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# Radial growth of $C^2$ functions satisfying Bloch type condition

*Dedicated to Professor Makoto Sakai on the occasion of his sixtieth birthday*

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**Abstract.** The aim of this paper is to give a simple proof of results by González-Koskela concerning the radial growth of $C^2$ functions satisfying Bloch type condition. Our results also give generalizations of their results.

## 1. Introduction

Denote by $\mathscr{B}$ the Bloch space of all holomorphic functions $f$ on the unit disk $U$ which satisfy

$$\|f\|_{\mathscr{B}} = |f(0)| + \sup_{z \in U}(1 - |z|^2)|f'(z)| < \infty.$$

The radial growth of Bloch functions was extensively discussed by Clunie-MacGregor [2], Korenblum [4], Makarov [5] and Pommerenke [7]. The law of the iterated logarithm of Makarov [5] states that if $f \in \mathscr{B}$, then

$$\limsup_{r \to 1} \frac{|f(r\zeta)|}{\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}}} \le C\|f\|_{\mathscr{B}} \tag{1}$$

for almost every $\zeta \in \partial U$, where $C$ is a universal constant. Pommerenke [7] proved that this inequality is true for $C = 1$ and this inequality is false for $C \le 0.685$. Recently, González and Koskela studied the radial growth of $C^2$ functions on the unit ball $\mathbf{B}^n$ of $\mathbf{R}^n$ which satisfy

$$|\nabla u(x)|^2 + |u(x)\Delta u(x)| \le \frac{c}{(1 - |x|)^2 \left(\log \frac{2}{1-|x|}\right)^{\gamma}} \tag{2}$$

for all $x \in \mathbf{B}^n$, where $c > 0$ and $\gamma \le 1$. They showed the following result ([3, Theorem 1.2]).

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THEOREM A. *Let $u$ be a $C^2$ function on $\mathbf{B}^n$ satisfying* (2). *Then, for almost all $\zeta$, $|\zeta| = 1$,*

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\left(\log \frac{1}{1-r}\right)^{1-\gamma} \log\log \frac{1}{1-r}}} \le c_1$$

*if $\gamma < 1$; and*

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\log\log \frac{1}{1-r}} \le c_2$$

*if $\gamma = 1$. Here the constants $c_1$ and $c_2$ depend only on $n, c, \gamma$.*

We denote by $B(x,r)$ and $S(x,r)$ the open ball and the sphere of center $x$ and radius $r$, respectively. We set $\mathbf{B}^n = B(0,1)$ and $\mathbf{S}^{n-1} = S(0,1)$. The Hausdorff measure with a measure function $h$ is written as $\mathscr{H}_h$. In case $h(r) = r^\alpha$, we write $\mathscr{H}_\alpha$ for $\mathscr{H}_h$.

Our first aim in the present note is to extend Theorem A by González-Koskela. For this purpose, let $\varphi$ be a positive, continuous and non-decreasing function on the interval $[0,1)$ satisfying

$$\varphi(1 - r/2) \le A\varphi(1 - r) \qquad \text{for every } r \in (0,1) \tag{3}$$

with a constant $A \ge 1$ and

$$\int_0^1 (1-t)\varphi(t)dt = \infty. \tag{4}$$

Set

$$\Phi(r) = \int_0^r (1-t)\varphi(t)dt.$$

THEOREM 1. *Let $u$ be a $C^2$ function on $\mathbf{B}^n$ with $u(0) = 0$ such that*

$$\mathscr{A}_u(x) = |\nabla u(x)|^2 + |u(x)\Delta u(x)| \le \varphi(|x|) \qquad \text{for all } x \in \mathbf{B}^n. \tag{5}$$

*Then for $\mathscr{H}_{n-1}$-a.e. $\zeta \in \mathbf{S}^{n-1}$,*

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\Phi(r) \log\log \frac{1}{1-r}}} \le \sqrt{A}.$$

REMARK 1. If we take $\varphi(r) = c(1-r)^{-2}\{\log(2/(1-r))\}^{-\gamma}$ for $c > 0$ and $\gamma \le 1$, then Theorem 1 gives Theorem A.

On the other hand, we have the lower limit result as follows:

THEOREM 2. *If $u$ is as in Theorem 1, then*

$$\liminf_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\Phi(r)\log\log\Phi(r)}} \le 2$$

*for* $\mathscr{H}_{n-1}$*-a.e.* $\zeta \in \mathbf{S}^{n-1}$.

By Theorems 1 and 2, we have the following corollary.

Corollary 1. *Let* $u$ *be a* $C^2$ *function on* $\mathbf{B}^n$ *satisfying*

$$\mathscr{A}_u(x) \le c(1-|x|)^{-2}\left(\log_{(1)}\frac{1}{1-|x|}\right)^{-1}\cdots\left(\log_{(\ell-1)}\frac{1}{1-|x|}\right)^{-1}\left(\log_{(\ell)}\frac{1}{1-|x|}\right)^{-\gamma},$$

*where* $c > 0$, $\gamma \le 1$ *and* $\log_{(k+1)}(t) = \log_{(k)} \circ \log_{(1)}(t)$ *with* $\log_{(1)}(t) = \log(e+t)$. *Then for* $\mathscr{H}_{n-1}$*-a.e.* $\zeta \in \mathbf{S}^{n-1}$,

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\left(\log_{(\ell)}\frac{1}{1-r}\right)^{1-\gamma}\log_{(2)}\frac{1}{1-r}}} \le c_1$$

*and*

$$\liminf_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\left(\log_{(\ell)}\frac{1}{1-r}\right)^{1-\gamma}\log_{(\ell+2)}\frac{1}{1-r}}} \le c_2$$

*when* $\gamma < 1$;

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\log_{(\ell+1)}\frac{1}{1-r}\log_{(2)}\frac{1}{1-r}}} \le c_3$$

*and*

$$\liminf_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\log_{(\ell+1)}\frac{1}{1-r}\log_{(\ell+3)}\frac{1}{1-r}}} \le c_4$$

*when* $\gamma = 1$. *Here* $c_1, c_2, c_3$ *and* $c_4$ *are constants depending only on* $c, \gamma$ *and* $\ell$.

## 2. Exponential integral

In this section, we present an exponential estimate for $C^2$ functions satisfying (5). For this we prepare the following lemma, which is a generalization of [3, Theorem 2.2].

Lemma 1. *Let* $\varphi$ *be a positive continuous function on* $[0,1)$, *and set*

$$\Phi(r) = \int_0^r (1-t)\varphi(t)\,dt.$$

*Let u be a $C^2$ function in $\mathbf{B}^n$ with $u(0) = 0$ which satisfies condition* (5). *Then*

$$\int_{\mathbf{S}^{n-1}} |u(r\zeta)|^{2k} dS(\zeta) \le \sigma_n 4^k k! [\Phi(r)]^k \tag{6}$$

*for all $k \in \{0, 1, 2, \dots\}$ and all $r \in (0, 1)$, where $\sigma_n$ denotes the surface measure of* $\mathbf{S}^{n-1}$.

Proof. First we show that

$$\frac{d}{dt} \int_{\mathbf{S}^{n-1}} v(t\zeta) dS(\zeta) = t^{1-n} \int_{B(0,t)} \Delta v(w) dw \tag{7}$$

for each $v \in C^2(\mathbf{B}^n)$. Using the divergence theorem, we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbf{S}^{n-1}} v(t\zeta) dS(\zeta) &= \int_{\mathbf{S}^{n-1}} \zeta \cdot \nabla v(t\zeta) dS(\zeta) \\ &= t^{1-n} \int_{S(0,t)} \frac{w}{t} \cdot \nabla v(w) dS(w) \\ &= t^{1-n} \int_{B(0,t)} \Delta v(w) dw. \end{aligned}$$

Thus (7) holds.

We prove this lemma by induction on $k$. Clearly, (6) holds for $k = 0$. Suppose that (6) holds for $k$. Using (7) and the assumption on induction, we obtain

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbf{S}^{n-1}} |u(t\zeta)|^{2(k+1)} dS(\zeta) \\ &\quad = 2(k+1) t^{1-n} \int_{B(0,t)} |u(w)|^{2k} (u(w) \Delta u(w) + (2k+1) |\nabla u(w)|^2) dw \\ &\quad \le 4(k+1)^2 t^{1-n} \int_{B(0,t)} |u(w)|^{2k} \mathscr{A}_u(w) dw \\ &\quad \le 4(k+1)^2 t^{1-n} \int_0^t \rho^{n-1} \varphi(\rho) \left( \int_{\mathbf{S}^{n-1}} |u(\rho z)|^{2k} dS(z) \right) d\rho \\ &\quad \le \sigma_n 4^{k+1} k! (k+1)^2 t^{1-n} \int_0^t \rho^{n-1} \varphi(\rho) [\Phi(\rho)]^k d\rho. \end{aligned}$$

Integrating both sides from 0 to $r$ and applying Fubini's theorem, we have

$$\begin{aligned}
\int_{\mathbf{S}^{n-1}} |u(r\zeta)|^{2(k+1)} dS(\zeta) &\le \sigma_n 4^{k+1} k!(k+1)^2 \int_0^r t^{1-n} \int_0^t \rho^{n-1}\varphi(\rho)[\Phi(\rho)]^k d\rho dt \\
&= \sigma_n 4^{k+1} k!(k+1)^2 \int_0^r \left(\int_\rho^r t^{1-n}\, dt\right) \rho^{n-1}\varphi(\rho)[\Phi(\rho)]^k d\rho \\
&\le \sigma_n 4^{k+1}(k+1)! \int_0^r (k+1)(1-\rho)\varphi(\rho)[\Phi(\rho)]^k d\rho \\
&= \sigma_n 4^{k+1}(k+1)! \int_0^r \frac{d}{d\rho}[\Phi(\rho)]^{k+1} d\rho \\
&= \sigma_n 4^{k+1}(k+1)! [\Phi(r)]^{k+1}.
\end{aligned}$$

Hence (6) also holds for $k+1$. The induction is completed.

Lemma 2. *Let $u$ be a function in $\mathbf{B}^n$ satisfying condition* (6). *Then for all $c$, $0 < c < 1/4$, and for all $r$, $0 < r < 1$,*

$$\int_{\mathbf{S}^{n-1}} \exp\left(\frac{c|u(r\zeta)|^2}{\Phi(r)}\right) dS(\zeta) \le \frac{\sigma_n}{1-4c}. \tag{8}$$

Proof. If $k$ is a non-negative integer, then, by (6), we have

$$\frac{1}{k!}\int_{\mathbf{S}^{n-1}} \left(\frac{c|u(r\zeta)|^2}{\Phi(r)}\right)^k dS(\zeta) \le (4c)^k \sigma_n$$

for $c > 0$. Hence it follows that

$$\begin{aligned}
\int_{\mathbf{S}^{n-1}} \exp\left(\frac{c|u(r\zeta)|^2}{\Phi(r)}\right) dS(\zeta) &= \int_{\mathbf{S}^{n-1}} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{c|u(r\zeta)|^2}{\Phi(r)}\right)^k dS(\zeta) \\
&\le \sigma_n \sum_{k=0}^{\infty} (4c)^k.
\end{aligned}$$

The series on the right converges if $0 < c < 1/4$ and thus our lemma is proved.

## 3. Proof of Theorem 1

Let $\varphi$ and $\Phi$ be as in the Introduction, and let $u$ be as in Theorem 1. To prove Theorem 1, we need the following two lemmas.

Lemma 3. *For every $0 < r < 1$,*

$$\Phi(1 - r/2) \le \frac{A}{4}\Phi(1-r) + \Phi(1/2).$$

Lemma 4. *Let $u$ be a $C^2$ function in $\mathbf{B}^n$ such that $|\nabla u(x)|^2 \le \varphi(|x|)$. Then for every $x \in \mathbf{B}^n \setminus \overline{B(0,1/2)}$,*

$$|u(y) - u(z)| \le A[\Phi(|x|)]^{1/2}$$

*whenever $y, z \in B(x, (1-|x|)/2)$.*

Proof. We see that

$$|u(y) - u(z)| \le (1-|x|)[\varphi((1+|x|)/2)]^{1/2} \le A^{1/2}(1-|x|)[\varphi(|x|)]^{1/2}$$

for all $y, z \in B(x, (1-|x|)/2)$. On the other hand, we have for $1/2 < t < 1$,

$$\Phi(t) \ge \int_{2t-1}^{t} (1-s)\varphi(s)ds \ge \varphi(2t-1)\int_{2t-1}^{t}(1-s)ds \ge A^{-1}(1-t)^2\varphi(t).$$

Thus Lemma 4 follows.

Proof of Theorem 1. From Lemma 2, we see that

$$\int_{\mathbf{B}^n} (1-|x|)^{-1}\left(\log \frac{2}{1-|x|}\right)^{-1-\delta} \exp\left(\frac{c|u(x)|^2}{\Phi(|x|)}\right)dx < \infty$$

for all $c$, $0 < c < 1/4$, and all $\delta > 0$. Then there exists a set $E \subset \mathbf{S}^{n-1}$ such that $\mathscr{H}_{n-1}(E) = 0$ and

$$\int_0^1 (1-r)^{-1}\left(\log \frac{2}{1-r}\right)^{-1-\delta} \exp\left(\frac{c|u(r\zeta)|^2}{\Phi(r)}\right)dr < \infty,$$

for each $\zeta \in \mathbf{S}^{n-1}\setminus E$, $0 < c < 1/4$ and $\delta > 0$, which implies that

$$\lim_{r\to 1}\int_r^{(1+r)/2} (1-t)^{-1}\left(\log \frac{2}{1-t}\right)^{-1-\delta} \exp\left(\frac{c|u(t\zeta)|^2}{\Phi(t)}\right)dt = 0. \tag{9}$$

Fix $\zeta \in \mathbf{S}^{n-1}\setminus E$. For $0 < r < 1$, define $I_r = [r, (1+r)/2)$. From (9), we obtain

$$\lim_{r\to 1}\left(\log \frac{1}{1-r}\right)^{-1-\delta} \exp\left(\frac{c \inf_{t\in I_r}|u(t\zeta)|^2}{\Phi((1+r)/2)}\right) = 0,$$

which implies that

$$\frac{c \inf_{t\in I_r}|u(t\zeta)|^2}{4^{-1}A\Phi(r) + \Phi(1/2)} \le (1+\delta)\log\log\frac{1}{1-r} \tag{10}$$

for $r$ near 1, by Lemma 3. Hence it follows from (10) and Lemma 4 that

$$\limsup_{r\to 1}\frac{|u(r\zeta)|}{\sqrt{\Phi(r)\log\log\frac{1}{1-r}}} \le \sqrt{\frac{A(1+\delta)}{4c}}.$$

Here, letting $c \to 1/4$ and $\delta \to 0$, we obtain

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\Phi(r)\log\log\frac{1}{1-r}}} \le \sqrt{A},$$

which proves Theorem 1.

## 4. Proof of Theorem 2

In this section we complete the proof of Theorem 2.

By Lemma 2, we see that

$$\int_{\mathbf{B}^n\setminus B(0,r_0)} (1-|x|)\varphi(|x|)\Phi(|x|)^{-1}(\log \Phi(|x|))^{-1-\delta} \exp\left(\frac{c|u(x)|^2}{\Phi(|x|)}\right) dx < \infty$$

for all $c$, $0 < c < 1/4$, and $\delta > 0$, where $r_0 = \Phi^{-1}(e)$. Consequently,

$$\lim_{r\to 1}\int_r^1 (1-t)\varphi(t)\Phi(t)^{-1}(\log \Phi(t))^{-1-\delta}\exp\left(\frac{c|u(t\zeta)|^2}{\Phi(t)}\right) dt = 0$$

for $\mathscr{H}_{n-1}$-a.e. $\zeta \in \mathbf{S}^{n-1}$, $0 < c < 1/4$ and $\delta > 0$. This implies that

$$\lim_{r\to 1}\int_r^1 (1-t)\varphi(t)\Phi(t)^{-1}(\log \Phi(t))^{-1-\delta}\exp\left(\frac{c g_r(\zeta)^2}{\Phi(t)}\right) dt = 0, \tag{11}$$

where $g_r(\zeta) = \inf_{r\le\rho<1}|u(\rho\zeta)|$. Since $e^{\delta x} \ge \delta x$ for $x > 0$, we have

$$\begin{aligned}
&\frac{d}{dt}\left(-(\log \Phi(t))^{-1-\delta}\exp\left(\frac{(1+\delta)^{-1}c g_r(\zeta)^2}{\Phi(t)}\right)\right)\\
&\quad = (1+\delta)(1-t)\varphi(t)\Phi(t)^{-1}(\log \Phi(t))^{-2-\delta}\exp\left(\frac{(1+\delta)^{-1}c g_r(\zeta)^2}{\Phi(t)}\right)\\
&\qquad + (\log \Phi(t))^{-1-\delta}(1-t)\varphi(t)\Phi(t)^{-1}\\
&\qquad \times \left(\frac{(1+\delta)^{-1}c g_r(\zeta)^2}{\Phi(t)}\right)\exp\left(\frac{(1+\delta)^{-1}c g_r(\zeta)^2}{\Phi(t)}\right)\\
&\quad \le (1+\delta+\delta^{-1})(1-t)\varphi(t)\Phi(t)^{-1}(\log \Phi(t))^{-1-\delta}\exp\left(\frac{c g_r(\zeta)^2}{\Phi(t)}\right)
\end{aligned}$$

for $r_0 < t < 1$. From (11), we obtain

$$\lim_{r\to 1}(\log \Phi(r))^{-1-\delta}\exp\left(\frac{(1+\delta)^{-1}c g_r(\zeta)^2}{\Phi(r)}\right) = 0,$$

which implies that

$$\frac{(1+\delta)^{-1}cg_r(\zeta)^2}{\Phi(r)} \le (1+\delta)\log\log\Phi(r)$$

for $r$ near 1. By letting $c \to 1/4$ and $\delta \to 0$, we have

$$\limsup_{r\to 1} \frac{g_r(\zeta)^2}{\Phi(r)\log\log\Phi(r)} \le 4,$$

which completes the proof of Theorem 2.

Corollary 2. *Let $\varphi$ and $\Phi$ be as in the Introduction. Let $u$ be a harmonic function on $\mathbf{B}^n$ satisfying*

$$|\nabla u(x)|^2 \le \varphi(|x|) \qquad \text{for all } x \in \mathbf{B}^n.$$

*Then*

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\Phi(r)\log\log\Phi(r)}} \le 2$$

*for $\mathscr{H}_{n-1}$-a.e. $\zeta \in \mathbf{S}^{n-1}$.*

Proof. Consider the radial maximal function of $u$ defined by

$$\mathscr{R}_u(r,\zeta) = \max_{0\le t\le r} |u(t\zeta)| \qquad \text{for } 0<r<1 \text{ and } \zeta \in \mathbf{S}^{n-1}.$$

By the Hardy-Littlewood maximal theorem [1, Chapter 6] and Lemma 1,

$$\int_{\mathbf{S}^{n-1}} |\mathscr{R}_u(r,\zeta)|^{2k} dS(\zeta) \le c_1 \int_{\mathbf{S}^{n-1}} |u(r\zeta)|^{2k} dS(\zeta) \le c_1\sigma_n 4^k k! [\Phi(r)]^k$$

for all $k$ and $0<r<1$, where $c_1$ is a constant depending only on $n$. As in the proof of Theorem 1, we have

$$\lim_{r\to 1} \int_r^1 (1-t)\varphi(t)\Phi(t)^{-1}(\log\Phi(t))^{-1-\delta} \exp\left(\frac{c_2|\mathscr{R}_u(t,\zeta)|^2}{\Phi(t)}\right) dt = 0$$

for $\mathscr{H}_{n-1}$-a.e. $\zeta \in \mathbf{S}^{n-1}$, all $c_2$, $0<c_2<1/4$, and $\delta>0$. Hence we see as in the proof of Theorem 2 that

$$\limsup_{r\to 1} \frac{\mathscr{R}_u(r,\zeta)}{\sqrt{\Phi(r)\log\log\Phi(r)}} \le 2$$

holds for $\mathscr{H}_{n-1}$-a.e. $\zeta \in \mathbf{S}^{n-1}$. Since $\mathscr{R}_u(r,\zeta) \ge |u(r\zeta)|$, we have

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\Phi(r)\,\log\log\Phi(r)}} \le 2$$

for $\mathscr{H}_{n-1}$-a.e. $\zeta \in \mathbf{S}^{n-1}$, which yields the required conclusion.

REMARK 2. Let $u$ be a harmonic function on $\mathbf{B}^n$ satisfying

$$\|u\| = \sup_{x\in\mathbf{B}^n}(1-|x|)|\nabla u(x)| < \infty.$$

Then Corollary 2 says that

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\sqrt{\log\frac{1}{1-r}\,\log\log\log\frac{1}{1-r}}} \le 2\|u\|$$

for $\mathscr{H}_{n-1}$-a.e. $\zeta \in \mathbf{S}^{n-1}$.

## 5. Hausdorff measures and radial growth

Let $\varphi$ and $\Phi$ be as in the Introduction. Take a positive non-decreasing function $\Psi$ on $[0,1)$ satisfying

$$\frac{\Phi(r)\,\log\log(1/(1-r))}{[\Psi(r)]^2} \to 0 \qquad \text{as } r \to 1.$$

For $\lambda > 0$, consider the measure function

$$h_\lambda(t) = t^{n-1}\exp\left(4^3A^{-4}\lambda^2\frac{[\Psi(1-t)]^2}{\Phi(1-t)}\right).$$

We finally establish the following result.

THEOREM 3. *If $\lambda > 0$ and $u$ is as in Theorem 1, then*

$$\limsup_{r\to 1}\frac{|u(r\zeta)|}{\Psi(r)} \le \lambda$$

*for $\mathscr{H}_{h_\lambda}$-a.e. $\zeta \in \mathbf{S}^{n-1}$.*

PROOF. In view of Lemma 2, we see that

$$\int_{\mathbf{B}^n}(1-|x|)^{-1}\left(\log\frac{2}{1-|x|}\right)^{-2}\exp\left(\frac{c_1|u(x)|^2}{\Phi(|x|)}\right)dx < \infty$$

for all $c_1$, $0 < c_1 < 1/4$. By the covering theorem, there exists a set $F \subset \mathbf{S}^{n-1}$ such that $\mathscr{H}_{h_\lambda}(F) = 0$ and

$$\lim_{t\to 0}[h_\lambda(5t)]^{-1}\int_{B(\zeta,t)\cap\mathbf{B}^n}(1-|x|)^{-1}\left(\log\frac{2}{1-|x|}\right)^{-2}\exp\left(\frac{c_1|u(x)|^2}{\Phi(|x|)}\right)dx=0 \quad (12)$$

for $\zeta\in\mathbf{S}^{n-1}\setminus F$ and $0<c_1<1/4$ (cf. [6, Lemma 5.8.2]).

Fix $\zeta\in\mathbf{S}^{n-1}\setminus F$. For $0<t<1$, write $D_t=B((1-t)\zeta+4^{-1}t\zeta,4^{-1}t)$. Since $D_t\subset B(\zeta,t)\cap\mathbf{B}^n$, we have

$$\begin{aligned}
&[h_\lambda(5t)]^{-1}\int_{B(\zeta,t)\cap\mathbf{B}^n}(1-|x|)^{-1}\left(\log\frac{2}{1-|x|}\right)^{-2}\exp\left(\frac{c_1|u(x)|^2}{\Phi(|x|)}\right)dx\\
&\quad\geq[h_\lambda(5t)]^{-1}\int_{D_t}(1-|x|)^{-1}\left(\log\frac{2}{1-|x|}\right)^{-2}\exp\left(\frac{c_1|u(x)|^2}{\Phi(|x|)}\right)dx\\
&\quad\geq[h_\lambda(5t)]^{-1}|D_t|t^{-1}\left(\log\frac{4}{t}\right)^{-2}\exp\left(\frac{c_1\inf_{x\in D_t}|u(x)|^2}{\Phi(1-t/2)}\right)\\
&\quad\geq c_2\exp\left(-4^3A^{-4}\lambda^2\frac{[\Psi(1-5t)]^2}{\Phi(1-5t)}-2\log\log(1/t)+\frac{c_1\inf_{x\in D_t}|u(x)|^2}{\Phi(1-t/2)}\right),
\end{aligned}$$

where $c_2$ is a positive constant. From (12), we obtain

$$\frac{c_1\inf_{x\in D_t}|u(x)|^2}{\Phi(1-t/2)}\leq 4^3A^{-4}\lambda^2\frac{[\Psi(1-5t)]^2}{\Phi(1-5t)}+2\log\log(1/t)$$

for sufficiently small $t>0$. By Lemma 3, we have

$$\frac{c_1\inf_{x\in D_t}|u(x)|^2}{4^{-1}A\Phi(1-t)+\Phi(1/2)}\leq A^{-1}\lambda^2\frac{\Psi(1-t)^2}{\Phi(1-t)-c_3}+2\log\log(1/t)$$

where $c_3=\Phi(1/2)((A/4)^3+(A/4)^2+(A/4))$, which implies that

$$\limsup_{t\to 0}\frac{\inf_{x\in D_t}|u(x)|^2}{\Psi(1-t)^2}\leq\frac{1}{4c_1}\lambda^2.$$

Hence it follows from Lemma 4 that

$$\limsup_{t\to 0}\frac{|u((1-t)\zeta)|}{\Psi(1-t)}\leq\frac{\lambda}{\sqrt{4c_1}}.$$

Letting $c_1\to 1/4$, we obtain

$$\limsup_{t\to 0}\frac{|u((1-t)\zeta)|}{\Psi(1-t)}\leq\lambda,$$

which proves Theorem 3.

REMARK 3. If we take $\varphi(r) = (1-r)^{-2}\{\log(2/(1-r))\}^{-\gamma}$, $\gamma < 1$, then the conclusion of Theorem 3 says that

$$\limsup_{r\to 1} \frac{|u(r\zeta)|}{\{\log(1/(1-r))\}^{\alpha}} \le \lambda$$

for $\mathscr{H}_h$-a.e. $\zeta \in \mathbf{S}^{n-1}$, where $2\alpha > 1-\gamma$ and

$$h(t) = t^{n-1}\exp[4^3 A^{-4}\lambda^2\{\log(1/t)\}^{2\alpha+\gamma-1}].$$

Thus Theorem 3 can not cover [3, Theorem 1.3] by González-Koskela.

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