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Nonunital decompositions of the matrix algebra of order three

Vsevolod Gubarev

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Abstract. All decompositions of $M_3(\mathbb{C})$ into a direct vector-space sum of two subalgebras such that none of the subalgebras contains the identity matrix are classified. Thus, the classification of all decompositions of $M_3(\mathbb{C})$ into a direct vector-space sum of two subalgebras as well as description of Rota–Baxter operators of nonzero weight on $M_3(\mathbb{C})$ is finished.

1. Introduction

Let R be a ring, suppose that $R = R_1 + R_2$, where R_1, R_2 are subrings (not necessarily ideals) of R. In this situation, we say that R decomposes into a sum of R_1, R_2. If $R_1 \cap R_2 = (0)$, then we call such decomposition as a direct one. The study of decompositions of associative rings and algebras started in 1963, when O. H. Kegel proved [10] that an associative algebra decomposed into a sum of two nilpotent subalgebras is itself nilpotent.

In 1995, K. I. Beidar and A. V. Mikhalev asked [3], if a sum of two PI-algebras is again a PI-algebra. In 2017, this problem was positively solved by M. Kępczyk [11]. The famous Köthe problem (If a ring R has no nonzero nil ideals, does it follow that R has no nonzero nil one-sided ideals?) is equivalent to a problem concerned decompositions [4]. In [12], all decompositions of 2- and 3-dimensional associative algebras were described.

It is natural to study decompositions involved matrix algebras. In 1999, Y. A. Bahturin and O. H. Kegel [1] described all algebras decomposed as a sum of two matrix algebras (it is Problem 3 from [12] posted twenty years later in 2019).

In [5], all direct decompositions of $M_2(\mathbb{C})$ were classified. The main goal of the current work is to finish a classification of direct decompositions of $M_3(\mathbb{C})$. In [8], all direct decompositions of $M_3(\mathbb{C})$ such that one of the subalgebras contains the identity matrix were classified (71 cases, some of them

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involve one or two parameters). Thus, it remains to describe direct decompositions of $M_3(\mathbb{C})$ such that none of the two subalgebras contains the identity matrix (we call them as nonunital ones).

One of the motivations and applications of the given problem is an area of Rota–Baxter operators. Given an algebra A and a scalar $\lambda \in F$, where F is a ground field, a linear operator $R : A \to A$ is called a Rota–Baxter operator of weight λ if the identity

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy)$$

holds for all $x, y \in A$.

Appeared in 1960 [2], such operators may be considered as an algebraic analogue of integral operator. They have different connections to Yang–Baxter equation, Loday algebras, double Lie algebras [6, 9]. Given an algebra A and its direct decomposition $A = A_1 \oplus A_2$, we get the canonical example of Rota–Baxter operator of weight -1 as the projection on one of A_j, e.g., $R(x_1 + x_2) = x_1$, here $x_i \in A_i$, $i = 1, 2$. In some cases like when A is the Grassmann algebra or the simple Jordan superalgebra K_3, there are no other Rota–Baxter operators of nonzero weight except such projections.

In [5], all non-projective Rota–Baxter operators of nonzero weight on $M_3(\mathbb{C})$ were classified. Therefore, joint with the articles [5, 8], we complete the description of all Rota–Baxter operators of nonzero weight on $M_3(\mathbb{C})$.

We split the problem of classification of a nonunital decomposition $M_3(\mathbb{C}) = S \oplus M$, where $\dim M > \dim S$, into the following cases: $\dim M = 6$ or $\dim M = 5$. In the first case, there is the only possibility for M [7]: it is the subalgebra of matrices with zero first column. In the second one, due to [8], we have either $M = \mathrm{Span}\{e_{11}, e_{12}, e_{13}, e_{22}, e_{23}\}$ or $M = \mathrm{Span}\{e_{11}, e_{12}, e_{13}, e_{23}, e_{33}\}$. We actively apply automorphisms of $M_3(\mathbb{C})$, which preserve the subalgebra M, it helps us to avoid computational difficulties.

The main results of the work are Theorems 2 and 3. In Theorem 2, we get three decompositions, when $\dim M = 6$. In Theorem 3, we obtain nine more decompositions, here $\dim M = 5$. Note that all twelve cases do not involve parameters. Therefore, due to [5, 8], we have 83 nontrivial decompositions of $M_3(\mathbb{C})$ and 119 nontrivial Rota–Baxter operators on $M_3(\mathbb{C})$.

In what follows, we will apply an automorphism Θ_{12} of $M_3(\mathbb{C})$, acting as follows, $\Theta_{12}(X) = T^{-1}XT$ for $T = e_{12} + e_{21} + e_{33}$. Analogously, we define $\Theta_{13}, \Theta_{23} \in \mathrm{Aut}(M_3(\mathbb{C}))$.

2. (6,3)-decompositions

Let M be a six-dimensional nonunital subalgebra of $M_3(\mathbb{C})$. Then M consists only of degenerate matrices, and then up to transpose M is isomorphic to the subalgebra having all zero elements in the first column [13]. Thus, we fix six-dimensional subalgebra $M = \mathrm{Span}\{e_{12}, e_{13}, e_{22}, e_{23}, e_{32}, e_{33}\}$.

Lemma 1 ([8]). *Let φ be an automorphism of $M_3(\mathbb{C})$ preserving the subalgebra $M = \mathrm{Span}\{e_{12}, e_{13}, e_{22}, e_{23}, e_{32}, e_{33}\}$. Then φ acts as follows,*

$$e_{11} \to \begin{pmatrix} 1 & \beta & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_{12} \to \begin{pmatrix} 0 & \kappa & \lambda \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_{13} \to \begin{pmatrix} 0 & \mu & \nu \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_{22} \to \frac{1}{\Delta} \begin{pmatrix} 0 & \kappa(\gamma\mu - \beta\nu) & \lambda(\gamma\mu - \beta\nu) \\ 0 & \kappa\nu & \lambda\nu \\ 0 & -\kappa\mu & -\lambda\mu \end{pmatrix},$$

$$e_{23} \to \frac{1}{\Delta} \begin{pmatrix} 0 & \mu(\gamma\mu - \beta\nu) & \nu(\gamma\mu - \beta\nu) \\ 0 & \mu\nu & \nu^2 \\ 0 & -\mu^2 & -\mu\nu \end{pmatrix},$$

$$e_{32} \to \frac{1}{\Delta} \begin{pmatrix} 0 & \kappa(\beta\lambda - \gamma\kappa) & \lambda(\beta\lambda - \gamma\kappa) \\ 0 & -\kappa\lambda & -\lambda^2 \\ 0 & \kappa^2 & \kappa\lambda \end{pmatrix}, \tag{1}$$

$$e_{33} \to \frac{1}{\Delta} \begin{pmatrix} 0 & \mu(\beta\lambda - \gamma\kappa) & \nu(\beta\lambda - \gamma\kappa) \\ 0 & -\lambda\mu & -\lambda\nu \\ 0 & \kappa\mu & \kappa\nu \end{pmatrix},$$

$$e_{21} \to \frac{1}{\Delta} \begin{pmatrix} \gamma\mu - \beta\nu & \beta(\gamma\mu - \beta\nu) & \gamma(\gamma\mu - \beta\nu) \\ \nu & \beta\nu & \gamma\nu \\ -\mu & -\beta\mu & -\gamma\mu \end{pmatrix},$$

$$e_{31} \to \frac{1}{\Delta} \begin{pmatrix} \beta\lambda - \gamma\kappa & \beta(\beta\lambda - \gamma\kappa) & \gamma(\beta\lambda - \gamma\kappa) \\ -\lambda & -\beta\lambda & -\gamma\lambda \\ \kappa & \beta\kappa & \gamma\kappa \end{pmatrix},$$

where $\Delta = \kappa\nu - \lambda\mu \neq 0$.

Theorem 2. *Every direct decomposition of $M_3(\mathbb{C})$ with two subalgebras of the dimensions 3 and 6 not containing the identity matrix, up to transpose and up to action of $\mathrm{Aut}(M_3(\mathbb{C}))$ is isomorphic to only one of the following cases:*

(A1) $\mathrm{Span}\{e_{11}, e_{21}, e_{31}\} \oplus \mathrm{Span}\{e_{12}, e_{13}, e_{22}, e_{23}, e_{32}, e_{33}\}$,

(A2) $\mathrm{Span}\{e_{11} + e_{22}, e_{21}, e_{31}\} \oplus \mathrm{Span}\{e_{12}, e_{13}, e_{22}, e_{23}, e_{32}, e_{33}\}$,

(A3) $\mathrm{Span}\{e_{11} + e_{22}, e_{21} + e_{22}, e_{31}\} \oplus \mathrm{Span}\{e_{12}, e_{13}, e_{22}, e_{23}, e_{32}, e_{33}\}$.

Proof. We study a decomposition $M_3(\mathbb{C}) = S \oplus M$, and S has a basis

$$v_1 = \begin{pmatrix} 0 & a & b \\ 1 & c & d \\ 0 & e & f \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 & r & s \\ 0 & t & u \\ 1 & x & y \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 1 & k & l \\ 0 & m & n \\ 0 & p & q \end{pmatrix}.$$

Note that $v_3^2 = v_3$, since S is a subalgebra. Hence, its submatrix $H = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$ is idempotent. We may conjugate H with appropriate nondegenerate matrix T such that $J = T^{-1}HT$ is in the Jordan form. We have the following variants for $J \in M_2(\mathbb{C})$:

1) $J = 0$,

2) $J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

3) $J = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$,

4) $J = E$.

Let us conjugate v_3 with the block-diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}$, such conjugation surely preserves M. In the case 4) involving the equality $v_3^2 = v_3$, we get $v_3 = E$, a contradiction.

The case 3) is conjugate to 2) under Θ_{23}. Thus, it remains to consider cases 1) and 2).

CASE 1: $J = 0$. Hence, $v_3 = e_{11} + ke_{12} + le_{13}$. Note that the automorphism φ defined by (1) with $\beta = k$ and $\gamma = l$ maps e_{11} to v_3. So, φ^{-1} maps v_3 to e_{11}, and we may assume that $v_3 = e_{11}$. Since $v_1e_{11}, v_2e_{11} \in S$, we get the decomposition (A1).

CASE 2: $J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The condition $v_3^2 = v_3$ implies that $v_3 = \begin{pmatrix} 1 & 0 & l \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Again, we apply φ defined by (1) with $\beta = \mu = \lambda = 0$ and $\gamma = l$, then $\varphi(e_{11} + e_{22}) = v_3$. We may assume that $v_3 = e_{11} + e_{22}$. Since $v_iv_3, v_3v_i \in S$ for $i = 1, 2$, we conclude that

$$v_1 = \begin{pmatrix} 0 & a & 0 \\ 1 & c & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & x & 0 \end{pmatrix}.$$

Taking φ defined by (1) with $\lambda = \mu = \gamma = 0$, $\nu = 1$, and $\beta = x$, then $\varphi(v_3) = v_3$ and $\varphi(e_{31}) = v_2$. Therefore, we may assume that $v_2 = e_{31}$. From $v_2v_1 \in S$, we get $a = 0$. If $c = 0$, then it is exactly the decomposition (A2). Otherwise, consider φ with $\lambda = \mu = \beta = \gamma = 0$, $\kappa = c$, $\nu = 1$, then $\varphi(v_2) = v_2$, $\varphi(v_3) = v_3$, and $\varphi(e_{21} + e_{22}) = (1/c)v_1$. Hence, we arrive at the decomposition (A3).

Decompositions (A1) and (A2) as well as (A1) and (A3) lie in different orbits under action (anti)automorphisms of $M_3(\mathbb{C})$ preserving M, since the 3-dimensional subalgebra from (A2) and (A3) but not from (A1) contains an idempotent of rank 2. Finally, decompositions (A2) and (A3) lie in different orbits too. Indeed, the dimensions of the semisimple parts of the subalgebra S from (A2) and (A3) do not equal.

3. (5,4)-decompositions

Let M be a nonunital 5-dimensional algebra, by [8, Lemma 5], we may assume up to transpose and action of $\mathrm{Aut}(M_3(\mathbb{C}))$ that either $M = \mathrm{Span}\{e_{11}, e_{12}, e_{13}, e_{22}, e_{23}\}$ or $M = \mathrm{Span}\{e_{11}, e_{12}, e_{13}, e_{23}, e_{33}\}$. Then the group $\mathrm{Aut}(M_3(\mathbb{C}))$ preserving M coincides with the group of automorphisms of $\mathrm{Aut}(M_3(\mathbb{C}))$ preserving the subalgebra of upper-triangular matrices. Thus, an automorphism $\psi \in \mathrm{Aut}(M_3(\mathbb{C}))$ preserving M has the form (1) considered with $\mu = 0$:

$$e_{11} \to \begin{pmatrix} 1 & \beta & \gamma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_{12} \to \begin{pmatrix} 0 & \delta & \varepsilon \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_{13} \to \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_{21} \to \frac{1}{\delta}\begin{pmatrix} -\beta & -\beta^2 & -\beta\gamma \\ 1 & \beta & \gamma \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_{22} \to \begin{pmatrix} 0 & -\beta & -\beta\varepsilon/\delta \\ 0 & 1 & \varepsilon/\delta \\ 0 & 0 & 0 \end{pmatrix}, \tag{2}$$

$$e_{23} \to \frac{1}{\delta}\begin{pmatrix} 0 & 0 & -\alpha\beta \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_{31} \to \frac{1}{\alpha\delta}\begin{pmatrix} \beta\varepsilon - \gamma\delta & \beta(\beta\varepsilon - \gamma\delta) & \gamma(\beta\varepsilon - \gamma\delta) \\ -\varepsilon & -\beta\varepsilon & -\gamma\varepsilon \\ \delta & \beta\delta & \gamma\delta \end{pmatrix},$$

$$e_{32} \to \frac{1}{\alpha}\begin{pmatrix} 0 & \beta\varepsilon - \gamma\delta & \varepsilon(\beta\varepsilon - \gamma\delta)/\delta \\ 0 & -\varepsilon & -\varepsilon^2/\delta \\ 0 & \delta & \varepsilon \end{pmatrix}, \qquad e_{33} \to \begin{pmatrix} 0 & 0 & \beta\varepsilon/\delta - \gamma \\ 0 & 0 & -\varepsilon/\delta \\ 0 & 0 & 1 \end{pmatrix},$$

where $\alpha, \delta \neq 0$.

THEOREM 3. *Every direct decomposition of $M_3(\mathbb{C})$ with two subalgebras of the dimensions 4 and 5 not containing the identity matrix, up to transpose and up to action of $\mathrm{Aut}(M_3(\mathbb{C}))$ is isomorphic to only one of the following cases:*

(B1) $\mathrm{Span}\{e_{21}, e_{31}, e_{32}, e_{33}\} \oplus \mathrm{Span}\{e_{11}, e_{12}, e_{13}, e_{22}, e_{23}\}$,
(B2) $\mathrm{Span}\{e_{11} + e_{21}, e_{31}, e_{32}, e_{33}\} \oplus \mathrm{Span}\{e_{11}, e_{12}, e_{13}, e_{22}, e_{23}\}$,
(B3) $\mathrm{Span}\{e_{21}, e_{31}, e_{32}, e_{22} + e_{33}\} \oplus \mathrm{Span}\{e_{11}, e_{12}, e_{13}, e_{22}, e_{23}\}$,
(B4) $\mathrm{Span}\{e_{21}, e_{31}, e_{32} + e_{23}, e_{22} + e_{33}\} \oplus \mathrm{Span}\{e_{11}, e_{12}, e_{13}, e_{22}, e_{23}\}$,
(B5) $\mathrm{Span}\{e_{21}, e_{31}, e_{32}, e_{11} + e_{33}\} \oplus \mathrm{Span}\{e_{11}, e_{12}, e_{13}, e_{22}, e_{23}\}$,

(B6) $\mathrm{Span}\{e_{21}, e_{31}, e_{32}, e_{22}\} \oplus \mathrm{Span}\{e_{11}, e_{12}, e_{13}, e_{23}, e_{33}\}$,
(B7) $\mathrm{Span}\{e_{21}, e_{11} + e_{31}, e_{12} + e_{32}, e_{22}\} \oplus \mathrm{Span}\{e_{11}, e_{12}, e_{13}, e_{23}, e_{33}\}$,
(B8) $\mathrm{Span}\{e_{21}, e_{31}, e_{32}, e_{22} + e_{33}\} \oplus \mathrm{Span}\{e_{11}, e_{12}, e_{13}, e_{23}, e_{33}\}$,
(B9) $\mathrm{Span}\{e_{21}, e_{31}, e_{32} + e_{23}, e_{22} + e_{33}\} \oplus \mathrm{Span}\{e_{11}, e_{12}, e_{13}, e_{23}, e_{33}\}$.

PROOF. Let us start with the case $M = \mathrm{Span}\{e_{11}, e_{12}, e_{13}, e_{22}, e_{23}\}$. Then the complement subalgebra S contains a basis

$$v_1 = \begin{pmatrix} a & b & c \\ 1 & d & e \\ 0 & 0 & 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} f & g & h \\ 0 & i & j \\ 1 & 0 & 0 \end{pmatrix},$$

$$v_3 = \begin{pmatrix} k & l & m \\ 0 & n & p \\ 0 & 1 & 0 \end{pmatrix}, \qquad v_4 = \begin{pmatrix} x & y & z \\ 0 & t & u \\ 0 & 0 & 1 \end{pmatrix}.$$

As in the proof of Theorem 2, we have $v_4^2 = v_4$. It is known [14] that an upper-triangular matrix from $M_3(\mathbb{C})$ is conjugate to its Jordan form with the help of some upper-triangular matrix, i.e. an automorphism preserving M. Thus, we have the following cases for the Jordan form J of v_4:

1) $J = e_{33}$,
2) $J = e_{22} + e_{33}$,
3) $J = e_{11} + e_{33}$.

CASE 1: $v_4 = e_{33}$. From $v_4 v_2 = v_2$ and $v_4 v_3 = v_3$, we conclude that $v_2 = e_{31}$ and $v_3 = e_{32}$. Also, $v_1 v_4 = 0$, hence, $c = e = 0$. Since $v_1^2 = (a+d)v_1$, we get $v_1 = \begin{pmatrix} a & ad & 0 \\ 1 & d & 0 \\ 0 & 0 & 0 \end{pmatrix}$. If $a + d = 0$, then we apply ψ defined by (2) with parameters $\gamma = \varepsilon = 0$, $\beta = d$. So, ψ^{-1} gives us the decomposition (B1). If $a + d \neq 0$, then consider ψ defined by (2) with parameters $\gamma = \varepsilon = 0$, $\beta = d$ and $\delta = a + d \neq 0$. It is easy to check that $\psi(e_{11} + e_{21}) = v_1$, however, $L(v_2, v_3, v_4)$ is ψ-invariant. It is the decomposition (B2).

CASE 2: $v_4 = e_{22} + e_{33}$. From the equalities

$$v_4 v_1 = v_1, \qquad v_1 v_4 = 0, \qquad v_4 v_2 = v_2, \qquad v_2 v_4 = 0, \qquad v_4 v_3 = v_3 v_4 = v_3,$$

we get $v_1 = e_{21}$, $v_2 = e_{31}$ and $v_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & n & p \\ 0 & 1 & 0 \end{pmatrix}$. If $p + n^2/4 = 0$, we take ψ defined by (2) with $\beta = \gamma = 0$, $\varepsilon/\delta = -n/2$ and get the decomposition (B3). Otherwise, we take ψ with $\beta = \gamma = 0$, $\varepsilon = -n/2$, $\delta = 1$, $\alpha = \sqrt{p + n^2/4}$, and get the decomposition (B4).

Case 3: $v_4 = e_{11} + e_{33}$. From the equalities

$$v_4 v_1 = 0, \qquad v_1 v_4 = v_1, \qquad v_4 v_2 = v_2 v_4 = v_2, \qquad v_4 v_3 = v_3, \qquad v_3 v_4 = 0,$$

we get

$$v_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & e \\ 0 & 0 & 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} f & 0 & h \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 & l & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Further, the relations $v_1 v_3 = 0$ and $v_3 v_1 = v_2 + e v_4$ imply $l + e = e + f - l = h - el = 0$. Thus, $l = -e$, $f = -2e$, $h = -e^2$. We apply ψ defined by (2) with $\beta = \varepsilon = 0$ and $\gamma = e$, and get the decomposition (B5) with the help of ψ^{-1}.

Now, we consider the case $M = \mathrm{Span}\{e_{11}, e_{12}, e_{13}, e_{23}, e_{33}\}$. The complement subalgebra S contains a basis

$$v_1 = \begin{pmatrix} a & b & c \\ 1 & 0 & d \\ 0 & 0 & e \end{pmatrix}, \qquad v_2 = \begin{pmatrix} f & g & h \\ 0 & 0 & i \\ 1 & 0 & j \end{pmatrix},$$

$$v_3 = \begin{pmatrix} k & l & m \\ 0 & 0 & p \\ 0 & 1 & n \end{pmatrix}, \qquad v_4 = \begin{pmatrix} x & y & z \\ 0 & 1 & u \\ 0 & 0 & t \end{pmatrix}.$$

We have $v_4^2 = v_4$, and there are the following variants of the Jordan form J of v_4:

1′) $J = e_{22}$,

2′) $J = e_{22} + e_{33}$,

3′) $J = e_{11} + e_{22}$.

The case 3′) is conjugate to the second one under Θ_{13}.

Case 1′: $v_4 = e_{22}$. From $v_2 v_4 = v_4 v_2 = 0$, $v_4 v_1 = v_1$ and $v_3 v_4 = v_3$ we get

$$v_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} f & 0 & h \\ 0 & 0 & 0 \\ 1 & 0 & j \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 & l & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

By $v_3 v_1 = v_2$, we express $v_2 = \begin{pmatrix} l & 0 & dl \\ 0 & 0 & 0 \\ 1 & 0 & d \end{pmatrix}$. If $d + l = 0$, then ψ (2) with parameters $\beta = \varepsilon = 0$, $\gamma = d$, $\alpha = \delta = 1$ acts as follows, $\psi(e_{21}) = v_1$, $\psi(e_{31}) = v_2$, $\psi(e_{32}) = v_3$, it is (B6).

If $d + l \neq 0$, then ψ defined by (2) with parameters $\beta = \varepsilon = 0$, $\gamma = d$, $\alpha = d + l$, $\delta = 1$ acts as follows, $\psi(e_{21}) = v_1$, $\psi(e_{12} + e_{32}) = v_2/(d+l)$, $\psi(e_{11} + e_{31}) = v_3/(d+l)$, we get (B7) with the help of ψ^{-1}.

Case 2′: $v_4 = e_{22} + e_{33}$. From the equations $v_1 v_4 = v_2 v_4 = 0$, $v_3 v_4 = v_3$, $v_4 v_i = v_i$, $i = 1, 2, 3$, we derive that $v_1 = e_{21}$, $v_2 = e_{31}$, and $v_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & p \\ 0 & 1 & n \end{pmatrix}$. Analogously to the case 2), we get either decomposition (B8), when $p + n^2/4 = 0$, or (B9), otherwise.

Finally, we show that all cases (B1)–(B9) lie in different orbits under action automorphisms or antiautomorphisms of $M_3(\mathbb{C})$. The group of decompositions (B1)–(B5) and the group of decompositions (B6)–(B9) have non-isomorphic biggest subalgebra M, thus, it is enough to show that decompositions lying in the same group are from different orbits. Inside the first group, decompositions (B2) and (B4) but not others have 2-dimensional radicals of corresponding subalgebras S. Further, the radical of S from (B4) but not from (B2) lies in one-sided annihilator of the whole S. The rank of the idempotent lying in S from (B1) equals 1, and the same parameter for (B3) and (B5) equals 2. Also, there exists an idempotent in S from (B5), which acts as unit on the square of its radical, and there are no such idempotents in S from the decomposition (B3).

For the second group, we have $S \cong M_2(\mathbb{C})$ only for (B7). The semisimple part of S is 2-dimensional only for (B9). Finally, the ranks of the idempotents lying in S from (B6) and (B8) are not equal.

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Vsevolod Gubarev
Laboratory of algebra
Sobolev Institute of Mathematics
Novosibirsk 630090 *RUSSIA*
Current address: Novosibirsk 630090 *RUSSIA*
E-mail: wsewolod89@gmail.com