for any integer $i \in \mathbf{N}$, $D_i(ab) = \sum_{j+l=i} D_j(a) D_l(b)$ for any $a, b \in R$.*
(2) *For any element a of R, there exists an integer $n \in \mathbf{N}$ such that $D_m(a) = 0$ for every integer $m \geq n$.*

Definition 5.2. *The kernel of a locally finite higher derivation $\mathcal{D}$ is the set $\{f \in R \mid D_i(f) = 0 \ (i \geq 1)\}$. We denote it by* Ker $\mathcal{D}$.

Definition 5.3. *An element $s \in R$ is called a slice of $\mathcal{D}$, a locally finite higher derivation, if $D_1(s) = 1$ and $D_i(s) = 0$ for every $i \geq 2$.*

Definition 5.4 ([12, 1.1]). *The locally finite higher derivation $\mathscr{D}$ on R is called* iterative *if $\mathscr{D}$ satisfies the additional condition:*

(1) $D_i D_j = \binom{i+j}{i} D_{i+j}$ *for all* $i, j \geq 0$.

Remark 5.4.1 ([12, 1.1]). Let $\mathscr{D}$ be iterative.

(1) When the characteristic $p = 0$, $D_i = \frac{1}{i!}(D_1)^i$ for every $i > 0$.

(2) When the characteristic $p > 0$, $D_i = \frac{(D_1)^{i_0}(D_p)^{i_1}\dots(D_{p^r})^{i_r}}{(i_0)!(i_1)!\dots(i_r)!}$ for every $i > 0$, where $i = i_0 + i_1 p + \cdots + i_r p^r$ is a p-adic expansion of i. In particular, $\operatorname{Ker} \mathscr{D} = \{f \in R \mid D_{p^l}(f) = 0 \ \ (l \geq 0)\}$.

Remark 5.4.2 ([5, §1], [12, 1.2]). The locally finite higher derivation $\mathscr{D}$ on R leads to a map $\sigma : R \to R[t]$ given by $\sigma(a) = \sum_{i \geq 0} D_i(a) t^i$, where t is an indeterminate. By the conditions in Definition 5.1, the map σ is a ring homomorphism. Moreover, when $\mathscr{D}$ is iterative, the map σ induces a $\mathbf{G}_a$-action on an affine algebraic variety Spec R, which is an algebraic group action of the additive group $\mathbf{G}_a$ of the field k.

By this correspondence, giving a locally finite iterative higher derivation $\mathscr{D}$ is equivalent to giving a $\mathbf{G}_a$-action on Spec R.

Definition 5.5. *Let $\mathscr{D}$ be a locally finite higher derivation of R and $a \in R$. We define a homomorphism $E_a\mathscr{D} : R \to R$ by*

$$E_a\mathscr{D}(f) = \sum_{l=0}^{\infty} D_l(f) a^l.$$

Remark 5.5.1. In the following, we will use only $E_{-s}\mathscr{D}$ with a slice s, as in section 2.

Proposition 5.6. *The homomorphism $E_a\mathscr{D}$ in Definition 5.5 is a ring homomorphism.*

Proof. Follows immediately from Definition 5.1, and left to the reader. □

Proposition 5.7. *We assume $p > 0$. Let $\mathscr{D}$ be a locally finite iterative higher derivation of R. We assume that $\mathscr{D}$ has a slice $s \in R$. Then* $\operatorname{Ker} \mathscr{D}$ *is the image of $E_{-s}\mathscr{D}$. In particular, when $R = k[a_1, \dots, a_n]$, we have*

$$\operatorname{Ker} \mathscr{D} = \operatorname{Im} E_{-s}\mathscr{D} = k[E_{-s}\mathscr{D}(a_1), \dots, E_{-s}\mathscr{D}(a_n)].$$

Remark 5.7.1. When we handle the kernel of a derivation or of a higher derivation in the zero characteristic case, we can extend the proposition by mechanically making a slice. But, for a higher derivation in the positive characteristic case, it is difficult to extend the proposition, because we cannot make a slice easily.

Lemma 5.7.2. *We have* $D_j((-s)^n) = \binom{n}{j}(-1)^n s^{n-j}$ *for* $n, j \in \mathbf{N}$, *where* $\binom{n}{j} = 0$ *for* $j > n$.

Proof. We use the ring homomorphism $\sigma : R \to R[t]$ of Remark 5.4.2. We have $\sigma((-s)^n) = (\sigma(-s))^n = (-s-t)^n$, since s is a slice of $\mathscr{D}$. So, because $\sigma((-s)^n) = \sum_{i \geq 0} D_i((-s)^n)t^i$ and t is an indeterminant, we deduce that

$$D_j((-s)^n) = \binom{n}{j}(-1)^j(-s)^{n-j} = \binom{n}{j}(-1)^n s^{n-j}. \qquad \square$$

Proof of Proposition 5.7. When $f \in \operatorname{Ker} \mathscr{D}$, then $f = E_{-s}\mathscr{D}(f) \in \operatorname{Im} E_{-s}\mathscr{D}$.

Conversely, we assume that $f = E_{-s}\mathscr{D}(g) \in \operatorname{Im} E_{-s}\mathscr{D}$, for some $g \in R$. By Remark 5.4.1, we only have to show that $D_{p^l}(E_{-s}\mathscr{D}(g)) = 0$ for every $l \geq 0$. In fact, using Lemma 5.7.2, we have

$$\begin{aligned}
&D_{p^l}(E_{-s}\mathscr{D}(g)) \\
&\quad = D_{p^l}\left(\sum_{n=0}^{\infty} D_n(g)(-s)^n\right) \\
&\quad = \sum_{n=0}^{\infty}\sum_{i+j=p^l} D_i(D_n(g))D_j((-s)^n) \\
&\quad = \sum_{n=0}^{\infty}\sum_{i+j=p^l} \binom{n+i}{i}\binom{n}{j}(-1)^n D_{n+i}(g)s^{n-j} \\
&\quad = \sum_{n=0}^{\infty}\left(\binom{n}{0}\binom{n}{p^l}(-1)^n D_n(g)s^{n-p^l} + \binom{n+p^l}{p^l}\binom{n}{0}(-1)^n D_{n+p^l}(g)s^n\right) \\
&\quad = \sum_{n=p^l}^{\infty}\binom{n}{p^l}(-1)^n D_n(g)s^{n-p^l} + \sum_{n=0}^{\infty}\binom{n+p^l}{p^l}(-1)^n D_{n+p^l}(g)s^n \\
&\quad = \sum_{n=p^l}^{\infty}((-1)^n + (-1)^{n-p^l})\binom{n}{p^l}D_n(g)s^{n-p^l} \\
&\quad = 0. \qquad \square
\end{aligned}$$

Appendix A. List—kernels of monomial derivations in two variables

In this section, we give a complete list of the kernel of monomial derivations of $k[x, y]$, the polynomial ring in two variables, in $p = 2$ and 3. A

k-derivation $D \in \mathrm{Der}_k k[x, y]$ is called a monomial k-derivation when $D = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}$ for some monomials $a, b \in k[x, y]$ with coefficients 1.

If a monomial derivation $D \neq 0$ satisfies $D(x) = 0$, then $\mathrm{Ker}\, D = k[x, y^p]$. So we treat a monomial derivation D with $D(x), D(y) \neq 0$. We note that such k-derivations are classified into three types $\frac{\partial}{\partial x} + x^a y^b \frac{\partial}{\partial y}$, $x^a \frac{\partial}{\partial x} + y^b \frac{\partial}{\partial y}$ and $y^a \frac{\partial}{\partial x} + x^b \frac{\partial}{\partial y}$, up to multiplication by monomials.

A.1. In case of $\mathrm{char}\, k = 2$**.** In this subsection, $\alpha \equiv \beta$ means $\alpha \equiv \beta \pmod 2$.

Example A.1. $D = \frac{\partial}{\partial x} + x^a y^b \frac{\partial}{\partial y}$ $(a, b \in \mathbf{Z}_{\geq 0})$

case	
$a \equiv 0,\ b \equiv 0$	$\mathrm{Ker}\, D = k[x^2, y^2, y + x^{a+1} y^b]$
$a \equiv 0,\ b \equiv 1$	$\mathrm{Ker}\, D = k[x^2, y^2]$
$a \equiv 1,\ b \equiv 0$	$\mathrm{Ker}\, D = k[x^2, y^2]$
$a \equiv 1,\ b \equiv 1$	$\mathrm{Ker}\, D = k[x^2, y^2]$

Example A.2. $D = x^a \frac{\partial}{\partial x} + y^b \frac{\partial}{\partial y}$ $(a, b \in \mathbf{Z}_{\geq 0})$

case	
$a \equiv 0,\ b \equiv 0$	$\mathrm{Ker}\, D = k[x^2, y^2, x^a y + x y^b]$
$a \equiv 0,\ b \equiv 1$	$\mathrm{Ker}\, D = k[x^2, y^2]$
$a \equiv 1,\ b \equiv 0$	$\mathrm{Ker}\, D = k[x^2, y^2]$
$a \equiv 1,\ b \equiv 1$ $(a, b) \neq (1, 1)$	$\mathrm{Ker}\, D = k[x^2, y^2]$
$(a, b) = (1, 1)$	$\mathrm{Ker}\, D = k[x^2, y^2, xy]$

Example A.3. $D = y^a \frac{\partial}{\partial x} + x^b \frac{\partial}{\partial y}$ $(a, b \in \mathbf{Z}_{\geq 0})$

case	
$a \equiv 0,\ b \equiv 0$	$\mathrm{Ker}\, D = k[x^2, y^2, x^{b+1} + y^{a+1}]$
$a \equiv 0,\ b \equiv 1$	$\mathrm{Ker}\, D = k[x^2, y^2]$
$a \equiv 1,\ b \equiv 0$	$\mathrm{Ker}\, D = k[x^2, y^2]$
$a \equiv 1,\ b \equiv 1$	$\mathrm{Ker}\, D = k[x^2, y^2]$

A.2. In case of char $k = 3$. In this subsection, $\alpha \equiv \beta$ means $\alpha \equiv \beta \pmod 3$.

EXAMPLE A.4. $D = \frac{\partial}{\partial x} + x^a y^b \frac{\partial}{\partial y}$ $(a, b \in \mathbf{Z}_{\geq 0})$

case	
$a \equiv 0,\ b \equiv 0$	$\operatorname{Ker} D = k[x^3, y^3, y - x^{a+1}y^b]$
$a \equiv 0,\ b \equiv 1$	$\operatorname{Ker} D = k[x^3, y^3]$
$a \equiv 0,\ b \equiv 2$	$\operatorname{Ker} D = k[x^3, y^3, y - x^{a+1}y^b + x^{2a+2}y^{2b-1}, y^2 + x^{a+1}y^{b+1}]$
$a \equiv 1,\ b \equiv 0$	$\operatorname{Ker} D = k[x^3, y^3, y + x^{a+1}y^b]$
$a \equiv 1,\ b \equiv 1$	$\operatorname{Ker} D = k[x^3, y^3]$
$a \equiv 1,\ b \equiv 2$	$\operatorname{Ker} D = k[x^3, y^3, y + x^{a+1}y^b + x^{2a+2}y^{2b-1}, y^2 - x^{a+1}y^{b+1}]$
$a \equiv 2,\ b \equiv 0$	$\operatorname{Ker} D = k[x^3, y^3]$
$a \equiv 2,\ b \equiv 1$	$\operatorname{Ker} D = k[x^3, y^3]$
$a \equiv 2,\ b \equiv 2$	$\operatorname{Ker} D = k[x^3, y^3]$

EXAMPLE A.5. $D = x^a \frac{\partial}{\partial x} + y^b \frac{\partial}{\partial y}$ $(a, b \in \mathbf{Z}_{\geq 0})$

case	
$a \equiv 0,\ b \equiv 0$	$\operatorname{Ker} D = k[x^3, y^3, xy^b - x^a y]$
$a \equiv 0,\ b \equiv 1$	$\operatorname{Ker} D = k[x^3, y^3]$
$a \equiv 0,\ b \equiv 2$	$\operatorname{Ker} D = k[x^3, y^3, xy^{b+1} + x^a y^2, x^2 y^{2b-1} - x^{a+1}y^b + x^{2a}y]$
$a \equiv 1,\ b \equiv 0$	$\operatorname{Ker} D = k[x^3, y^3]$
$a \equiv 1,\ b \equiv 1$ $(a, b) \neq (1, 1)$	$\operatorname{Ker} D = k[x^3, y^3]$
$(a, b) = (1, 1)$	$\operatorname{Ker} D = k[x^3, y^3, x^2 y, xy^2]$
$a \equiv 1,\ b \equiv 2$	$\operatorname{Ker} D = k[x^3, y^3]$
$a \equiv 2,\ b \equiv 0$	$\operatorname{Ker} D = k[x^3, y^3, x^2 y^b + x^{a+1}y, xy^{2b} - x^a y^{b+1} + x^{2a-1}y^2]$
$a \equiv 2,\ b \equiv 1$	$\operatorname{Ker} D = k[x^3, y^3]$
$a \equiv 2,\ b \equiv 2$	$\operatorname{Ker} D = k[x^3, y^3, x^2 y^{b+1} - x^{a+1}y^2, xy^{2b-1} + x^a y^b + x^{2a-1}y]$

EXAMPLE A.6. $D = y^a \frac{\partial}{\partial x} + x^b \frac{\partial}{\partial y}$ $(a, b \in \mathbf{Z}_{\geq 0})$

case	
$a \equiv 0,\ b \equiv 0$	$\mathrm{Ker}\, D = k[x^3, y^3, y^{a+1} - x^{b+1}]$
$a \equiv 0,\ b \equiv 1$	$\mathrm{Ker}\, D = k[x^3, y^3, y^{a+1} + x^{b+1}]$
$a \equiv 0,\ b \equiv 2$	$\mathrm{Ker}\, D = k[x^3, y^3]$
$a \equiv 1,\ b \equiv 0$	$\mathrm{Ker}\, D = k[x^3, y^3, y^{a+1} + x^{b+1}]$
$a \equiv 1,\ b \equiv 1$	$\mathrm{Ker}\, D = k[x^3, y^3, y^{a+1} - x^{b+1}]$
$a \equiv 1,\ b \equiv 2$	$\mathrm{Ker}\, D = k[x^3, y^3]$
$a \equiv 2,\ b \equiv 0$	$\mathrm{Ker}\, D = k[x^3, y^3]$
$a \equiv 2,\ b \equiv 1$	$\mathrm{Ker}\, D = k[x^3, y^3]$
$a \equiv 2,\ b \equiv 2$	$\mathrm{Ker}\, D = k[x^3, y^3]$

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