

## An umbilical point on a non-real-analytic surface

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**ABSTRACT.** Let  $F$  be a smooth function of two variables which is zero at  $(0, 0)$  and positive on a punctured neighborhood of  $(0, 0)$ . Then the function  $\exp(-1/F)$  is smoothly extended to  $(0, 0)$  and then the origin  $o$  of  $\mathbf{R}^3$  is an umbilical point of its graph. In this paper, we shall study the behavior of the principal distributions around  $o$  on condition that the norm of the gradient vector field of  $\log F$  is bounded from below by a positive constant on a punctured neighborhood of  $(0, 0)$ .

### 1. Introduction

Let  $S$  be a surface in  $\mathbf{R}^3$  and  $p_0$  an isolated umbilical point of  $S$ . Then the *index* of  $p_0$  on  $S$  is defined by the index of  $p_0$  with respect to a principal distribution.

It is known that if  $S$  is a surface with constant mean curvature and if  $S$  is connected and not totally umbilical, then each umbilical point of  $S$  is isolated and its index is negative ([Ho, p139]); if  $S$  is a special Weingarten surface, then the same result is obtained ([HaW]).

It has been expected that the index of an isolated umbilical point on a surface is not more than one. We call this conjecture the *index conjecture*. In relation to the index conjecture, the following two conjectures are known: Carathéodory's conjecture and Loewner's conjecture. *Carathéodory's conjecture* asserts that there exist at least two umbilical points on a compact, strictly convex surface in  $\mathbf{R}^3$ . If the index conjecture is true, then we see from Hopf-Poincaré's theorem that there exist at least two umbilical points on a compact, orientable surface of genus zero, and this immediately gives the affirmative answer to Carathéodory's conjecture. Let  $F$  be a real-valued, smooth function of two real variables  $x, y$ , and set  $\partial_{\bar{z}} := (\partial/\partial x + \sqrt{-1}\partial/\partial y)/2$ . Then *Loewner's conjecture* for a positive integer  $n \in \mathbf{N}$  asserts that if a vector field

$$\operatorname{Re}(\partial_{\bar{z}}^n F) \frac{\partial}{\partial x} + \operatorname{Im}(\partial_{\bar{z}}^n F) \frac{\partial}{\partial y}$$

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has an isolated zero point, then its index with respect to this vector field is not more than  $n$  ([K], [T]). Loewner's conjecture for  $n = 1$  is affirmatively solved; Loewner's conjecture for  $n = 2$  is equivalent to the index conjecture. We may find [B], [GSaB], [SX1], [SX2] and [SX3] as recent papers in relation to Carathéodory's and Loewner's conjectures.

Let  $g$  be a homogeneous polynomial of degree  $k$  in two variables such that on its graph, the origin  $o := (0, 0, 0)$  of  $\mathbf{R}^3$  is an isolated umbilical point. In [A1], we studied the behavior of the principal distributions around  $o$  on the graph of  $g$ . In particular, we showed that the index of  $o$  is an element of  $\{1 - k/2 + i\}_{i=0}^{\lfloor k/2 \rfloor}$ . In [A2], we studied the behavior of the principal distributions around each of an isolated umbilical point of the graph of  $g$  and the point added to the graph by the one-point compactification for the graph.

In [A3], we have studied the behavior of the principal distributions around an isolated umbilical point on a real-analytic surface. Let  $F$  be a real-analytic function on a neighborhood of  $(0, 0)$  satisfying

$$F(0, 0) = \frac{\partial F}{\partial x}(0, 0) = \frac{\partial F}{\partial y}(0, 0) = 0$$

and the condition that  $o$  is an isolated umbilical point of its graph. Then there exists a real number  $a_F \in \mathbf{R}$  satisfying

$$F(x, y) = a_F(x^2 + y^2)/2 + o(x^2 + y^2). \quad (1)$$

Let  $\sigma_F$  be a function defined by

$$\sigma_F := \begin{cases} 0 & \text{if } a_F = 0, \\ \frac{1}{a_F} - \frac{|a_F|}{a_F} \sqrt{\frac{1}{a_F^2} - (x^2 + y^2)} & \text{if } a_F \neq 0. \end{cases}$$

Then we obtain  $F \not\equiv \sigma_F$  and we see that there exist an integer  $k_F \geq 3$  and a nonzero homogeneous polynomial  $g_F$  of degree  $k_F$  such that all the partial derivatives of  $F - \sigma_F - g_F$  of order less than  $k_F + 1$  vanish at  $(0, 0)$ . In [A3], we have proved that if  $o$  is an isolated umbilical point of the graph of  $g_F$ , then the index of  $o$  on the graph of  $F$  is not less than the index of  $o$  on the graph of  $g_F$  and not more than one. In addition, we have proved that if the graph of  $F$  is special Weingarten, then the index of  $o$  is equal to  $1 - k_F/2$ . In [A4], we have described a similar discussion on the graph of a smooth function with such coefficients as the above  $F$  has in Taylor's formula.

On the other hand, there exists a nonconstant smooth function such that its coefficients are not helpful. For example, a smooth function

$$\exp(-1/(x^2 + y^2))$$

defined on  $\mathbf{R}^2 \setminus \{(0,0)\}$  is smoothly extended to  $(0,0)$  and then we see that all the coefficients at  $(0,0)$  vanish and that  $o$  is an umbilical point of its graph.

Let  $l$  denote a positive integer or  $\infty$  and  $\mathcal{C}_o^{(\infty,l)}$  the set of the smooth functions on a connected neighborhood of  $(0,0)$  in  $\mathbf{R}^2$  such that all the coefficients of order less than  $l$  vanish at  $(0,0)$ . The following hold:

$$\mathcal{C}_o^{(\infty,l)} \supset \mathcal{C}_o^{(\infty,l+1)} \supset \mathcal{C}_o^{(\infty,\infty)} \cong \{0\},$$

where  $l \in \mathbf{N}$ . For  $l \in \mathbf{N} \cup \{\infty\}$ , let  $\mathcal{C}_{o,+}^{(\infty,l)}$  be the subset of  $\mathcal{C}_o^{(\infty,l)}$  such that each element of  $\mathcal{C}_{o,+}^{(\infty,l)}$  is positive on a punctured neighborhood of  $(0,0)$ . For a positive number  $a > 0$ , we set

$$E_0(a) := a, \quad E_1(a) := \exp(-1/a).$$

In addition, we set

$$E_v(a) := E_1(E_{v-1}(a))$$

for each  $v \in \mathbf{N}$  inductively. Then for  $F \in \mathcal{C}_{o,+}^{(\infty,l)}$  and  $v \in \mathbf{N}$ , the smooth function  $E_v(F)$  defined on a punctured neighborhood of  $(0,0)$  is smoothly extended to  $(0,0)$  and then we see that  $E_v(F)$  is an element of  $\mathcal{C}_{o,+}^{(\infty,\infty)}$  and that  $o$  is an umbilical point of its graph. The purpose of this paper is to study the behavior of the principal distributions around  $o$  on the graph of  $E_v(F)$  for  $F \in \mathcal{C}_{o,+}^{(\infty,l)}$  such that the norm of the gradient vector field of  $\log F$  is bounded from below by a positive constant on a punctured neighborhood of  $(0,0)$ .

We set

$$|\text{grad}_F| = \sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2}.$$

We shall prove

**THEOREM 1.1.** *Let  $F$  be an element of  $\mathcal{C}_{o,+}^{(\infty,l)}$  such that for each  $c > 0$ , there exists a punctured neighborhood of  $(0,0)$  on which  $|\text{grad}_F|/F > c$  holds. Then  $o$  is an isolated umbilical point with index one on the graph of  $E_v(aF)$  for any  $v \in \mathbf{N}$  and any  $a > 0$ .*

**EXAMPLE 1.2.** For  $l \in \mathbf{N}$ , let  $\mathcal{P}^l$  be the set of the homogeneous polynomials of degree  $l$  in two variables and set  $\mathcal{P}_+^l := \mathcal{P}^l \cap \mathcal{C}_{o,+}^{(\infty,l)}$ . For example,  $x^2 + y^2$  is an element of  $\mathcal{P}_+^2$ . Let  $F$  be an element of  $\mathcal{C}_{o,+}^{(\infty,l)}$  for  $l \in \mathbf{N}$  such that there exists an element  $g \in \mathcal{P}_+^l$  satisfying  $F - g \in \mathcal{C}_o^{(\infty,l+1)}$ . Then for each  $c > 0$ , we may find a punctured neighborhood of  $(0,0)$  on which  $|\text{grad}_F|/F > c$  holds. Therefore we see from Theorem 1.1 that  $o$  is an isolated umbilical point with index one on the graph of  $E_v(aF)$  for any  $v \in \mathbf{N}$  and any  $a > 0$ .

EXAMPLE 1.3. Let  $F$  be an element of  $\mathcal{C}_{o,+}^{(\infty,l)}$  such that its graph is locally strictly convex at each point, where a surface  $S$  is called *locally strictly convex* at a point  $p \in S$  if there exists a punctured neighborhood of  $p$  in  $S$  which does not share any point with the tangent plane at  $p$ . For each  $\theta \in \mathbf{R}$ , we set

$$\Phi_{(F;\theta)}(r) := F(r \cos \theta, r \sin \theta)$$

for each  $r \in (-r_0, r_0)$ , where  $r_0$  is some positive number. Then  $d\Phi_{(F;\theta)}/dr$  is increasing. Therefore for any  $r \in (0, r_0)$ , the following hold:

$$\Phi_{(F;\theta)}(r) = \int_0^r \frac{d\Phi_{(F;\theta)}}{ds}(s) ds < r \frac{d\Phi_{(F;\theta)}}{dr}(r) \leq r |\text{grad}_F(r \cos \theta, r \sin \theta)|.$$

Then we see that for each  $c > 0$ , there exists a punctured neighborhood of  $(0, 0)$  on which  $|\text{grad}_F|/F > c$  holds. Therefore we see that  $o$  is an isolated umbilical point with index one on the graph of  $E_\nu(aF)$  for any  $\nu \in \mathbf{N}$  and any  $a > 0$ .

REMARK 1.4. Our discussions in [A1]~[A4] crucially depend on coefficients of functions; in order to show that the function in Example 1.2 satisfies the assumption in Theorem 1.1, we again depend on coefficients of the function. However, in Example 1.3, we do not depend on coefficients of the function.

EXAMPLE 1.5. Let  $F$  be an element of  $\mathcal{C}_{o,+}^{(\infty,l)}$  such that  $|\text{grad}_F|/F > C_0$  holds for some  $C_0 > 0$  on a punctured neighborhood of  $(0, 0)$ . Then noticing

$$\frac{|\text{grad}_{E_1(aF)}|}{E_1(aF)} = \frac{1}{aF} \times \frac{|\text{grad}_F|}{F}$$

for any  $a > 0$ , we see that for each  $c > 0$ , there exists a punctured neighborhood of  $(0, 0)$  on which  $|\text{grad}_{E_1(aF)}|/E_1(aF) > c$  holds. Therefore we see by Theorem 1.1 that  $o$  is an isolated umbilical point with index one on the graph of  $E_{\nu+1}(aF)$  for any  $\nu \in \mathbf{N}$  and any  $a > 0$ . We also see that Theorem 1.1 for  $\nu = 1$  implies Theorem 1.1 for any  $\nu \in \mathbf{N}$ .

In addition, we shall prove

THEOREM 1.6. Let  $F$  be an element of  $\mathcal{C}_{o,+}^{(\infty,l)}$  such that  $|\text{grad}_F|/F > C_0$  holds for some  $C_0 > 0$  on a punctured neighborhood of  $(0, 0)$ .

(a) If  $l \geq 3$ , then  $o$  is an isolated umbilical point with index one on the graph of  $E_1(aF)$  for any  $a > 0$ ;

(b) If there exists a nonzero  $g \in \mathcal{P}^2$  satisfying  $g \geq 0$  on  $\mathbf{R}^2$ ,  $g \notin \mathcal{P}_+^2$  and  $F - g \in \mathcal{C}_o^{(\infty,3)}$ , then there exists a positive number  $a_0 > 0$  such that  $o$  is an isolated umbilical point with index one on the graph of  $E_1(aF)$  for any  $a \in (0, a_0)$ .

EXAMPLE 1.7. Let  $F$  be an element of  $\mathcal{C}_{0,+}^{(\infty,l)}$  such that  $|\text{grad}_F|/F > C_0$  holds for some  $C_0 > 0$  on a punctured neighborhood of  $(0,0)$ . If

$$F(x, y) = y^4 + o((x^2 + y^2)^2),$$

then we see from (a) of Theorem 1.6 that  $o$  is an isolated umbilical point with index one on the graph of  $E_1(aF)$  for any  $a > 0$ . If

$$F(x, y) = y^2 + o(x^2 + y^2),$$

then we see from (b) of Theorem 1.6 that there exists a positive number  $a_0 > 0$  such that  $o$  is an isolated umbilical point with index one on the graph of  $E_1(aF)$  for any  $a \in (0, a_0)$ ; we have not succeeded to grasp the behavior of the principal distributions around  $o$  on the graph of  $E_1(aF)$  for a large  $a > 0$  yet.

After Section 2 for preliminaries, we shall prove Theorems 1.1 and 1.6 in Section 3. In [A2] and [A3], we have studied the limit of each principal distribution toward an isolated umbilical point along the intersection of a surface with each normal plane at this point. In Section 4, we shall make a similar study on the graph of  $E_\nu(aF)$  and find another phenomenon than those which appear in [A2] and [A3].

## 2. Preliminaries

Let  $f$  be a smooth function of two variables  $x, y$  and  $\mathbf{G}_f$  the graph of  $f$ . We set  $f_x := \partial f / \partial x$ ,  $f_y := \partial f / \partial y$  and

$$E_f := 1 + f_x^2, \quad F_f := f_x f_y, \quad G_f := 1 + f_y^2.$$

The *first fundamental form* of  $\mathbf{G}_f$  is a symmetric tensor field  $\mathbf{I}_f$  on  $\mathbf{G}_f$  of type  $(0, 2)$  represented in terms of the coordinates  $(x, y)$  as

$$\mathbf{I}_f := E_f dx^2 + 2F_f dx dy + G_f dy^2,$$

where

$$dx^2 := dx \otimes dx, \quad dx dy := \frac{1}{2}(dx \otimes dy + dy \otimes dx), \quad dy^2 := dy \otimes dy.$$

We set  $f_{xx} := \partial^2 f / \partial x^2$ ,  $f_{xy} := \partial^2 f / \partial x \partial y$ ,  $f_{yy} := \partial^2 f / \partial y^2$  and

$$L_f := \frac{f_{xx}}{\sqrt{\det(\mathbf{I}_f)}}, \quad M_f := \frac{f_{xy}}{\sqrt{\det(\mathbf{I}_f)}}, \quad N_f := \frac{f_{yy}}{\sqrt{\det(\mathbf{I}_f)}},$$

where  $\det(\mathbf{I}_f) := E_f G_f - F_f^2$ . The *Weingarten map* of  $\mathbf{G}_f$  is a tensor field  $\mathbf{W}_f$  on  $\mathbf{G}_f$  of type  $(1, 1)$  satisfying

$$\left[ \mathbf{W}_f \left( \frac{\partial}{\partial x} \right), \mathbf{W}_f \left( \frac{\partial}{\partial y} \right) \right] = \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right] \mathbf{W}_f,$$

where

$$\mathbf{W}_f := \begin{pmatrix} E_f & F_f \\ F_f & G_f \end{pmatrix}^{-1} \begin{pmatrix} L_f & M_f \\ M_f & N_f \end{pmatrix}.$$

A *principal direction* of  $\mathbf{G}_f$  is a one-dimensional eigenspace of  $\mathbf{W}_f$ . A point of  $\mathbf{G}_f$  is called *umbilical* if at the point,  $\mathbf{W}_f$  is represented by the identity transformation up to a constant. By the symmetry of  $\mathbf{W}_f$  with respect to  $\mathbf{I}_f$ , we see that at each non-umbilical point, there exist just two principal directions, which are perpendicular to each other with respect to  $\mathbf{I}_f$ .

Let  $\mathbf{PD}_f$  be a symmetric tensor field on  $\mathbf{G}_f$  of type  $(0, 2)$  represented in terms of the coordinates  $(x, y)$  as

$$\mathbf{PD}_f := \frac{1}{\sqrt{\det(\mathbf{I}_f)}} \{ A_f dx^2 + 2B_f dx dy + C_f dy^2 \},$$

where

$$A_f := E_f M_f - F_f L_f, \quad 2B_f := E_f N_f - G_f L_f, \quad C_f := F_f N_f - G_f M_f.$$

Then for tangent vectors  $\mathbf{v}_1, \mathbf{v}_2$ ,

$$\frac{1}{2} \sum_{\{i,j\}=\{1,2\}} \mathbf{v}_i \wedge \mathbf{W}_f(\mathbf{v}_j) = \frac{\mathbf{PD}_f(\mathbf{v}_1, \mathbf{v}_2)}{\sqrt{\det(\mathbf{I}_f)}} \left( \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \right).$$

Therefore we obtain

**PROPOSITION 2.1** ([A3]). *A tangent vector  $\mathbf{v}_0$  to  $\mathbf{G}_f$  is in a principal direction if and only if  $\mathbf{PD}_f(\mathbf{v}_0, \mathbf{v}_0) = 0$  holds.*

Let  $\mathbf{D}_f, \mathbf{N}_f$  be symmetric tensor fields on  $\mathbf{G}_f$  of type  $(0, 2)$  represented in terms of the coordinates  $(x, y)$  as

$$\begin{aligned} \mathbf{D}_f &:= f_{xy} dx^2 + (f_{yy} - f_{xx}) dx dy - f_{xy} dy^2, \\ \mathbf{N}_f &:= (f_{xy} f_x^2 - f_x f_y f_{xx}) dx^2 \\ &\quad + (f_{yy} f_x^2 - f_{xx} f_y^2) dx dy + (f_x f_y f_{yy} - f_{xy} f_y^2) dy^2. \end{aligned}$$

Then  $\det(\mathbf{I}_f) \mathbf{PD}_f = \mathbf{D}_f + \mathbf{N}_f$ . For a tangent vector  $\mathbf{v}$ , we set

$$\begin{aligned} \tilde{\mathbf{D}}_f(\mathbf{v}) &:= \mathbf{D}_f(\mathbf{v}, \mathbf{v}), & \tilde{\mathbf{N}}_f(\mathbf{v}) &:= \mathbf{N}_f(\mathbf{v}, \mathbf{v}), \\ \widetilde{\mathbf{PD}}_f(\mathbf{v}) &:= \mathbf{PD}_f(\mathbf{v}, \mathbf{v}). \end{aligned}$$

We set

$$\mathbf{grad}_f := \begin{pmatrix} f_x \\ f_y \end{pmatrix}, \quad \mathbf{grad}_f^\perp := \begin{pmatrix} -f_y \\ f_x \end{pmatrix}, \quad \mathbf{Hess}_f := \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

For  $\phi \in \mathbf{R}$ , we set

$$u_\phi := \begin{pmatrix} \cos \phi \\ \sin \phi \end{pmatrix}, \quad \mathbf{U}_\phi := \cos \phi \frac{\partial}{\partial x} + \sin \phi \frac{\partial}{\partial y}.$$

Let  $\langle \cdot, \cdot \rangle$  be the scalar product in  $\mathbf{R}^2$ . Then we obtain

LEMMA 2.2 ([A3]). *For any  $\phi \in \mathbf{R}$ ,*

$$\tilde{\mathbf{D}}_f(\mathbf{U}_\phi) = \langle \mathbf{Hess}_f u_\phi, u_{\phi+\pi/2} \rangle,$$

$$\tilde{\mathbf{N}}_f(\mathbf{U}_\phi) = \langle \mathbf{grad}_f, u_\phi \rangle \langle \mathbf{grad}_f^\perp, \mathbf{Hess}_f u_\phi \rangle.$$

### 3. Proof of Theorems 1.1 and 1.6

Let  $F$  be an element of  $\mathcal{C}_{o,+}^{(\infty,l)}$ . We set

$$\chi_v(F) := \begin{cases} 1 & \text{if } v = 0, \\ \prod_{i=1}^v E_i(F) & \text{if } v \in \mathbf{N}. \end{cases}$$

Then by induction with respect to  $v \in \mathbf{N}$ , we obtain

LEMMA 3.1. *For any  $v \in \mathbf{N}$ ,*

$$\begin{aligned} \mathbf{grad}_{E_v(F)} &= \frac{E_v(F)}{F^2 \chi_{v-1}(F)} \mathbf{grad}_F, \\ \mathbf{Hess}_{E_v(F)} &= \frac{E_v(F)}{F^2 \chi_{v-1}(F)} \mathbf{Hess}_F \\ &\quad + \frac{E_v(F)}{F^4 \chi_{v-1}(F)} \left( \sum_{i=0}^{v-1} \frac{1 - 2E_i(F)}{\chi_i(F)} \right) \begin{pmatrix} F_x^2 & F_x F_y \\ F_x F_y & F_y^2 \end{pmatrix}. \end{aligned}$$

By Lemma 2.2 together with Lemma 3.1, we obtain

LEMMA 3.2. *For any  $v \in \mathbf{N}$ ,*

$$\begin{aligned} \tilde{\mathbf{D}}_{E_v(F)}(\mathbf{U}_\phi) &= \frac{E_v(F)}{F^2 \chi_{v-1}(F)} \tilde{\mathbf{D}}_F(\mathbf{U}_\phi) \\ &\quad + \frac{E_v(F)}{F^4 \chi_{v-1}(F)} \left( \sum_{i=0}^{v-1} \frac{2E_i(F) - 1}{\chi_i(F)} \right) \langle \mathbf{grad}_F, u_\phi \rangle \langle \mathbf{grad}_F^\perp, u_\phi \rangle, \\ \tilde{\mathbf{N}}_{E_v(F)}(\mathbf{U}_\phi) &= \left( \frac{E_v(F)}{F^2 \chi_{v-1}(F)} \right)^3 \tilde{\mathbf{N}}_F(\mathbf{U}_\phi). \end{aligned}$$

Suppose that there exist positive numbers  $C_0, r_0 > 0$  satisfying

$$|\text{grad}_F| > C_0 F > 0$$

on  $\{0 < x^2 + y^2 < r_0^2\}$ . Let  $\psi_F$  be a continuous function on  $(0, r_0) \times \mathbf{R}$  such that  $\text{grad}_F(r \cos \theta, r \sin \theta)$  is represented by  $u_{\psi_F(r, \theta)}$  up to a positive constant for any  $(r, \theta) \in (0, r_0) \times \mathbf{R}$ . By Lemma 3.2, we see that for any  $(r, \theta) \in (0, r_0) \times \mathbf{R}$ ,

$$\begin{aligned} \det(\mathbf{I}_{E_v(aF)}) \widetilde{\text{PD}}_{E_v(aF)}(\mathbf{U}_\phi) &= \widetilde{\text{D}}_{E_v(aF)}(\mathbf{U}_\phi) + \widetilde{\text{N}}_{E_v(aF)}(\mathbf{U}_\phi) \\ &= \frac{E_v(aF)}{a^2 F^2 \chi_{v-1}(aF)} \left\{ a \widetilde{\text{D}}_F(\mathbf{U}_\phi) + \frac{1}{a} \left( \frac{E_v(aF)}{F^2 \chi_{v-1}(aF)} \right)^2 \widetilde{\text{N}}_F(\mathbf{U}_\phi) \right. \\ &\quad \left. + \frac{|\text{grad}_F|^2}{2F^2 \chi_{v-1}(aF)} Y_{v-1}(aF) \sin 2(\phi - \psi_F(r, \theta)) \right\} \end{aligned}$$

holds at  $(r \cos \theta, r \sin \theta)$ , where  $v \in \mathbf{N}$ ,  $a > 0$  and

$$Y_{v-1}(aF) := \sum_{i=0}^{v-1} \frac{\chi_{v-1}(aF)}{\chi_i(aF)} (2E_i(aF) - 1).$$

Therefore we see that at  $(r \cos \theta, r \sin \theta)$ ,  $\mathbf{U}_\phi$  is in a principal direction of  $\mathbf{G}_{E_v(aF)}$  if and only if the following holds:

$$\begin{aligned} a \chi_{v-1}(aF) \widetilde{\text{D}}_F(\mathbf{U}_\phi) + \frac{\chi_{v-1}(aF)}{a} \left( \frac{E_v(aF)}{F^2 \chi_{v-1}(aF)} \right)^2 \widetilde{\text{N}}_F(\mathbf{U}_\phi) \\ + \frac{1}{2} \frac{|\text{grad}_F|^2}{F^2} Y_{v-1}(aF) \sin 2(\phi - \psi_F(r, \theta)) = 0. \end{aligned} \quad (2)$$

We notice the following:

- (a) There exists a positive constant  $C_1 > 0$  satisfying

$$\chi_{v-1}(aF) |\widetilde{\text{D}}_F(\mathbf{U}_\phi)| < C_1$$

on a neighborhood of  $(0, 0)$  and for any  $\phi \in \mathbf{R}$ ;

- (b) for each positive number  $\varepsilon > 0$ , there exists a neighborhood of  $(0, 0)$  on which the following hold:

$$\frac{E_v(aF)}{F^2 \chi_{v-1}(aF)} < \varepsilon,$$

$$|Y_{v-1}(aF) + 1| < \varepsilon.$$

**PROOF OF THEOREM 1.1.** Suppose that for each  $c > 0$ , there exists a punctured neighborhood of  $(0, 0)$  on which  $|\text{grad}_F|/F > c$  holds. Then noticing

(2) and the above (a), (b), we see that for each positive number  $\varepsilon_0 > 0$ , there exists a number  $\rho_0 \in (0, r_0)$  such that for each  $(\rho, \theta) \in (0, \rho_0) \times \mathbf{R}$ , there exists the only one number  $\phi_{E_v(aF)}(\rho, \theta)$  satisfying

$$|\phi_{E_v(aF)}(\rho, \theta) - \psi_F(\rho, \theta)| < \varepsilon_0 \quad (3)$$

and the condition that at  $(\rho \cos \theta, \rho \sin \theta)$ ,  $\mathbf{U}_{\phi_{E_v(aF)}(\rho, \theta)}$  is in a principal direction of  $\mathbf{G}_{E_v(aF)}$ . Therefore we see that  $o$  is an isolated umbilical point on  $\mathbf{G}_{E_v(aF)}$  and that  $\phi_{E_v(aF)}$  is continuous on  $(0, \rho_0) \times \mathbf{R}$ . Roughly speaking, around  $o$ , a principal distribution is approximated by the gradient vector field of  $F$ . The index  $\text{ind}_o(\mathbf{G}_{E_v(aF)})$  of  $o$  on  $\mathbf{G}_{E_v(aF)}$  is represented as follows:

$$\text{ind}_o(\mathbf{G}_{E_v(aF)}) = \frac{\phi_{E_v(aF)}(\rho, \theta + 2\pi) - \phi_{E_v(aF)}(\rho, \theta)}{2\pi}.$$

Noticing (3) and that the index is represented as the half of an integer, we obtain

$$\text{ind}_o(\mathbf{G}_{E_v(aF)}) = \frac{\psi_F(\rho, \theta + 2\pi) - \psi_F(\rho, \theta)}{2\pi}. \quad (4)$$

We set

$$m(\rho_0) := \min_{\{x^2+y^2=\rho_0^2\}} F(x, y).$$

Then  $m(\rho_0) > 0$ . Noticing that  $\text{grad}_F$  does not vanish on  $\{0 < x^2 + y^2 < \rho_0^2\}$ , we see that the set

$$C_m := \{(x, y) \in \mathbf{R}^2; x^2 + y^2 < \rho_0^2, F(x, y) = m\}$$

for  $m \in (0, m(\rho_0))$  is a connected and simple closed curve such that the bounded domain contains  $(0, 0)$ . Therefore by (4), we obtain

$$\text{ind}_o(\mathbf{G}_{E_v(aF)}) = 1.$$

Hence we obtain Theorem 1.1.  $\square$

**PROOF OF THEOREM 1.6.** Suppose  $l \geq 3$ . Then for each positive number  $\varepsilon > 0$ , we obtain

$$|\tilde{\mathbf{D}}_F(\mathbf{U}_\phi)| < \varepsilon$$

on a neighborhood of  $(0, 0)$  and for any  $\phi \in \mathbf{R}$ . Therefore noticing (2) and the above-mentioned (b), we obtain the same result as in Theorem 1.1. Suppose that there exists a nonzero  $g \in \mathcal{P}^2$  satisfying  $g \geq 0$  on  $\mathbf{R}^2$ ,  $g \notin \mathcal{P}_+^2$  and  $F - g \in \mathcal{C}_o^{(\infty, 3)}$ . Then  $\text{Hess}_F$  is not represented by the unit matrix up to any constant at  $(0, 0)$ . Therefore noticing (2) and the above-mentioned (a), (b), we

may find a positive number  $a_0 > 0$  such that for any  $a \in (0, a_0)$ ,  $o$  is an isolated umbilical point on  $\mathbf{G}_{E_1(aF)}$ . In addition, we may obtain  $\text{ind}_o(\mathbf{G}_{E_1(aF)}) = 1$  for any  $a \in (0, a_0)$  in the same way as in the proof of Theorem 1.1. Hence we obtain Theorem 1.6.  $\square$

#### 4. On representation of the index

The purpose of this section is to study the limit of each principal distribution toward  $o$  along the intersection of the graph of  $E_v(aF)$  with each normal plane at  $o$ . In order to do this, we shall firstly introduce some terms.

Let  $\rho_0$  be a positive number and  $\zeta$  a continuous function on  $(0, \rho_0) \times \mathbf{R}$ . Then  $\zeta$  is called *admissible* if there exists a number  $\text{ind}(\zeta)$  satisfying

$$\text{ind}(\zeta) = \frac{\zeta(\rho, \theta + 2\pi) - \zeta(\rho, \theta)}{2\pi}$$

for any  $(\rho, \theta) \in (0, \rho_0) \times \mathbf{R}$  and if there exists a discrete subset  $\Sigma$  of  $\mathbf{R}$  satisfying the following:

- (a) For each  $\theta_0 \in \mathbf{R} \setminus \Sigma$ , there exists a number  $\zeta_o(\theta_0)$  satisfying

$$\lim_{\rho \rightarrow 0} \zeta(\rho, \theta_0) = \zeta_o(\theta_0);$$

- (b) For each  $\theta_0 \in \mathbf{R}$ , there exist numbers  $\zeta_o(\theta_0 + 0)$ ,  $\zeta_o(\theta_0 - 0)$  satisfying

$$\lim_{\theta \rightarrow \theta_0 \pm 0} \zeta_o(\theta) = \zeta_o(\theta_0 \pm 0)$$

and in addition, if  $\theta_0 \in \mathbf{R} \setminus \Sigma$ , then the following hold:

$$\zeta_o(\theta_0 + 0) = \zeta_o(\theta_0 - 0) = \zeta_o(\theta_0). \quad (5)$$

Suppose that  $\zeta$  is admissible. Then the number  $\text{ind}(\zeta)$  is called the *index* of  $\zeta$ . The minimum of such sets as  $\Sigma$  is denoted by  $\Sigma_\zeta$  and each element of  $\Sigma_\zeta$  is called a *singular argument* of  $\zeta$ . A singular argument  $\theta_0$  of  $\zeta$  is just a number for which (5) does not hold. Noticing the definition of  $\text{ind}(\zeta)$ , we see that if  $\theta_0 \in \Sigma_\zeta$ , then  $\theta_0 + 2n\pi \in \Sigma_\zeta$  for any  $n \in \mathbf{Z}$ . From  $\zeta$ , we may obtain a continuous function  $\beta_\zeta$  on  $\mathbf{R}$  such that on each connected component of  $\mathbf{R} \setminus \Sigma_\zeta$ ,  $\beta_\zeta - \zeta_o$  is constant. Such a function as  $\beta_\zeta$  is called a *base function* of  $\zeta$ . There exists a number  $\text{ind}(\beta_\zeta)$  satisfying

$$\text{ind}(\beta_\zeta) = \frac{\beta_\zeta(\theta + 2\pi) - \beta_\zeta(\theta)}{2\pi}$$

for any  $\theta \in \mathbf{R}$ . The number  $\text{ind}(\beta_\zeta)$  is called the *index* of  $\beta_\zeta$ . For each  $\theta_0 \in \Sigma_\zeta$ ,

$$\Gamma_{\xi,o}(\theta_0) := \xi_o(\theta_0 + 0) - \xi_o(\theta_0 - 0)$$

is called the *gap* of  $\xi$  in  $\theta_0$ . The index  $\text{ind}(\xi)$  is represented as follows:

$$\text{ind}(\xi) = \text{ind}(\beta_\xi) + \frac{1}{2\pi} \sum_{\theta_0 \in \Sigma_\xi \cap [\theta, \theta + 2\pi)} \Gamma_{\xi,o}(\theta_0). \quad (6)$$

EXAMPLE 4.1. Let  $g$  be an element of  $\mathcal{P}^l$  and  $\eta_g$  a continuous function on  $\mathbf{R}$  such that  $u_{\eta_g(\theta)}$  is an eigenvector of  $\text{Hess}_g(\cos \theta, \sin \theta)$  for any  $\theta \in \mathbf{R}$ . Suppose that on  $\mathbf{G}_g$ ,  $o$  is an isolated umbilical point. Let  $\rho_0$  be a positive number such that on  $\{0 < x^2 + y^2 < \rho_0^2\}$ , there exists no umbilical point of  $\mathbf{G}_g$  and  $\phi_g$  a continuous function on  $(0, \rho_0) \times \mathbf{R}$  such that for any  $(\rho, \theta) \in (0, \rho_0) \times \mathbf{R}$ ,  $\mathbf{U}_{\phi_g(\rho, \theta)}$  is in a principal direction of  $\mathbf{G}_g$  at  $(\rho \cos \theta, \rho \sin \theta)$ . Suppose that  $g$  depends only on  $x^2 + y^2$ . Then each of two vector fields  $x\partial/\partial x + y\partial/\partial y$ ,  $-y\partial/\partial x + x\partial/\partial y$  is in a principal direction at any point of  $\mathbf{G}_g$ . Therefore  $\phi_g$  is admissible; a base function of  $\phi_g$  is given by  $\theta$  and there exists no singular argument of  $\phi_g$ . Suppose that  $g$  does not depend only on  $x^2 + y^2$ . Then  $l \geq 3$ , and  $\phi_g$  is admissible; a base function of  $\phi_g$  is given by  $\eta_g$ ; a number  $\theta_0$  is a singular argument of  $\phi_g$  if and only if  $\text{Hess}_g(\cos \theta_0, \sin \theta_0)$  is represented by the unit matrix up to a constant; the gap  $\Gamma_{\phi_g,o}(\theta_0)$  for each  $\theta_0 \in \Sigma_{\phi_g}$  is equal to  $-\pi/2$  ([A2]), and the index of  $\phi_g$  is just the index of  $o$  on  $\mathbf{G}_g$ .

EXAMPLE 4.2. Let  $F$  be a real-analytic function on a neighborhood of  $(0, 0)$  satisfying

$$F(0, 0) = \frac{\partial F}{\partial x}(0, 0) = \frac{\partial F}{\partial y}(0, 0) = 0$$

and the condition that on  $\mathbf{G}_F$ ,  $o$  is an isolated umbilical point. Let  $\rho_0$  be a positive number such that on  $\{0 < x^2 + y^2 < \rho_0^2\}$ , there exists no umbilical point of  $\mathbf{G}_F$  and  $\phi_F$  a continuous function on  $(0, \rho_0) \times \mathbf{R}$  such that for any  $(\rho, \theta) \in (0, \rho_0) \times \mathbf{R}$ ,  $\mathbf{U}_{\phi_F(\rho, \theta)}$  is in a principal direction of  $\mathbf{G}_F$  at  $(\rho \cos \theta, \rho \sin \theta)$ . Then  $\phi_F$  is admissible; a base function of  $\phi_F$  is given by  $\eta_{g_F}$ , where  $g_F$  is as in Section 1; at any singular argument  $\theta_0$  of  $\phi_F$ ,  $\text{Hess}_{g_F}(\cos \theta_0, \sin \theta_0)$  is represented by the unit matrix up to a constant; if  $o$  is an isolated umbilical point of  $\mathbf{G}_{g_F}$ , then the gap  $\Gamma_{\phi_F,o}(\theta_0)$  for each  $\theta_0 \in \Sigma_{\phi_F}$  is equal to  $-\pi/2$ ,  $0$  or  $\pi/2$  ([A3]), and the index of  $\phi_F$  is just the index of  $o$  on  $\mathbf{G}_F$ . Suppose that  $o$  is not any isolated umbilical point of  $\mathbf{G}_{g_F}$ . Then it is in general very difficult to grasp  $\Gamma_{\phi_F,o}(\theta_0)$  for  $\theta_0 \in \Sigma_{\phi_F}$ . The author considers that this difficulty is just the thing which prevents us from solving the index conjecture (see Section 1) on a real-analytic surface. In relation to the index conjecture, the following holds: if  $\Gamma_{\phi_F,o}(\theta_0) \leq \pi$  for any  $\theta_0 \in \Sigma_{\phi_F}$ , then  $\text{ind}(\phi_F) \leq 1$  ([A3]).

EXAMPLE 4.3. Let  $F$  be an element of  $\mathcal{C}_o^{(\infty, 2)}$  such that on  $\mathbf{G}_F$ ,  $o$  is an isolated umbilical point. Then there exists a real number  $a_F \in \mathbf{R}$  satisfying (1). We obtain  $F \not\equiv \sigma_F$ . If  $F - \sigma_F \notin \mathcal{C}_o^{(\infty, \infty)}$ , then  $F$  has properties as in Example 4.2 ([A4]).

REMARK 4.4. In [A3], we presented one way of computing the index  $\text{ind}(\eta_g)$  of  $\eta_g$  for each  $g \in \mathcal{P}^l$  and  $l \geq 3$ .

REMARK 4.5. Noticing that at any non-umbilical point of a surface, the two principal directions are perpendicular to each other with respect to the first fundamental form, we see that if a principal distribution around an isolated umbilical point on a surface is given by an admissible function, then the other principal distribution is also given by another admissible function.

EXAMPLE 4.6. Let  $F$  be an element of  $\mathcal{C}_{o,+}^{(\infty, l)}$  such that  $\text{grad}_F$  does not vanish on  $\{0 < x^2 + y^2 < \rho_0^2\}$  for some  $\rho_0 > 0$  and  $\psi_F$  a continuous function on  $(0, \rho_0) \times \mathbf{R}$  such that  $\text{grad}_F(\rho \cos \theta, \rho \sin \theta)$  is represented by  $u_{\psi_F(\rho, \theta)}$  up to a positive constant for any  $(\rho, \theta) \in (0, \rho_0) \times \mathbf{R}$ . Suppose that there exists a non-zero  $g \in \mathcal{P}^l$  satisfying  $F - g \in \mathcal{C}_o^{(\infty, l+1)}$ . Then  $g \geq 0$ . Let  $\psi_g$  be a continuous function on  $\mathbf{R}$  such that  $\text{grad}_g(\cos \theta, \sin \theta)$  is represented by  $u_{\psi_g(\theta)}$  up to a constant for any  $\theta \in \mathbf{R}$  and  $Z_g$  the set of the numbers such that  $g(\cos \theta_0, \sin \theta_0) = 0$  holds for each  $\theta_0 \in Z_g$ . We see the following:  $\psi_F$  is admissible; a base function of  $\psi_F$  is given by  $\psi_g$ ; the index of  $\psi_g$  is represented as

$$\text{ind}(\psi_g) = 1 - \#\{Z_g \cap [\theta, \theta + \pi)\}, \quad (7)$$

and any singular argument of  $\psi_F$  is an element of  $Z_g$ . Let  $\theta_0$  be an element of  $Z_g$ . We set  $\theta_0 = 0$ . Then  $\psi_g(0) \in \{(2i+1)\pi/2\}_{i \in \mathbf{Z}}$ . Therefore noticing  $g \geq 0$ , we see that there exist integers  $n_+, n_- \in \mathbf{Z}$  satisfying

$$\psi_{F,o}(\pm 0) = (4n_{\pm} \pm 1)\pi/2.$$

Let  $\varepsilon_0$  be a positive number satisfying

$$Z_g \cap [-\varepsilon_0, \varepsilon_0] = \{0\}$$

and  $\delta_+, \delta_-$  elements of  $(0, \rho_0)$  satisfying

$$F(\delta_+ \cos \varepsilon_0, \delta_+ \sin \varepsilon_0) = F(\delta_- \cos \varepsilon_0, -\delta_- \sin \varepsilon_0) (=: m_0).$$

As we have seen in the proof of Theorem 1.1, if  $m_0 < m(\rho_0)$ , then the contour line  $C_{m_0}$  is a connected, simple closed curve such that the bounded domain contains  $(0, 0)$ . Therefore noticing that  $C_{m_0}$  contains the two points  $(\delta_+ \cos \varepsilon_0, \delta_+ \sin \varepsilon_0)$ ,  $(\delta_- \cos \varepsilon_0, -\delta_- \sin \varepsilon_0)$  and that  $f_x \partial/\partial x + f_y \partial/\partial y$  is normal to  $C_{m_0}$  at any point of  $C_{m_0}$ , we obtain  $n_+ = n_-$  and  $0 \in \Sigma_{\psi_F}$  (see Figure 1). Therefore we obtain  $\Sigma_{\psi_F} = Z_g$  and

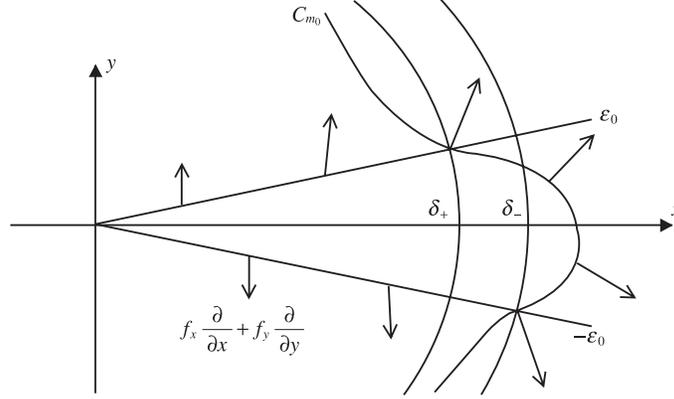


Fig. 1. EXAMPLE 4.6

$$\Gamma_{\psi_F, o}(\theta_0) = \pi \quad (8)$$

for any  $\theta_0 \in \Sigma_{\psi_F}$ . By (6), (7) and (8), we obtain  $\text{ind}(\psi_F) = 1$ . This means that the index of  $(0, 0)$  with respect to  $f_x \partial / \partial x + f_y \partial / \partial y$  is equal to one and does not contradict the proof of Theorem 1.1.

Let  $F$  be an element of  $\mathcal{C}_{o, +}^{(\infty, l)}$  satisfying  $|\text{grad}_F|/F > C_0$  for some  $C_0 > 0$  on a punctured neighborhood of  $(0, 0)$ . Suppose  $l \geq 3$ . Then Theorem 1.6 says that  $o$  is an isolated umbilical point of  $\mathbf{G}_{E_1(aF)}$  for any  $a > 0$ . Let  $\rho_0$  be a positive number such that on  $\{0 < x^2 + y^2 < \rho_0^2\}$ , there exists no umbilical point of  $\mathbf{G}_{E_1(aF)}$  and  $\phi_{E_1(aF)}$  a continuous function on  $(0, \rho_0) \times \mathbf{R}$  such that for any  $(\rho, \theta) \in (0, \rho_0) \times \mathbf{R}$ ,  $\mathbf{U}_{\phi_{E_1(aF)}(\rho, \theta)}$  is in a principal direction of  $\mathbf{G}_{E_1(aF)}$  at  $(\rho \cos \theta, \rho \sin \theta)$ . Referring to the proof of Theorem 1.6 and Example 4.6, we obtain

**PROPOSITION 4.7.** *Suppose that there exists a nonzero  $g \in \mathcal{P}^l$  satisfying  $F - g \in \mathcal{C}_o^{(\infty, l+1)}$ . Then the following hold:*

- (a)  $\phi_{E_1(aF)}$  is admissible;
- (b) a base function of  $\phi_{E_1(aF)}$  is given by  $\psi_g$ ;
- (c)  $\Sigma_{\phi_{E_1(aF)}} = Z_g$ ;
- (d) the gap  $\Gamma_{\phi_{E_1(aF)}, o}(\theta_0)$  is equal to  $\pi$  for any  $\theta_0 \in \Sigma_{\phi_{E_1(aF)}}$ .

**REMARK 4.8.** Suppose  $l = 2$ . Then for a suitable  $a > 0$ ,  $\phi_{E_1(aF)}$  is admissible;  $\Sigma_{\phi_{E_1(aF)}} = Z_g$ , and  $\Gamma_{\phi_{E_1(aF)}, o}(\theta_0) = \pi$  for any  $\theta_0 \in \Sigma_{\phi_{E_1(aF)}}$ . However, a base function of  $\phi_{E_1(aF)}$  may not be given by  $\psi_g$  for any  $a > 0$ .

**REMARK 4.9.** For any  $l \geq 2$ , any  $v \in \mathbf{N}$  and any  $a > 0$ ,  $\phi_{E_{v+1}(aF)}$  has properties as (a)~(d) in Proposition 4.7.

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### References

- [A1] N. Ando, An isolated umbilical point of the graph of a homogeneous polynomial, *Geom. Dedicata*, **82** (2000), 115–137.
- [A2] N. Ando, The behavior of the principal distributions on the graph of a homogeneous polynomial, *Tohoku Math. J.*, **54** (2002), 163–177.
- [A3] N. Ando, The behavior of the principal distributions on a real-analytic surface, to appear in *J. Math. Soc. Japan*.
- [A4] N. Ando, The index of an isolated umbilical point on a surface, (a survey), *Contemporary Mathematics* **308** “Differential Geometry and Integrable Systems” by Amer. Math. Soc., 1–11.
- [B] L. Bates, A weak counterexample to the Carathéodory’s conjecture, *Diff. Geom. and its Appl.*, **15** (2001), 79–80.
- [GSaB] C. Gutierrez and F. Sanchez-Bringas, Planer vector field versions of Carathéodory’s and Loewner’s conjectures, *Publ. Mat.*, **41** (1997), no. 1 169–179.
- [HaW] P. Hartman and A. Wintner, Umbilical points and W-surfaces, *Amer. J. Math.*, **76** (1954), 502–508.
- [Ho] H. Hopf, Lectures on differential geometry in the large, *Lecture Notes in Math.* vol. 1000, Springer-Verlag, Berlin-NewYork, 1989.
- [K] T. Klotz, On Bol’s proof of Carathéodory’s conjecture, *Comm. Pure Appl. Math.*, **12** (1959), 277–311.
- [SX1] B. Smyth and F. Xavier, A sharp geometric estimate for the index of an umbilic on a smooth surface, *Bull. London Math. Soc.*, **24** (1992), 176–180.
- [SX2] B. Smyth and F. Xavier, Real solvability of the equation  $\partial_{\bar{z}}^2\omega = \rho g$  and the topology of isolated umbilics, *J. Geom. Anal.*, **8** (1998), 655–671.
- [SX3] B. Smyth and F. Xavier, Eigenvalue estimates and the index of Hessian fields, *Bull. London Math. Soc.*, **33** (2001), 109–112.
- [T] C. J. Titus, A proof of a conjecture of Loewner and of the conjecture of Carathéodory on umbilic points, *Acta Math.*, **131** (1973), 43–77.

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