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# On paths in the $\mu$-constant and $\mu^*$-constant strata

Christophe EYRAL and Mutsuo OKA

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**ABSTRACT.** Let $f$ and $f'$ be non-constant, complex, polynomial function-germs at the origin $\mathbf{0} \in \mathbb{C}^n$ such that $f(\mathbf{0}) = f'(\mathbf{0}) = 0$. We suppose that $f$ and $f'$ have an isolated singularity at $\mathbf{0}$ and the same Milnor number (respectively, the same Teissier $\mu^*$-sequence). We show that if $f$ and $f'$ can be connected by a *continuous* path in the $\mu$-constant (respectively, $\mu^*$-constant) stratum, then they can also be connected by a *piecewise complex-analytic path* (defined in the relevant natural way) in this stratum.

## 1. Introduction

Let $\mathbf{z} = (z_1, \dots, z_n)$ be coordinates for $\mathbb{C}^n$ $(n \geq 2)$, and let $f(\mathbf{z})$ and $f'(\mathbf{z})$ be non-constant polynomial function-germs at $\mathbf{0} \in \mathbb{C}^n$ such that $f(\mathbf{0}) = f'(\mathbf{0}) = 0$. We suppose that $f$ and $f'$ have an isolated singularity at $\mathbf{0}$ and the same Milnor number (respectively, the same Teissier $\mu^*$-sequence). In this short note, we show that if $f$ and $f'$ can be connected by a *continuous* path in the $\mu$-constant (respectively, $\mu^*$-constant) stratum, then they can also be connected by a *piecewise complex-analytic path* (see Definition 5.1) in this stratum (see Theorem 5.4). This property is probably known to some specialists but to our knowledge it has never been written explicitly. It may be quite useful. For instance, by [3] (respectively, [6]), we know that if $f$ and $f'$ are connected by a $\mu$-constant (respectively, $\mu^*$-constant) piecewise complex-analytic path $\{f_s\}$, $0 \leq s \leq 1$, then the monodromy zeta-function (respectively, the local embedded topology of the link) of $f_s$ at $\mathbf{0}$ is independent of $s$. The above property implies that this conclusion remains valid whenever $f$ and $f'$ lie in the same path-connected component of the $\mu$-constant (respectively, $\mu^*$-constant) stratum.

Hereafter, we introduce and study the properties of two constructible sets, $W(n, m, \mu)$ and $W^*(n, m, \mu^*)$, which are key objects to prove our main result.

## 2. The $\mu$-constant and $\mu^*$-constant strata

Let $\mathcal{O}_n \equiv \mathbb{C}\{z_1, \dots, z_n\}$ $(n \geq 1)$ be the ring of convergent power series at the origin, and let $\mathfrak{M} := \{f \in \mathcal{O}_n \mid f(\mathbf{0}) = 0\}$ be its maximal ideal. It is

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well known that for a given $f \in \mathfrak{M}$, if $H$ is a generic linear $i$-plane of $\mathbb{C}^n$ $(1 \le i \le n)$, then the Milnor number

$$\mu_{\mathbf{0}}^{(i)}(f) := \mu_{\mathbf{0}}(f|_H)$$

of the restriction of $f$ to $H$ depends only on $i$ and $f$. (Note that for a non-generic linear $i$-plane $L$, we have $\mu_{\mathbf{0}}^{(i)}(f) \le \mu_{\mathbf{0}}(f|_L)$.) In [7], Teissier introduced the $\mu^*$-sequence of $f$ at $\mathbf{0}$ as the $n$-tuple

$$\mu_{\mathbf{0}}^*(f) := (\mu_{\mathbf{0}}^{(n)}(f), \mu_{\mathbf{0}}^{(n-1)}(f), \dots, \mu_{\mathbf{0}}^{(1)}(f)).$$

Note that $\mu_{\mathbf{0}}^{(n)}(f)$ is nothing but the Milnor number $\mu_{\mathbf{0}}(f)$ of $f$ at $\mathbf{0}$, and $\mu_{\mathbf{0}}^{(1)}(f)$ is the multiplicity of $f$ at $\mathbf{0}$ minus 1.

By definition, if $\mu$ is a non-negative integer, then the $\mu$-constant stratum $\mathfrak{M}(\mu)$ of $\mathfrak{M}$ consists of all function-germs $f \in \mathfrak{M}$ such that the Milnor number $\mu_{\mathbf{0}}(f)$ of $f$ at $\mathbf{0}$ is equal to $\mu$. Similarly, if $\mu^{(n)}, \dots, \mu^{(1)}$ are non-negative integers and if $\mu^*$ denotes the $n$-tuple $(\mu^{(n)}, \mu^{(n-1)}, \dots, \mu^{(1)})$, then the $\mu^*$-constant stratum $\mathfrak{M}(\mu^*)$ of $\mathfrak{M}$ consists of all function-germs $f \in \mathfrak{M}$ such that the $\mu^*$-sequence $\mu_{\mathbf{0}}^*(f)$ of $f$ at $\mathbf{0}$ is given by the $n$-tuple $\mu^*$.

## 3. The constructible set $W(n, m, \mu)$

Now, let $m$ be a positive integer and let

$$P(n, m) := \{f \in \mathbb{C}[z_1, \dots, z_n] \mid \deg f \le m\}.$$

It is well known that $P(n, m)$ is a vector space of dimension $N := \binom{n+m}{n}$, putting an order on the basis' monomials $1 = M_1 < M_2 < \cdots < M_N$. Hereafter, we identify $P(n, m)$ with $\mathbb{C}^N$. Let

$$\pi_m : \mathcal{O}_n \to P(n, m)$$

be the natural projection obtained by deleting all terms of degree greater than $m$ (if any). For any $f \in \mathcal{O}_n$, we write $J(f)$ for the Jacobian ideal of $f$ (i.e., the ideal of $\mathcal{O}_n$ generated by the partial derivatives of $f$) and we put

$$J_m(f) := \pi_m(J(f)).$$

An element of $J_m(f)$ is written as $\pi_m(\sum_{i=1}^n h_i(\partial f/\partial z_i))$, $h_i \in \mathcal{O}_n$, and we easily check that $J_m(f)$ is the subspace of $P(n, m) \equiv \mathbb{C}^N$ generated by the following set of $nN$ vectors:

$$B_m(f) := \left\{ \pi_m\left( M_j \frac{\partial f}{\partial z_i} \right) \,\middle|\, 1 \le i \le n,\ 1 \le j \le N \right\}.$$

Hereafter, we identify $B_m(f)$ with an $N \times nN$ matrix. Then for any $\mu \in \mathbb{N}$, we consider the algebraic variety[1]

$$V(n,m,\mu) := \{f \in P(n,m) \mid \text{all } (N-\mu)\times(N-\mu) \text{ minors of } B_m(f) \text{ vanish}\},$$

and we define

$$W(n,m,\mu) := V(n,m,\mu-1) \setminus V(n,m,\mu). \tag{3.1}$$

Clearly, $W(n,m,\mu)$ is a constructible set, and for any $f \in P(n,m)$, the following equivalences hold true:

$$f \in W(n,m,\mu) \Leftrightarrow \text{rk } B_m(f) = N - \mu \Leftrightarrow \dim P(m,n)/J_m(f) = \mu. \tag{3.2}$$

The next proposition is a crucial step in the proof of Theorem 5.4, the main result of this paper. To state it, let us consider the set

$$W(n,\mu) := \{f \in \mathcal{O}_n \mid \dim \mathcal{O}_n/J(f) = \mu\}.$$

Remark 3.1. *The $\mu$-constant stratum $\mathfrak{M}(\mu)$ of $\mathfrak{M}$ is nothing but $\mathfrak{M} \cap W(n,\mu)$.*

Proposition 3.2. *Let $f \in \mathfrak{M}$ (i.e., $f(\mathbf{0}) = 0$).*
(1) *If $f \in W(n,\mu)$, then $\pi_m(f) \in W(n,m,\mu)$ for any $m \geq \mu$.*
(2) *If $f \in W(n,m,\mu)$ for some $m \geq \mu$, then $f \in W(n,\mu)$.*

Proof. Item (1) is easy. Take $f \in W(n,\mu)$. Then $\dim \mathcal{O}_n/J(f) = \mu$, and it is well known that this implies $\mathfrak{M}^\mu \subseteq J(f)$. Thus for any $m \geq \mu$, we have $\mathfrak{M}^{m+1} \subseteq \mathfrak{M}^\mu \subseteq J(f)$, and hence,

$$\mathcal{O}_n/J(f) = \mathcal{O}_n/(J(f) + \mathfrak{M}^{m+1}) = P(n,m)/J_m(f).$$

It follows that $\dim P(n,m)/J_m(f) = \mu$, and we conclude with (3.2).

Let us now prove item (2). Writing $\overline{\mathfrak{M}}$ for the canonical image of $\mathfrak{M}$ in

$$\mathcal{O}_n/(J(f) + \mathfrak{M}^{m+1}) = P(n,m)/J_m(f),$$

we look at the canonical decomposition

$$P(n,m)/J_m(f) = \mathcal{O}_n/(J(f) + \mathfrak{M}^{m+1}) = \bigoplus_{r=0}^{m} \overline{\mathfrak{M}}^r/\overline{\mathfrak{M}}^{r+1}.$$

Clearly, there exists $0 \leq r_0 \leq \mu$ such that $\overline{\mathfrak{M}}^{r_0}/\overline{\mathfrak{M}}^{r_0+1} = 0$, as otherwise $\dim P(n,m)/J_m(f) > \mu$, which is a contradiction. In particular, this implies

$$\mathfrak{M}^{r_0} \subseteq (J(f) + \mathfrak{M}^{m+1}) + \mathfrak{M}^{r_0+1}.$$

[1] By identifying $f = \sum_{j=1}^N a_j M_j$ with its coordinates $(a_1, \ldots, a_N) \in \mathbb{C}^N$ with respect to the basis $\{M_1, \ldots, M_N\}$, we immediately see that $V(n,m,\mu)$ is an algebraic variety.

Now, since $0 \le r_0 \le \mu \le m$, we also have $\mathfrak{M}^{m+1} \subseteq \mathfrak{M}^{r_0+1}$, a new inclusion which, combined with the above one, shows that

$$\mathfrak{M}^{r_0} \subseteq \mathfrak{M}^{r_0+1} + J(f). \tag{3.3}$$

Clearly, (3.3) implies that for any $k \ge r_0$ the following equality holds:

$$\mathfrak{M}^k + J(f) = \mathfrak{M}^{r_0} + J(f). \tag{3.4}$$

This equality, in turn, shows that $f$ has an isolated singularity at $\mathbf{0}$ (i.e., there exists $\mu' > 0$ such that $f \in W(n, \mu')$). Indeed, if not, then

$$\dim \mathcal{O}_n/(J(f) + \mathfrak{M}^k) \to \infty \qquad \text{as } k \to \infty. \tag{3.5}$$

However, by (3.4), for any $k \ge r_0$, we have

$$\mathcal{O}_n/(J(f) + \mathfrak{M}^k) \overset{(3.4)}{=} \mathcal{O}_n/(J(f) + \mathfrak{M}^{r_0}) \overset{(3.4)}{=} \mathcal{O}_n/(J(f) + \mathfrak{M}^{m+1}) = P(n,m)/J_m(f),$$

and therefore

$$\dim \mathcal{O}_n/(J(f) + \mathfrak{M}^k) = \dim P(n,m)/J_m(f) = \mu,$$

which contradicts (3.5). Now, since $f$ has an isolated singularity at $\mathbf{0}$, it follows from the Rückert Nullstelensatz [5] that there exists $\ell > 0$ such that $\mathfrak{M}^\ell \subseteq J(f)$. Clearly, we can assume $\ell \ge m$. Then,

$$\begin{aligned} \mu' &= \dim \mathcal{O}_n/J(f) = \dim \mathcal{O}_n/(\mathfrak{M}^\ell + J(f)) \\ &\overset{(3.4)}{=} \dim \mathcal{O}_n/(\mathfrak{M}^{m+1} + J(f)) = \dim P(n,m)/J_m(f) = \mu. \end{aligned}$$

In other words, $f \in W(n, \mu)$. □

The following statement is an immediate consequence of Proposition 3.2.

COROLLARY 3.3. *For any $m' \ge m \ge \mu$, the following inclusions are homotopy equivalences:*

$$W(n,m,\mu) \cap \mathfrak{M} \hookrightarrow W(n,m',\mu) \cap \mathfrak{M} \hookrightarrow W(n,\mu) \cap \mathfrak{M}.$$

## 4. The constructible set $W^*(n,m,\mu_n^*)$

Let $\mu^{(n)}, \dots, \mu^{(1)}$ be non-negative integers, and let $\mu_n^* := (\mu^{(n)}, \mu^{(n-1)}, \dots, \mu^{(1)})$. Hereafter, we are going to define a constructible set $W^*(n,m,\mu_n^*) \subseteq P(n,m)$ by induction on $n$. For that purpose, we consider the natural projection

$$\mathrm{pr}_1 : P(n,m) \times \mathbf{C}^{n-1} \to P(n,m)$$

onto the first factor and we introduce the map

$$\phi_n : P(n,m) \times \mathbb{C}^{n-1} \to P(n-1,m)$$

which associates to any $(f, b_1, \dots, b_{n-1}) \in P(n,m) \times \mathbb{C}^{n-1}$ the polynomial function defined by

$$(z_1, \dots, z_{n-1}) \mapsto f(z_1, \dots, z_{n-1}, b_1 z_1 + \dots + b_{n-1} z_{n-1}).$$

The induction starts at $n = 1$ in which case we set

$$W^*(1, m, \mu_1^*) := W(1, m, \mu^{(1)}), \tag{4.1}$$

where $W(1, m, \mu^{(1)})$ is the constructible set defined in (3.1). Now, suppose that for any $n \geq 2$ we have defined a constructible subset $W^*(n-1, m, \mu_{n-1}^*) \subseteq P(n-1, m)$, and let us define a new constructible subset $W^*(n, m, \mu_n^*) \subseteq P(n, m)$ by the relation

$$W^*(n, m, \mu_n^*) := A(n, m, \mu_n^*) \backslash B(n, m, \mu_n^*), \tag{4.2}$$

where

$$A(n, m, \mu_n^*) := \mathrm{pr}_1(\phi_n^{-1}(W^*(n-1, m, \mu_{n-1}^*)) \cap (W(n, m, \mu^{(n)}) \times \mathbb{C}^{n-1})),$$

$$B(n, m, \mu_n^*) := \mathrm{pr}_1\left(\phi_n^{-1}\left(\bigcup_{s < \mu^{(n-1)}} W(n-1, m, s)\right)\right).$$

Again, $W(n, m, \mu^{(n)})$ and $W(n-1, m, s)$ are the constructible sets defined in (3.1). Note that $f \in A(n, m, \mu_n^*)$ means $f \in W(n, m, \mu^{(n)})$ and there exists $(b_1, \dots, b_{n-1}) \in \mathbb{C}^{n-1}$ such that if

$$H := \{\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n \mid z_n = b_1 z_1 + \dots + b_{n-1} z_{n-1}\}$$

denotes the corresponding hyperplane, then

$$\phi_n(f, b_1, \dots, b_{n-1}) = f|_H \in W^*(n-1, m, \mu_{n-1}^*).$$

Saying $f \notin B(n, m, \mu_n^*)$ means that the above hyperplane $H$ is generic. That $W^*(n, m, \mu_n^*)$ is a constructible set follows from the Chevalley theorem (see [1, Théorème 3] and [2, Théorème (1.8.4)]).

The proposition below is a consequence of Proposition 3.2. It also plays a crucial role in the proof of Theorem 5.4. Let

$$W^*(n, \mu_n^*) := \{f \in \mathcal{O}_n \mid \mu_{\mathbf{0}}^*(f - f(\mathbf{0})) = \mu_n^*\}.$$

Note that for $n = 1$, we have $W^*(1, \mu_1^*) = W(1, \mu^{(1)})$.

Remark 4.1. *The $\mu^*$-constant stratum $\mathfrak{M}(\mu^*)$ of $\mathfrak{M}$ is nothing but $\mathfrak{M} \cap W^*(n, \mu_n^*)$.*

Proposition 4.2. *Put* $\mu^{\max} := \max\{\mu^{(n)}, \ldots, \mu^{(1)}\}$ *and pick* $f \in \mathfrak{M}$ *(i.e.,* $f(\mathbf{0}) = 0$*).*

(1) *If* $f \in W^*(n, \mu_n^*)$*, then* $\pi_m(f) \in W^*(n, m, \mu^*)$ *for any* $m \geq \mu^{\max}$*.*

(2) *If* $f \in W^*(n, m, \mu_n^*)$ *for some* $m \geq \mu^{\max}$*, then* $f \in W^*(n, \mu_n^*)$*.*

Proof. Let us first show item (1). We argue by induction on $n$. By Proposition 3.2, if $f \in W^*(1, \mu_1^*) = W(1, \mu^{(1)})$, then $\pi_m(f) \in W(1, m, \mu^{(1)}) =: W^*(1, m, \mu_1^*)$. For the inductive step, we assume that the following implication holds true:

$$f \in W^*(n-1, \mu_{n-1}^*) \Rightarrow \left| \begin{array}{l} \pi_m(f) \in W^*(n-1, m, \mu_{n-1}^*) \\ \text{for any } m \geq \max\{\mu^{(n-1)}, \ldots, \mu^{(1)}\}. \end{array} \right.$$

Now take $f \in W^*(n, \mu_n^*)$. We want to show that $\pi_m(f) \in W(n, m, \mu^{(n)})$, and for $H$ generic, $\pi_m(f)|_H \in W^*(n-1, m, \mu_{n-1}^*)$. We have:

$$f \in W^*(n, \mu_n^*) \Rightarrow f \in W(n, \mu^{(n)}) \overset{\text{Prop. 3.2}}{\Rightarrow} \pi_m(f) \in W(n, m, \mu^{(n)}).$$

Also, for any $m \geq \mu^{\max}$ we have:

$$\begin{aligned} f \in W^*(n, \mu_n^*) &\Rightarrow \text{for any } H \text{ generic, } f|_H \in W^*(n-1, \mu_{n-1}^*) \\ &\overset{\text{induction}}{\Rightarrow} \pi_m(f|_H) = \pi_m(f)|_H \in W^*(n-1, m, \mu_{n-1}^*). \end{aligned}$$

Let us now prove item (2). Again, we argue by induction on $n$. By Proposition 3.2, if $f \in W^*(1, m, \mu_1^*) := W(1, m, \mu^{(1)})$ for some $m \geq \mu^{(1)}$, then $f \in W(1, \mu^{(1)}) = W^*(1, \mu_1^*)$. For the inductive step, we assume that the following implication holds true:

$$f \in W^*(n-1, m, \mu_{n-1}^*) \text{ with } m \geq \max\{\mu^{(n-1)}, \ldots, \mu^{(1)}\} \Rightarrow f \in W^*(n-1, \mu_{n-1}^*).$$

Now take $f \in W^*(n, m, \mu_n^*)$ with $m \geq \mu^{\max}$. Then, by definition, $f \in W(n, m, \mu^{(n)})$, and for $H$ generic, $f|_H \in W^*(n-1, m, \mu_{n-1}^*)$. Thus, by the induction hypothesis, $f|_H \in W^*(n-1, \mu_{n-1}^*)$. Altogether, $f \in W^*(n, \mu_n^*)$. □

The following statement is an immediate corollary of Proposition 4.2.

Corollary 4.3. *For any* $m' \geq m \geq \mu$*, the following inclusions are homotopy equivalences:*

$$W^*(n, m, \mu_n^*) \cap \mathfrak{M} \hookrightarrow W^*(n, m', \mu_n^*) \cap \mathfrak{M} \hookrightarrow W^*(n, \mu_n^*) \cap \mathfrak{M}.$$

## 5. From continuous to piecewise complex-analytic paths

We first give a precise definition of a piecewise complex-analytic path in the $\mu$-constant (respectively, $\mu^*$-constant) stratum, and then we state and prove our main result.

**5.1. Piecewise complex-analytic paths.** Let $\mu \in \mathbb{N}$ and $\mu^* := (\mu^{(n)}, \mu^{(n-1)}, \dots, \mu^{(1)}) \in \mathbb{N}^n$ $(n \geq 1)$. Again, put $\mu^{\max} := \max\{\mu^{(n)}, \dots, \mu^{(1)}\}$.

Piecewise complex-analytic paths in $\mathfrak{M}(\mu)$ are defined as follows.

Definition 5.1. Let $f$ and $f'$ be non-constant polynomial function-germs at $\mathbf{0} \in \mathbb{C}^n$ such that $f(\mathbf{0}) = f'(\mathbf{0}) = 0$. Suppose that $f$ and $f'$ are in the same path-connected component of the $\mu$-constant stratum $\mathfrak{M}(\mu)$ of $\mathfrak{M}$. We say that $f$ and $f'$ can be joined by a *piecewise complex-analytic path* in $\mathfrak{M}(\mu)$ if there exists a continuous path

$$\gamma : [0,1] \to W(n,m,\mu) \cap \mathfrak{M}$$

for some integer $m \geq \mu$ such that:

(1) $\gamma(0) = f$ and $\gamma(1) = f'$ (in particular, this implies $f, f' \in W(n,m,\mu)$);

(2) there is a partition $0 = s_0 < s_1 < \cdots < s_{q_0} = 1$ of $[0,1]$, and for each $0 \leq q \leq q_0 - 1$, there exists an open subset $U_q \subseteq \mathbb{C}$ containing the interval $[s_q, s_{q+1}]$, together with a complex-analytic map

$$\tilde{\gamma}_q : U_q \to W(n,m,\mu) \cap \mathfrak{M},$$

such that

$$\tilde{\gamma}_q|_{[s_q, s_{q+1}]} = \gamma|_{[s_q, s_{q+1}]}.$$

A path $\gamma$ as above is called a *piecewise complex-analytic path* between $f$ and $f'$. Note that if $f$ and $f'$ can be joined by a piecewise complex-analytic path

$$\gamma : [0,1] \to W(n,m,\mu) \cap \mathfrak{M},$$

then, by Proposition 3.2, the Milnor number $\mu_{\mathbf{0}}(\gamma(s))$ is independent of $s \in [0,1]$. This justifies the terminology that $\gamma$ is a path *in the $\mu$-constant stratum* $\mathfrak{M}(\mu)$.

Piecewise complex-analytic paths in $\mathfrak{M}(\mu^*)$ are defined similarly, replacing $W(n,m,\mu)$ by $W^*(n,m,\mu^*)$ and changing the inequality $m \geq \mu$ into $m \geq \mu^{\max}$ in Definition 5.1. In this case, if $f$ and $f'$ can be joined by a piecewise complex-analytic path

$$\gamma : [0,1] \to W^*(n,m,\mu^*) \cap \mathfrak{M},$$

then, by Proposition 4.2, the $\mu^*$-sequence of $\gamma(s)$ is independent of $s \in [0,1]$.

**5.2. A key observation.** The following proposition is also a fundamental step in the proof of Theorem 5.4.

Proposition 5.2. *If $f$ and $f'$ are in the same path-connected component of $\mathfrak{M}(\mu)$, and if there exist an integer $m \geq \mu$ and a continuous map*

$$\varrho : [0,1] \to W(n,m,\mu) \cap \mathfrak{M}$$

*with* $\varrho(0) = f$ *and* $\varrho(1) = f'$, *then there also exists a continuous map*

$$\gamma : [0,1] \to W(n,m,\mu) \cap \mathfrak{M}$$

*satisfying the conditions* (1) *and* (2) *of Definition* 5.1.

A similar statement also holds true if we replace $\mathfrak{M}(\mu)$ and $W(n,m,\mu)$ by $\mathfrak{M}(\mu^*)$ and $W^*(n,m,\mu^*)$ and if we change the inequality $m \geq \mu$ into $m \geq \mu^{\max}$ both in Proposition 5.2 and Definition 5.1. The proof is similar to that of Proposition 5.2.

PROOF (of Proposition 5.2). Clearly, the assertion is true if the constructible set

$$W(n,m,\mu) \cap \mathfrak{M}$$

is smooth. If it is singular, then we can reduce the proof to the smooth case by the following argument. First, observe that each point $x$ of the image $\mathrm{im}(\varrho)$ has an open neighbourhood $U_x \subseteq P(n,m) \equiv \mathbf{C}^N$ such that the intersection $\mathrm{im}(\varrho) \cap U_x$ is contained in an irreducible $k$-dimensional algebraic subvariety $V_x$ of $W(n,m,\mu) \cap \mathfrak{M}$ (for some integer $k$). By the Noether normalization theorem, for each point $y$ of such a variety $V_x$, there is an open neighbourhood $O_y \subseteq \mathbf{C}^N$ and a finite branched covering

$$\pi_{x,y} : V_x \cap O_y \to U \subseteq \mathbf{C}^k,$$

where $U$ is an open ball of $\mathbf{C}^k$. Using the compactness of $\mathrm{im}(\varrho)$, we choose a sufficiently fine partition $0 = s_0 < s_1 < \cdots < s_{q_0} = 1$ of $[0,1]$ so that for each $q$ there exist $x, y$ with

$$\varrho([s_q, s_{q+1}]) \subseteq V_{x,y} := V_x \cap O_y.$$

Let $\varrho_q$ be the restriction of $\varrho$ to $[s_q, s_{q+1}]$, and let $L \subseteq U$ be (the trace on $U$ of) a complex line through $\pi_{x,y} \circ \varrho_q(s_q)$ and $\pi_{x,y} \circ \varrho_q(s_{q+1})$. The inverse image $\pi_{x,y}^{-1}(L)$ of $L$ by $\pi_{x,y}$ is an algebraic variety of complex dimension 1, and we easily show that $\varrho_q$ is homotopic to a path contained in $\pi_{x,y}^{-1}(L)$ by a homotopy leaving the ends $\varrho(s_q)$ and $\varrho(s_{q+1})$ fixed. We still denote by $\varrho_q$ the path of $\pi_{x,y}^{-1}(L)$ obtained in this way, and we consider a normalization

$$\tau : N(\pi_{x,y}^{-1}(L)) \to \pi_{x,y}^{-1}(L).$$

Then $\varrho_q$ can be lifted to a path $\varsigma_q$ in $N(\pi_{x,y}^{-1}(L))$, and since $N(\pi_{x,y}^{-1}(L))$ is smooth and the problem is solved in this case, we can find an open subset $U_q \subseteq \mathbf{C}$ containing $[s_q, s_{q+1}]$ together with a complex-analytic map $\tilde{\varsigma}_q : U_q \to N(\pi_{x,y}^{-1}(L))$ such that

$$\tilde{\varsigma}_q|_{[s_q, s_{q+1}]} = \varsigma_q.$$

The desired complex-analytic map $\tilde{\gamma}_q$ is given by the composite $\tilde{\gamma}_q := \tau \circ \tilde{\varsigma}_q$, and $\gamma$ is the continuous path defined on each $[s_q, s_{q+1}]$ by the restriction $\tilde{\gamma}_q|_{[s_q,s_{q+1}]}$. □

Remark 5.3. *The above argument actually shows that if two points in a constructible set C can be connected by a continuous path in C, then they can also be connected by a piecewise complex-analytic path in C. This assertion (and hence Proposition* 5.2*) can be also viewed, more simply, as a consequence of the following observation of Ramanujam:* "Any two points of an irreducible variety can be connected by an irreducible curve". *This observation appears in* [4] *with the following proof:*

> "To prove this one notices that by Chow's lemma, one may assume that the variety in question is projective. Then one could blow up the two points and in any projective imbedding of this blow-up, take a generic hyperplane section which is irreducible of lower dimension. Since this hyperplane meets the two exceptional divisors, the problem reduces to one of lower dimension and hence proves the assertion by induction."

*Now, coming back to the assertion about C, since the image of a continuous path connecting two points in the same path-connected component of C is in a union of finitely many irreducible components of the Zariski closure of C, it is enough to consider the case when this closure is irreducible, and this case is clear from Ramanujam's observation.*

**5.3. Statement and proof of the main theorem.** We can now give a precise statement and prove the main result of this note.

Theorem 5.4. *Let* $\mu \in \mathbb{N}$ *and* $\mu^* := (\mu^{(n)}, \mu^{(n-1)}, \dots, \mu^{(1)}) \in \mathbb{N}^n$ $(n \geq 1)$, *and let* $f$ *and* $f'$ *be non-constant polynomial function-germs at* $\mathbf{0} \in \mathbb{C}^n$ *such that* $f(\mathbf{0}) = f'(\mathbf{0}) = 0$.

(1) *If* $f$ *and* $f'$ *are in the same path-connected component of the* $\mu$*-constant stratum* $\mathfrak{M}(\mu)$, *then they can be joined by a piecewise complex-analytic path*

$$\gamma : [0,1] \to W(n,m,\mu) \cap \mathfrak{M}$$

*for any integer* $m \geq \max\{\deg f, \deg f', \mu\}$.

(2) *Similarly, if* $f$ *and* $f'$ *are in the same path-connected component of the* $\mu^*$*-constant stratum* $\mathfrak{M}(\mu^*)$, *then they can be joined by a piecewise complex-analytic path*

$$\gamma : [0,1] \to W^*(n,m,\mu^*) \cap \mathfrak{M}$$

*for any integer* $m \geq \max\{\deg f, \deg f', \mu^{(n)}, \dots, \mu^{(1)}\}$.

PROOF. To show the first item, let $f$ and $f'$ be polynomial function-germs lying in the same path-connected component of the $\mu$-constant stratum of $\mathfrak{M}$. Then there is a continuous path $\varrho : [0,1] \to \mathfrak{M}$, $s \mapsto \varrho(s)$, such that $\varrho(0) = f$, $\varrho(1) = f'$ and $\mu_{\mathbf{0}}(\varrho(s)) = \mu_{\mathbf{0}}(f) = \mu_{\mathbf{0}}(f') =: \mu$. In other words, $\varrho(s) \in W(n,\mu) \cap \mathfrak{M} = \mathfrak{M}(\mu)$. Take any $m \geq \max\{\deg f, \deg f', \mu\}$. Then Proposition 3.2 shows that $\pi_m(\varrho(s)) \in W(n,m,\mu) \cap \mathfrak{M}$ for any $s \in [0,1]$, and since

$$\pi_m(\varrho(0)) = \pi_m(f) = f \qquad \text{and} \qquad \pi_m(\varrho(1)) = \pi_m(f') = f',$$

we have that $s \mapsto \pi_m(\varrho(s))$ is a path in $W(n,m,\mu) \cap \mathfrak{M}$ from $f$ to $f'$. Thus, by Proposition 5.2, there is also a path

$$\gamma : [0,1] \to W(n,m,\mu) \cap \mathfrak{M}$$

satisfying the conditions (1) and (2) of Definition 5.1 (i.e., $f$ and $f'$ can be joined by a piecewise complex-analytic path in $\mathfrak{M}(\mu)$).

To prove the second item, let $f$ and $f'$ be polynomial function-germs lying in the same path-connected component of the $\mu^*$-constant stratum of $\mathfrak{M}$. Then there is a continuous path $\varrho : [0,1] \to \mathfrak{M}$, $s \mapsto \varrho(s)$, such that $\varrho(0) = f$, $\varrho(1) = f'$ and the $\mu^*$-sequence of $\varrho(s)$ is given by $\mu^* \equiv \mu_n^* := (\mu^{(n)}, \ldots, \mu^{(1)})$ for any $s \in [0,1]$, where $\mu^{(i)} := \mu_{\mathbf{0}}^{(i)}(f) = \mu_{\mathbf{0}}^{(i)}(f')$. In other words, $\varrho(s) \in W^*(n,\mu^*) \cap \mathfrak{M} = \mathfrak{M}(\mu^*)$. Take any $m \geq \max\{\deg f, \deg f', \mu^{(n)}, \ldots, \mu^{(1)}\}$. Then Proposition 4.2 shows that $\pi_m(\varrho(s)) \in W^*(n,m,\mu^*) \cap \mathfrak{M}$ for any $s \in [0,1]$, and since

$$\pi_m(\varrho(0)) = \pi_m(f) = f \qquad \text{and} \qquad \pi_m(\varrho(1)) = \pi_m(f') = f',$$

we have that $s \mapsto \pi_m(\varrho(s))$ is a path in $W^*(n,m,\mu^*) \cap \mathfrak{M}$ from $f$ to $f'$. Thus, by the $\mu^*$ version of Proposition 5.2 (see the comment after it), there is also a path

$$\gamma : [0,1] \to W^*(n,m,\mu^*) \cap \mathfrak{M}$$

satisfying the conditions (1) and (2) of Definition 5.1 with $\mathfrak{M}(\mu^*)$ instead of $\mathfrak{M}(\mu)$ and $W^*(n,m,\mu^*)$ instead of $W(n,m,\mu)$ (i.e., $f$ and $f'$ can be joined by a piecewise complex-analytic path in $\mathfrak{M}(\mu^*)$). □

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*Christophe Eyral*
*Institute of Mathematics*
*Polish Academy of Sciences*
*ul. Śniadeckich* 8
00-656 *Warsaw, Poland*
*E-mail: cheyral@impan.pl*

*Mutsuo Oka*
3-19-8 *Nakaochiai*
*Shinjuku-ku*
*Tokyo* 161-0032*, Japan*
*(Professor Emeritus of Tokyo Institute of Technology)*
*E-mail: okamutsuo@gmail.com*