Hiroshima Math. J., **54** (2024), 133–155
doi:10.32917/h2022020

Asymptotic behavior of a viscoelastic problem with long-term memory and Tresca friction law

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(Received August 20, 2022)
(Revised November 2, 2023)

Abstract. This paper examines the asymptotic behavior of solutions of the three dimensional viscoelastic problem with long-term memory and Tresca friction law in a thin domain Σ^{ε}. We study the asymptotic behavior of this problem when the thickness ε tends to zero and we prove a convergence theorem for the displacement and velocity in appropriate functional spaces. Besides, the limit problem with the limit of Tresca free boundary conditions and a specific Reynolds limit equation is obtained.

1. Introduction

A multitude of physical and engineering problems requires an actual study of models that take into account effects such as hardening and memory materials. It is well known that viscoelastic materials have a wide application in natural sciences as they show a natural damping, which is due to the special property of these materials to keep memory of their history. This property attracted the attention of many mathematicians, where the memory effects of these materials are modeled by integro- differential equations. In mathematical literature, several studies of viscosity models have been conducted with different boundary conditions. Lions [11] gave some models for the behavior viscosity laws, he then studied the existence and uniqueness of solution for some viscosity problems with the friction of Tresca. The asymptotic stability of the equations of linear viscoelasticity with Dirichlet boundary condition is studied in [8]. In [6], a study is conducted of the existence and uniform decay rates for the solutions of the semi-linear viscosity problem. In addition existence and uniqueness of regular and weak solutions are then proved. Sofonea et al. [19] studied the existence and uniqueness of the weak solution of an antiplane contact problem for viscoelastic materials with long-term memory and Tresca friction. Furthermore, a numerical analysis of some

2020 *Mathematics Subject Classification*. 35C20.
Key words and phrases. viscoelastic problem, Tresca friction law, long-term memory, weak solution, thin domain.

viscosity problems with long-term memory and friction of the Tresca type can be found in [16] and [17].

More recently, many authors have applied asymptotic methods to problems of three-dimensional viscosity and elasticity in order to derive new models reduced to other one-dimensional or two-dimensional models. We can find the study of the asymptotic analysis for a family of shells in viscoelastic material with long-term memory without friction to justify the two-dimensional limit problems in [12] and [13]. Dilmi and Benseridi in [4] used the asymptotic analysis to justify the convergence results and the two-dimensional problem limit of a dynamical problem of isothermal elasticity with the non-linear friction of Tresca. The asymptotic convergence of a dynamical problem of a non-isothermal linear elasticity with friction was studied in [18]. An asymptotic analysis of the boundary value problems for linear elastic with a non-linear term in a stationary regime has been studied in [5] and [9]. In [15] Paumier studied the asymptotic modeling of a thin elastic plate in a unilateral contact with a friction against a rigid obstacle (Signorini problem with friction), where he proved that any family of solutions of the three-dimensional problem of Signorini with a friction converges strongly towards a unique solution of a two-dimensional problem of plate of the Signorini type without friction. Bayada and Lhalouani [2] investigated the asymptotic and numerical analysis for a unilateral contact problem with Coulomb's friction between an elastic body and a thin elastic soft layer. Benseghir et al. [3] studied the theoretical analysis of a frictionless contact between two general elastic bodies in a stationary regime in a three-dimensional thin domain with Tresca friction law.

Concerning the asymptotic analysis for incompressible fluids in a thin domain, with friction boundary conditions for the stationary case we direct the reader to see [1, 10].

In this work, we analyse the asymptotic behavior of the three-dimensional displacement field of a body made of a viscoelastic long-term memory material in the presence of a Tresca friction law on a boundary part, and condition of Dirichlet on other part, as the thickness approaches zero. This model takes into account the history of previous deformations or stresses experienced by the material. The memory effect results in a precise damping mechanism that controls the stability of dynamic systems. Recently, a large number of current physical and technical problems have shown that it becomes necessary to study models that take into account the effect of thickness and memory on a material and its stability. The purpose of this work is to justify mathematically the two-dimensional model of the three-dimensional linear viscosity problem endowed with long-term memory and Tresca friction law. This work gives generalization and investigation of some of the results obtained in the previously mentioned papers. The first strong point is that this study takes into account materials with the term memory. The second point is what kind of conditions can be applied to obtain estimates that allow us to reach the limit problem. In the current study, we will have to solve the set of difficulties that we encounter due to the change of context. The first difficulty is formulating the proof of some estimates on the displacement and the velocity (Theorem 2). The second difficulty is to know what the asymptotic behavior of material will be when the thickness of the thin film is very small (Theorems 4–5).

This article is organized as follows. In Section 2, we give the viscosity problem with its variational formulation, then we give the theorem of the existence and uniqueness of the weak solution. In Section 3, we study the asymptotic analysis of the problem, when the thickness of the domain tends to zero. After using the change of scale technique, we establish some estimates independent of the thickness ε for the displacement and velocity. The obtained estimates allow us to give the convergence results as well as the unique limit problem in Section 4.

2. Problem statement and weak formulation

To formulate the system of the linear viscosity with long-term memory, we begin with the following notations. Let Σ^ε be a bounded domain in $\mathbb{R}^3$, where $0 < \varepsilon < 1$ is a positive real number that tends to zero. The boundary of Σ^ε will be noted $\partial\overline{\Sigma}^\varepsilon = \partial\overline{\Sigma}_1^\varepsilon \cup \partial\overline{\Sigma}_L^\varepsilon \cup \overline{\Lambda}$, where $\partial\overline{\Sigma}_1^\varepsilon$ is the upper surface of the equation $x_3 = \varepsilon q(x_1, x_2)$, $\partial\overline{\Sigma}_L^\varepsilon$ is the lateral boundary, Λ is a bounded domain in $\mathbb{R}^3$ of the equation $x_3 = 0$ which constitutes the bottom of the domain Σ^ε and $q(.)$ is a supposed function of class C^1 defined on Λ such that

$$0 < \underline{q} = q_{min} \leq q(x_1, x_2) \leq q_{max} = \bar{q}, \qquad \forall (x_1, x_2) \in \overline{\Lambda}.$$

For all $x = (x', x_3) \in \mathbb{R}^3$ where $x' = (x_1, x_2) \in \mathbb{R}^2$, the physical domain Σ^ε is given by

$$\Sigma^\varepsilon = \{(x', x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \Lambda,\ 0 < x_3 < \varepsilon q(x')\}.$$

Let $u^\varepsilon(x,t)$ be the displacement at a point x at the moment $t \in [0, T]$. We denote by $r(.)$ the strain tensor given by

$$r_{ij}(u^\varepsilon) = \frac{1}{2}\left(\frac{\partial u_i^\varepsilon}{\partial x_j} + \frac{\partial u_j^\varepsilon}{\partial x_i}\right), \qquad 1 \leq i, j \leq 3.$$

The viscosity behavior law with the long-term memory is given by

$$\chi_{ij}^\varepsilon(u^\varepsilon) = 2\mu r_{ij}(u^\varepsilon) + 2\int_0^t g(t-s) r_{ij}(u^\varepsilon)(s)\,ds, \qquad 1 \leq i, j \leq 3,$$

where μ is the Lamé coefficient and $g(.)$ is a relaxation function which satisfies the following conditions:

1) $g : \mathbb{R}_+ \to \mathbb{R}_-$ is a function of class C^2 such that

$$0 < l \le \frac{\mu}{2} + \int_0^t g(s)ds, \qquad \forall t \in [0, T].$$

2) $g(.)$ is increasing, and there are positive constants G_1 and G_2 such that

$$0 < g'(t) \le G_1 \qquad \text{and} \qquad |g''(t)| \le G_2, \qquad \forall t \in [0, T].$$

For more details on the function $g(.)$, we present the explanations given by Kendra Cherry in [7]: "Long-term memory is usually divided into two types explicit and implicit", where "explicit memories, also known as declarative memories, include all of the memories that are available in consciousness and implicit memories are those that are mostly unconscious". For the good understanding of the problem presented in this work with the long-term memory, we give some examples on this term which satisfy the conditions (1) and (2): $g_1(s) = \frac{-\mu}{4}e^{-s}$, $g_2(s) = \frac{-\mu}{4\pi(1+s^2)}$ and $g_3(s) = \frac{-\mu}{4T}e^{-s^2}$, $T > 0$. These can appear in models of continuum mechanics, in particular for the fluid Newtonian and non-Newtonian as Maxwell's fluid, Bingham fluid and others.

The viscosity equation with long-term memory is given as follows

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} - div(\chi^\varepsilon(u^\varepsilon)) = p^\varepsilon \qquad in\ \Sigma^\varepsilon \times]0, T[, \tag{1}$$

where $p^\varepsilon = (p_1^\varepsilon, p_2^\varepsilon, p_3^\varepsilon)$ is the external force.

Let n be the unit outward normal vector to $\partial\Sigma^\varepsilon$. The normal and the tangential components of u^ε on the boundary are given by

$$u_n^\varepsilon = u^\varepsilon \cdot n, \qquad u_\tau^\varepsilon = u^\varepsilon - u_n^\varepsilon n.$$

Similarly, we denote by χ_n^ε and χ_τ^ε the normal and tangential components of χ^ε

$$\chi_n^\varepsilon = (\chi^\varepsilon \cdot n) \cdot n, \qquad \chi_\tau^\varepsilon = \chi^\varepsilon \cdot n - (\chi_n^\varepsilon)n.$$

The boundary conditions are:

- The displacement u^ε is known on $\partial\Sigma_1^\varepsilon \times]0, T[$ and on $\partial\Sigma_L^\varepsilon \times]0, T[$, also it satisfies the Dirichlet condition

$$u^\varepsilon = 0. \tag{2}$$

• On $\Lambda \times]0, T[$, the velocity is supposed to be unknown, and it satisfies the following condition

$$\frac{\partial u^{\varepsilon}}{\partial t} \cdot n = 0. \tag{3}$$

• An existence of a friction on $\Lambda \times]0, T[$ is modeled by the nonlinear Tresca law (see [11, 14])

$$\begin{cases} |\chi_{\tau}^{\varepsilon}| < \kappa^{\varepsilon} \Rightarrow \left(\frac{\partial u^{\varepsilon}}{\partial t}\right)_{\tau} = 0, \\ |\chi_{\tau}^{\varepsilon}| = \kappa^{\varepsilon} \Rightarrow \exists \beta > 0 \text{ such that } \left(\frac{\partial u^{\varepsilon}}{\partial t}\right)_{\tau} = -\beta \chi_{\tau}^{\varepsilon}, \end{cases} \tag{4}$$

where $|.|$ denotes the Euclidean norm of $\mathbb{R}^3$ and κ^{ε} is a given function.

The problem (1)–(4) is supplemented by initial conditions given by

$$u^{\varepsilon}(x,0) = u_0(x), \qquad \frac{\partial u^{\varepsilon}}{\partial t}(x,0) = u_1(x), \qquad \forall x \in \Sigma^{\varepsilon}. \tag{5}$$

To obtain the variational formulation of the problem (1)–(5), we use the following notations. We denote by (u, v) the scalar product in $L^2(\Sigma^{\varepsilon})$, i.e., $(u, v) = \int_{\Sigma^{\varepsilon}} uv \, dx$. The symbol $H^1(\Sigma^{\varepsilon})^3$ is the Sobolev space given by

$$H^1(\Sigma^{\varepsilon})^3 = \left\{ u \in L^2(\Sigma^{\varepsilon})^3 : \frac{\partial u_i}{\partial x_j} \in L^2(\Sigma^{\varepsilon}),\ i, j = 1, 2, 3 \right\},$$

we call $H_0^1(\Sigma^{\varepsilon})^3$ the closure of $D(\Sigma^{\varepsilon})^3$ in $H^1(\Sigma^{\varepsilon})^3$, and $H^{-1}(\Sigma^{\varepsilon})^3$ denotes the dual space of $H_0^1(\Sigma^{\varepsilon})^3$. Let X be a Banach space endowed with the norm $\|.\|_X$. We denote by $L^2(0, T; X)$ the space of functions $u :]0, T[\to X$ such that $u(t)$ is measurable for dt. This space is a Banach space endowed with the norm

$$\|u\|_{L^2(0,T;X)} = \left(\int_0^T \|u(s)\|_X^2 ds \right)^{1/2}.$$

The symbol $L^{\infty}(0, T; X)$ denotes the space of functions $u :]0, T[\to X$ which are measurable and $u \in L^{\infty}(0, T)$. This space is a Banach space endowed with the norm

$$\|u\|_{L^{\infty}(0,T;X)} = ess \sup_{s \in]0,T[} \|u(s)\|_X.$$

By multiplying the equation (1) by $\left(\varphi - \frac{\partial u^{\varepsilon}}{\partial t}\right)$, integrating over Σ^{ε} and using integration by parts and the boundary conditions (2)–(4), we get the following weak problem:

$$\begin{cases} \textit{Find } u^\varepsilon \in W^\varepsilon, \textit{ with } \frac{\partial u^\varepsilon}{\partial t} \in W^\varepsilon, \ \forall t \in [0,T], \textit{ such that} \\ \left(\frac{\partial^2 u^\varepsilon}{\partial t^2}, \varphi - \frac{\partial u^\varepsilon}{\partial t}\right) + \mu \breve{a}\left(u^\varepsilon, \varphi - \frac{\partial u^\varepsilon}{\partial t}\right) \\ \quad + \int_0^t g(t-s)\breve{a}\left(u^\varepsilon(s), \varphi - \frac{\partial u^\varepsilon}{\partial t}(t)\right) ds \\ \quad + \jmath^\varepsilon(\varphi) - \jmath^\varepsilon\left(\frac{\partial u^\varepsilon}{\partial t}\right) \geq \left(p^\varepsilon, \varphi - \frac{\partial u^\varepsilon}{\partial t}\right), \quad \forall \varphi \in W^\varepsilon, \\ u^\varepsilon(x,0) = u_0(x), \quad \frac{\partial u^\varepsilon}{\partial t}(x,0) = u_1(x), \end{cases} \tag{6}$$

where

$$W^\varepsilon = \{v \in H^1(\Sigma^\varepsilon)^3 : v = 0 \text{ on } \partial\Sigma_1^\varepsilon \cup \partial\Sigma_L^\varepsilon, \ v \cdot n = 0 \text{ on } \Lambda\},$$

$$\breve{a}(u,v) = 2\int_{\Sigma^\varepsilon} r(u) r(v) dx, \qquad \forall u, v \in H^1(\Sigma^\varepsilon)^3,$$

$$\jmath^\varepsilon(v) = \int_\Lambda \kappa^\varepsilon |v| dx', \qquad \forall v \in H^1(\Sigma^\varepsilon)^3,$$

and

$$(p,v) = \int_{\Sigma^\varepsilon} p \cdot v \, dx, \qquad \forall v \in H^1(\Sigma^\varepsilon)^3,$$

where

$$r(u)r(v) = \sum_{i,j=1}^3 r_{ij}(u) r_{ij}(v).$$

The existence and uniqueness are stated as follows.

Theorem 1. *Under the assumptions*

$$p^\varepsilon, \frac{\partial p^\varepsilon}{\partial t}, \frac{\partial^2 p^\varepsilon}{\partial t^2} \in L^2(0,T; L^2(\Sigma^\varepsilon)^3),$$

$$\kappa^\varepsilon \in C_0^\infty(\Lambda), \quad \kappa^\varepsilon > 0, \quad \textit{is independent of } t,$$

$$u_0(x) \in H^1(\Sigma^\varepsilon)^3, \quad u_1(x) \in H^1(\Sigma^\varepsilon)^3, \quad (u_1)_\tau = 0, \tag{7}$$

there exists a unique solution u^ε *of* (6) *such that*

$$u^\varepsilon, \frac{\partial u^\varepsilon}{\partial t} \in L^\infty(0,T; H^1(\Sigma^\varepsilon)^3), \qquad \frac{\partial^2 u^\varepsilon}{\partial t^2} \in L^\infty(0,T; L^2(\Sigma^\varepsilon)^3).$$

Proof. The proof is similar to the one in [11].

3. A priori estimates

For the asymptotic analysis of the problem we use the scaling $z = x_3/\varepsilon$. This method consists of transposing the problem initially posed in the domain Σ^ε in an equivalent problem posed in a domain Σ independent of ε, where

$$\Sigma = \{(x', z) \in \mathbb{R}^3 : (x', 0) \in \Lambda,\ 0 < z < q(x')\},$$

and $\partial\overline{\Sigma} = \partial\overline{\Sigma}_1 \cup \partial\overline{\Sigma}_L \cup \Lambda$ is its boundary. We define now on Σ the following functions

$$\begin{cases} \hat{u}_i^\varepsilon(x', z, t) = u_i^\varepsilon(x', x_3, t), \quad i = 1, 2, \\ \hat{u}_3^\varepsilon(x', z, t) = \varepsilon^{-1} u_3^\varepsilon(x', x_3, t). \end{cases}$$

For the data of problem (6), we suppose that they depend on ε in the following manner

$$\begin{cases} \hat{p}(x', z, t) = \varepsilon^2 p^\varepsilon(x', x_3, t), \\ \hat{\kappa} = \varepsilon \kappa^\varepsilon, \end{cases}$$

with $\hat{p}$ and $\hat{\kappa}$ independent of ε.

We introduce now the functional framework on Σ as follows

$$W = \{\varphi \in H^1(\Sigma)^3 : \varphi = 0 \ \ on\ \ \partial\overline{\Sigma}_1 \cup \partial\overline{\Sigma}_L \ \ and\ \ \varphi \cdot n = 0 \ \ on\ \ \Lambda\},$$

$$\Pi(W) = \{\varphi \in H^1(\Sigma)^2 : \varphi = (\varphi_1, \varphi_2),\ \varphi_i = 0 \ \ on\ \ \partial\overline{\Sigma}_1 \cup \partial\overline{\Sigma}_L \ \ for\ \ i = 1, 2\},$$

$$P_z = \left\{ v = (v_1, v_2) \in L^2(\Sigma)^2 : \frac{\partial v_i}{\partial z} \in L^2(\Sigma),\ i = 1, 2 \ \ and\ \ v = 0 \ \ on\ \ \partial\overline{\Sigma}_1 \right\}.$$

The set P_z is a Banach space endowed with the norm

$$\|v\|_{P_z} = \left(\sum_{i=1}^{2} \left(\|v_i\|^2_{L^2(\Sigma)} + \left\| \frac{\partial v_i}{\partial z} \right\|^2_{L^2(\Sigma)} \right) \right)^{1/2}.$$

Multiplying (6) by ε and while passing to the fixed domain Σ, we show that the variational problem is equivalent to the following problem

$$\begin{cases} Find\ \ \hat{u}^\varepsilon \in W,\ \ with\ \ \frac{\partial \hat{u}^\varepsilon}{\partial t} \in W,\ \ \forall t \in]0, T[\ \ such\ \ that \\ \sum_{i=1}^{2} \varepsilon^2 \left(\frac{\partial^2 \hat{u}_i^\varepsilon}{\partial t^2}, \hat{\varphi}_i - \frac{\partial \hat{u}_i^\varepsilon}{\partial t} \right) + \varepsilon^4 \left(\frac{\partial^2 \hat{u}_3^\varepsilon}{\partial t^2}, \hat{\varphi}_3 - \frac{\partial \hat{u}_3^\varepsilon}{\partial t} \right) + \mu \hat{a}\left(\hat{u}^\varepsilon, \hat{\varphi} - \frac{\partial \hat{u}^\varepsilon}{\partial t} \right) \\ \quad + \int_0^t g(t-s) \hat{a}\left(\hat{u}^\varepsilon(s), \hat{\varphi} - \frac{\partial \hat{u}^\varepsilon}{\partial t}(t) \right) ds + \hat{J}(\hat{\varphi}) - \hat{J}\left(\frac{\partial \hat{u}^\varepsilon}{\partial t} \right) \\ \quad \geq \sum_{i=1}^{2} \left(\hat{p}_i, \hat{\varphi}_i - \frac{\partial \hat{u}_i^\varepsilon}{\partial t} \right) + \varepsilon \left(\hat{p}_3, \hat{\varphi}_3 - \frac{\partial \hat{u}_3^\varepsilon}{\partial t} \right), \quad \forall \hat{\varphi} \in W, \\ \hat{u}^\varepsilon(0) = \hat{u}_0, \quad \frac{\partial \hat{u}^\varepsilon}{\partial t}(0) = \hat{u}_1, \end{cases} \tag{8}$$

where

$$\hat{J}(\hat{\varphi}) = \int_\Lambda \hat{\kappa} |\hat{\varphi}| dx',$$

and

$$\hat{a}(\hat{\psi},\hat{\varphi}) = \varepsilon^2 \sum_{i,j=1}^{2} \int_{\Sigma} \left(\frac{\partial \hat{\psi}_i}{\partial x_j} + \frac{\partial \hat{\psi}_j}{\partial x_i} \right) \frac{\partial \hat{\varphi}_i}{\partial x_j}\, dx'dz$$
$$+ \sum_{i=1}^{2} \int_{\Sigma} \left(\frac{\partial \hat{\psi}_i}{\partial z} + \varepsilon^2 \frac{\partial \hat{\psi}_3}{\partial x_i} \right)\left(\frac{\partial \hat{\varphi}_i}{\partial z} + \varepsilon^2 \frac{\partial \hat{\varphi}_3}{\partial x_i} \right) dx'dz$$
$$+ 2\varepsilon^2 \int_{\Sigma} \frac{\partial \hat{\psi}_3}{\partial z} \frac{\partial \hat{\varphi}_3}{\partial z}\, dx'dz.$$

In the next, we will obtain estimates for the displacement $\hat{u}^\varepsilon$ and velocity $\frac{\partial \hat{u}^\varepsilon}{\partial t}$.

THEOREM 2. *Under the assumptions of Theorem* 1, *there exists a positive constant C independent of ε, such that the following estimates hold:*

$$\sum_{i=1}^{2} \left[\left\| \frac{\partial \hat{u}_i^\varepsilon}{\partial z} \right\|_{L^2(\Sigma)}^2 + \left\| \varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial t} \right\|_{L^2(\Sigma)}^2 + \left\| \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \right\|_{L^2(\Sigma)}^2 \right]$$
$$+ \sum_{i,j=1}^{2} \left\| \varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \right\|_{L^2(\Sigma)}^2 + \left\| \varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \right\|_{L^2(\Sigma)}^2 + \left\| \varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial t} \right\|_{L^2(\Sigma)}^2 \le C, \tag{9}$$

and

$$\sum_{i=1}^{2} \left[\left\| \frac{\partial}{\partial z}\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial t} \right) \right\|_{L^2(\Sigma)}^2 + \left\| \varepsilon \frac{\partial^2 \hat{u}_i^\varepsilon}{\partial t^2} \right\|_{L^2(\Sigma)}^2 + \left\| \varepsilon^2 \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{u}_3^\varepsilon}{\partial t} \right) \right\|_{L^2(\Sigma)}^2 \right]$$
$$+ \sum_{i,j=1}^{2} \left\| \varepsilon \frac{\partial}{\partial x_j}\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial t} \right) \right\|_{L^2(\Sigma)}^2 + \left\| \varepsilon \frac{\partial^2 \hat{u}_3^\varepsilon}{\partial t^2} \right\|_{L^2(\Sigma)}^2 + \left\| \varepsilon \frac{\partial}{\partial z}\left(\frac{\partial \hat{u}_3^\varepsilon}{\partial t} \right) \right\|_{L^2(\Sigma)}^2 \le C. \tag{10}$$

PROOF. Let u^ε be the solution of problem (6). By choosing $\varphi = 0$, we find

$$\left(\frac{\partial^2 u^\varepsilon}{\partial t^2}, \frac{\partial u^\varepsilon}{\partial t} \right) + \mu \breve{a}\left(u^\varepsilon, \frac{\partial u^\varepsilon}{\partial t} \right) + \int_0^t g(t-s)\breve{a}\left(u^\varepsilon(s), \frac{\partial u^\varepsilon}{\partial t}(t) \right) ds + \jmath^\varepsilon\left(\frac{\partial u^\varepsilon}{\partial t} \right)$$
$$\le \left(p^\varepsilon, \frac{\partial u^\varepsilon}{\partial t} \right).$$

Since we know that $\jmath^\varepsilon(.)$ is positive (since $\kappa^\varepsilon > 0$), we have

$$\frac{1}{2}\frac{d}{dt}\left[\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|_{L^2(\Sigma^\varepsilon)}^2 + \mu \breve{a}(u^\varepsilon, u^\varepsilon) \right] + \int_0^t g(t-s)\breve{a}\left(u^\varepsilon(s), \frac{\partial u^\varepsilon}{\partial t}(t) \right) ds$$
$$\le \left(p^\varepsilon, \frac{\partial u^\varepsilon}{\partial t} \right). \tag{11}$$

On the other hand, we have

$$
\begin{aligned}
&\frac{1}{2}\int_0^t g(t-s)\breve{a}\left(u^\varepsilon(s),\frac{\partial u^\varepsilon}{\partial t}(t)\right)ds \\
&\quad= \int_0^t g(t-s)\int_{\Sigma^\varepsilon} r(u^\varepsilon(s))r\left(\frac{\partial u^\varepsilon}{\partial t}(t)\right)dxds \\
&\quad= \int_0^t g(t-s)\int_{\Sigma^\varepsilon} r\left(\frac{\partial u^\varepsilon}{\partial t}(t)\right)(r(u^\varepsilon(s))-r(u^\varepsilon(t)))dxds \\
&\qquad+ \int_0^t g(t-s)\int_{\Sigma^\varepsilon} r\left(\frac{\partial u^\varepsilon}{\partial t}(t)\right)r(u^\varepsilon(t))dxds \\
&\quad= -\frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\int_{\Sigma^\varepsilon}(r(u^\varepsilon(s))-r(u^\varepsilon(t)))^2dxds \\
&\qquad+ \frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\int_{\Sigma^\varepsilon}(r(u^\varepsilon(t)))^2dxds \\
&\quad= -\frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\|r(u^\varepsilon(s))-r(u^\varepsilon(t))\|^2_{L^2(\Sigma^\varepsilon)}ds \\
&\qquad+ \frac{1}{2}\int_0^t g(t-s)\frac{d}{dt}\|r(u^\varepsilon(t))\|^2_{L^2(\Sigma^\varepsilon)}ds \\
&\quad= -\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(t-s)\|r(u^\varepsilon(s))-r(u^\varepsilon(t))\|^2_{L^2(\Sigma^\varepsilon)}ds\right] \\
&\qquad+ \frac{1}{2}\int_0^t g'(t-s)\|r(u^\varepsilon(s))-r(u^\varepsilon(t))\|^2_{L^2(\Sigma^\varepsilon)}ds \\
&\qquad+ \frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)ds\|r(u^\varepsilon(t))\|^2_{L^2(\Sigma^\varepsilon)}\right] - \frac{1}{2}g(t)\|r(u^\varepsilon(t))\|^2_{L^2(\Sigma^\varepsilon)}. \qquad (12)
\end{aligned}
$$

By inserting (12) in (11), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Sigma^\varepsilon)} + \mu\breve{a}(u^\varepsilon,u^\varepsilon) - 2(g\star r(u^\varepsilon))(t)\right] \\
&\qquad+ \frac{d}{dt}\left[\int_0^t g(s)ds\|r(u^\varepsilon)(t)\|^2_{L^2(\Sigma^\varepsilon)}\right] \\
&\qquad+ (g'\star r(u^\varepsilon))(t) - g(t)\|r(u^\varepsilon)(t)\|^2_{L^2(\Sigma^\varepsilon)} \\
&\quad\le \left(p^\varepsilon,\frac{\partial u^\varepsilon}{\partial t}\right), \qquad (13)
\end{aligned}
$$

where

$$(g \star v^\varepsilon)(t) = \int_0^t g(t-s)\|v^\varepsilon(s) - v^\varepsilon(t)\|^2_{L^2(\Sigma^\varepsilon)} ds, \qquad \forall v^\varepsilon \in L^2(\Sigma^\varepsilon).$$

We integrate the inequality (13) over $(0,t)$ to obtain

$$\begin{aligned}&\left[\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Sigma^\varepsilon)} + \mu\breve{a}(u^\varepsilon, u^\varepsilon) - 2(g \star r(u^\varepsilon))(t) + 2\int_0^t g(s)ds\|r(u^\varepsilon)(t)\|^2_{L^2(\Sigma^\varepsilon)}\right] \\ &\quad + 2\int_0^t (g' \star r(u^\varepsilon))(\eta)d\eta - 2\int_0^t g(\eta)\|r(u^\varepsilon)(\eta)\|^2_{L^2(\Sigma^\varepsilon)} d\eta \\ &\leq \|u_1\|^2_{L^2(\Sigma^\varepsilon)} + 2\mu\|\nabla u_0\|^2_{L^2(\Sigma^\varepsilon)} + 2(p^\varepsilon(t), u^\varepsilon(t)) \\ &\quad - 2(p^\varepsilon(0), u^\varepsilon(0)) - 2\int_0^t \left(\frac{\partial p^\varepsilon}{\partial t}(s), u^\varepsilon(s)\right) ds.\end{aligned}$$

Taking in account that $g(.)$ is negative and increasing, thus

$$(g \star r(u^\varepsilon))(t) \leq 0, \qquad \text{and} \qquad 0 \leq (g' \star r(u^\varepsilon))(t),$$

so, we get

$$\begin{aligned}&\left[\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Sigma^\varepsilon)} + 2\left(\mu + \int_0^t g(s)ds\right)\|r(u^\varepsilon)(t)\|^2_{L^2(\Sigma^\varepsilon)}\right] \\ &\leq \|u_1\|^2_{L^2(\Sigma^\varepsilon)} + 2\mu\|\nabla u_0\|^2_{L^2(\Sigma^\varepsilon)} + 2(p^\varepsilon(t), u^\varepsilon(t)) \\ &\quad - 2(p^\varepsilon(0), u^\varepsilon(0)) - 2\int_0^t \left(\frac{\partial p^\varepsilon}{\partial t}(s), u^\varepsilon(s)\right) ds.\end{aligned}$$

According to Korn's inequality ($\|r(u^\varepsilon)\|^2_{L^2(\Sigma^\varepsilon)} \geq C_W\|\nabla u^\varepsilon\|^2_{L^2(\Sigma^\varepsilon)}$, where C_W is a constant independent of ε), we find

$$\begin{aligned}&\left[\left\|\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Sigma^\varepsilon)} + 2C_W\left(\mu + \int_0^t g(s)ds\right)\|\nabla u^\varepsilon\|^2_{L^2(\Sigma^\varepsilon)}\right] \\ &\leq \|u_1\|^2_{L^2(\Sigma^\varepsilon)} + 2\mu\|\nabla u_0\|^2_{L^2(\Sigma^\varepsilon)} + 2(p^\varepsilon(t), u^\varepsilon(t)) \\ &\quad - 2(p^\varepsilon(0), u^\varepsilon(0)) - 2\int_0^t \left(\frac{\partial p^\varepsilon}{\partial t}(s), u^\varepsilon(s)\right) ds. \qquad (14)\end{aligned}$$

On the other hand, the last term of (14) can be estimated by Poincaré's inequality

$$\|v^\varepsilon\|_{L^2(\Sigma^\varepsilon)} \leq \varepsilon\bar{q}\|\nabla v^\varepsilon\|_{L^2(\Sigma^\varepsilon)}, \qquad \forall v^\varepsilon \in W^\varepsilon,$$

and Young's inequality

$$ab \le \frac{a^2}{2} + \frac{b^2}{2}, \qquad \forall a, b \in \mathbb{R},$$

to get

$$-2\int_0^t \left(\frac{\partial p^\varepsilon}{\partial t}(s), u^\varepsilon(s) \right) ds$$
$$\le \frac{4(\varepsilon\bar{q})^2}{\mu C_W} \int_0^t \left\| \frac{\partial p^\varepsilon}{\partial t}(s) \right\|^2_{L^2(\Sigma^\varepsilon)} ds + \mu C_W \int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(\Sigma^\varepsilon)} ds, \tag{15}$$

consequently

$$\left[\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|^2_{L^2(\Sigma^\varepsilon)} + 2C_W \left(\mu + \int_0^t g(s)ds \right) \|\nabla u^\varepsilon\|^2_{L^2(\Sigma^\varepsilon)} \right]$$
$$\le \|u_1\|^2_{L^2(\Sigma^\varepsilon)} + 2(\mu+1)\|\nabla u_0\|^2_{L^2(\Sigma^\varepsilon)} + \mu C_W \|\nabla u^\varepsilon\|^2_{L^2(\Sigma^\varepsilon)}$$
$$+ \mu C_W \int_0^t \|\nabla u^\varepsilon(s)\|^2_{L^2(\Sigma^\varepsilon)} ds + 4(\varepsilon\bar{q})^2 \|p^\varepsilon(0)\|^2_{L^2(\Sigma^\varepsilon)}$$
$$+ \frac{4(\varepsilon\bar{q})^2}{\mu C_W} \|p^\varepsilon(t)\|^2_{L^2(\Sigma^\varepsilon)} + \frac{4(\varepsilon\bar{q})^2}{\mu C_W} \int_0^t \left\| \frac{\partial p^\varepsilon}{\partial t}(s) \right\|^2_{L^2(\Sigma^\varepsilon)} ds. \tag{16}$$

As

$$\varepsilon^2 \|p^\varepsilon\|^2_{L^2(\Sigma^\varepsilon)} = \varepsilon^{-1} \|\hat{p}\|^2_{L^2(\Sigma)},$$

we multiply (16) by ε to deduce that

$$\varepsilon \left[\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|^2_{L^2(\Sigma^\varepsilon)} + 2C_W l \|\nabla u^\varepsilon\|^2_{L^2(\Sigma^\varepsilon)} \right]$$
$$\le A + \mu C_W \int_0^t \varepsilon \left[\left\| \frac{\partial u^\varepsilon}{\partial t} \right\|^2_{L^2(\Sigma^\varepsilon)} + \|\nabla u^\varepsilon(s)\|^2_{L^2(\Sigma^\varepsilon)} \right] ds, \tag{17}$$

where A is a constant independent of ε with

$$A = 2(\mu+1)\|\nabla \hat{u}_0\|^2_{L^2(\Sigma)} + \|\hat{u}_1\|^2_{L^2(\Sigma)} + 4\bar{q}^2 \|\hat{p}(0)\|^2_{L^2(\Sigma)}$$
$$+ \frac{4\bar{q}^2}{\mu C_W} \|\hat{p}(t)\|^2_{L^2(\Sigma)} + \frac{4\bar{q}^2}{\mu C_W} \int_0^t \left\| \frac{\partial \hat{p}}{\partial t}(s) \right\|^2_{L^2(\Sigma)} ds.$$

Thus, through employing Gronwall's lemma in the inequality (17), then we get (9).

The functional $\jmath^{\varepsilon}(.)$ is convex but non-differentiable. To overcome this difficulty, we adopt the following approach. Let $\jmath_{\zeta}^{\varepsilon}(.)$ be a functional defined by

$$\jmath_{\zeta}^{\varepsilon}(v) = \int_{\Sigma} \kappa^{\varepsilon}(x')\phi_{\zeta}(|v_{\tau}|^2)dx' \text{ where } \phi_{\zeta}(\lambda) = \frac{1}{1+\zeta}|\lambda|^{(1+\zeta)}, \qquad \zeta > 0.$$

We consider the approximate equation

$$\left(\frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}, \varphi\right) + \mu\breve{a}(u_{\zeta}^{\varepsilon}, \varphi) + \int_0^t g(t-s)\breve{a}(u_{\zeta}^{\varepsilon}(s), \varphi)ds + \left((\jmath_{\zeta}^{\varepsilon})'\left(\frac{\partial u_{\zeta}^{\varepsilon}}{\partial t}\right), \varphi\right)$$
$$= (p^{\varepsilon}, \varphi), \qquad \forall \varphi \in W^{\varepsilon}, \tag{18}$$

with

$$u_{\zeta}^{\varepsilon}(x', 0) = u_0(x'), \qquad \frac{\partial u_{\zeta}^{\varepsilon}(x', 0)}{\partial t} = u_1(x').$$

In order to show the priori estimate (10), we differentiate (18) with respect to t and we take $\varphi = \frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}$, then we get

$$\left(\frac{\partial^3 u_{\zeta}^{\varepsilon}}{\partial t^3}, \frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}\right) + \mu\breve{a}\left(\frac{\partial u_{\zeta}^{\varepsilon}}{\partial t}, \frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}\right) + \int_0^t g'(t-s)\breve{a}\left(u_{\zeta}^{\varepsilon}(s), \frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}(t)\right)ds$$
$$+ g(0)\breve{a}\left(u_{\zeta}^{\varepsilon}(t), \frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}(t)\right) + \left(\frac{\partial}{\partial t}(\jmath_{\zeta}^{\varepsilon})'\left(\frac{\partial u_{\zeta}^{\varepsilon}}{\partial t}\right), \frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}\right)$$
$$= \left(\frac{\partial p^{\varepsilon}}{\partial t}, \frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}\right).$$

As $\left(\frac{\partial}{\partial t}(\jmath_{\zeta}^{\varepsilon})'\left(\frac{\partial u_{\zeta}^{\varepsilon}}{\partial t}\right), \frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}\right) \geq 0$, we obtain

$$\frac{d}{dt}\left[\left\|\frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}\right\|_{L^2(\Sigma^{\varepsilon})}^2 + 2\mu C_W \left\|\nabla \frac{\partial u_{\zeta}^{\varepsilon}}{\partial t}\right\|_{L^2(\Sigma^{\varepsilon})}^2\right]$$
$$+ 2\int_0^t g'(t-s)\breve{a}\left(u_{\zeta}^{\varepsilon}(s), \frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}(t)\right)ds$$
$$+ 2g(0)\breve{a}\left(u_{\zeta}^{\varepsilon}(t), \frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}(t)\right)$$
$$\leq 2\left(\frac{\partial p^{\varepsilon}}{\partial t}, \frac{\partial^2 u_{\zeta}^{\varepsilon}}{\partial t^2}\right).$$

On the other hand, we have

$$\int_0^t g'(t-s)\breve{a}\left(u_\zeta^\varepsilon(s), \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(t)\right)ds$$

$$= -g'(0)\breve{a}\left(u_\zeta^\varepsilon(t), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right) + \frac{d}{dt}\int_0^t g'(t-s)\breve{a}\left(u_\zeta^\varepsilon(s), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right)ds$$

$$- \int_0^t g''(t-s)\breve{a}\left(u_\zeta^\varepsilon(s), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right)ds,$$

also

$$g(0)\breve{a}\left(u_\zeta^\varepsilon(t), \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(t)\right)$$

$$= g(0)\frac{d}{dt}\breve{a}\left(u_\zeta^\varepsilon(t), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right) - g(0)\breve{a}\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}(t), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right).$$

In the end we have

$$\frac{d}{dt}\left[\left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right\|^2_{L^2(\Sigma^\varepsilon)} + 2\mu C_W\left\|\nabla\frac{\partial u_\zeta^\varepsilon}{\partial t}\right\|^2_{L^2(\Sigma^\varepsilon)}\right]$$

$$+ 2\frac{d}{dt}\int_0^t g'(t-s)\breve{a}\left(u_\zeta^\varepsilon(s), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right)ds$$

$$- 2g'(0)\breve{a}\left(u_\zeta^\varepsilon(t), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right) - 2\int_0^t g''(t-s)\breve{a}\left(u_\zeta^\varepsilon(s), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right)ds$$

$$+ 2g(0)\frac{d}{dt}\breve{a}\left(u_\zeta^\varepsilon(t), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right) - 2g(0)\breve{a}\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}(t), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right)$$

$$\leq 2\left(\frac{\partial p^\varepsilon}{\partial t}, \frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right).$$

We integrate this inequality over $(0, t)$ to obtain

$$\left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right\|^2_{L^2(\Sigma^\varepsilon)} + 2\mu C_W\left\|\nabla\frac{\partial u_\zeta^\varepsilon}{\partial t}\right\|^2_{L^2(\Sigma^\varepsilon)} + 2\int_0^t g'(t-s)\breve{a}\left(u_\zeta^\varepsilon(s), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right)ds$$

$$+ 2g(0)\breve{a}\left(u_\zeta^\varepsilon(t), \frac{\partial u_\zeta^\varepsilon}{\partial t}(t)\right) - 2g(0)\breve{a}\left(u_\zeta^\varepsilon(0), \frac{\partial u_\zeta^\varepsilon}{\partial t}(0)\right)$$

$$
\begin{aligned}
&- 2g(0)\int_0^t \breve{a}\left(\frac{\partial u_\zeta^\varepsilon}{\partial t}(s), \frac{\partial u_\zeta^\varepsilon}{\partial t}(s)\right) ds - 2g'(0)\int_0^t \breve{a}\left(u_\zeta^\varepsilon(s), \frac{\partial u_\zeta^\varepsilon}{\partial t}(s)\right) ds \\
&- 2\int_0^t\int_0^\tau g''(\tau - s)\breve{a}\left(u_\zeta^\varepsilon(s), \frac{\partial u_\zeta^\varepsilon}{\partial t}(\tau)\right) ds d\tau \\
\le{}& \left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0)\right\|_{L^2(\Sigma^\varepsilon)}^2 + (2\mu + 1)\|\nabla u_1\|_{L^2(\Sigma^\varepsilon)}^2 + \frac{\mu C_W}{2}\left\|\nabla \frac{\partial u_\zeta^\varepsilon(t)}{\partial t}\right\|_{L^2(\Sigma^\varepsilon)}^2 \\
&+ \mu C_W \int_0^t \left\|\nabla \frac{\partial u_\zeta^\varepsilon(s)}{\partial t}\right\|_{L^2(\Sigma^\varepsilon)}^2 ds + 4(\varepsilon\bar{q})^2\left\|\frac{\partial p^\varepsilon(0)}{\partial t}\right\|_{L^2(\Sigma^\varepsilon)}^2 \\
&+ \frac{8(\varepsilon\bar{q})^2}{\mu C_W}\left\|\frac{\partial p^\varepsilon(t)}{\partial t}\right\|_{L^2(\Sigma^\varepsilon)}^2 + \frac{4(\varepsilon\bar{q})^2}{\mu C_W}\int_0^t \left\|\frac{\partial^2 p^\varepsilon(s)}{\partial t^2}\right\|_{L^2(\Sigma^\varepsilon)}^2 ds. \qquad (19)
\end{aligned}
$$

It is necessary to estimate $\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0)$. To this end, we deduce from (18) and (7) that

$$
\left(\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0), \varphi\right) = (p^\varepsilon(0), \varphi) - \mu\breve{a}(u_0, \varphi), \qquad \forall \varphi \in W^\varepsilon,
$$

which gives

$$
\left|\left(\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0), \varphi\right)\right| \le (\varepsilon\bar{q}\|p^\varepsilon(0)\|_{L^2(\Sigma^\varepsilon)} + 2\mu\|u_0\|_{H^1(\Sigma^\varepsilon)})\|\varphi\|_{H^1(\Sigma^\varepsilon)}.
$$

Multiplying this inequality by $\sqrt{\varepsilon}$ we get

$$
\sqrt{\varepsilon}\left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0)\right\|_{L^2(\Sigma^\varepsilon)} \le C', \qquad (20)
$$

where $C' = \bar{q}\|\hat{p}(0)\|_{L^2(\Sigma)} + 2\mu\|\hat{u}_0\|_{H^1(\Sigma)}$ is independent of ε.

By a standard calculation based on Cauchy-Schwarz and Young's inequalities, (19) becomes as follows:

$$
\begin{aligned}
&\left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right\|_{L^2(\Sigma^\varepsilon)}^2 + \frac{\mu C_W}{2}\left\|\nabla \frac{\partial u_\zeta^\varepsilon}{\partial t}\right\|_{L^2(\Sigma^\varepsilon)}^2 \\
&\quad \le \left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}(0)\right\|_{L^2(\Sigma^\varepsilon)}^2 + \int_0^t \left[\left\|\frac{\partial^2 u_\zeta^\varepsilon}{\partial t^2}\right\|_{L^2(\Sigma^\varepsilon)}^2 + 4\mu C_W\left\|\nabla \frac{\partial u_\zeta^\varepsilon}{\partial t}\right\|_{L^2(\Sigma^\varepsilon)}^2\right]
\end{aligned}
$$

$$+\frac{8g(0)^2}{\mu C_W}\|\nabla u_\zeta^\varepsilon\|^2_{L^2(\Sigma^\varepsilon)}+\frac{32G_1^2}{\mu C_W}\int_0^t\|\nabla u_\zeta^\varepsilon(s)\|^2_{L^2(\Sigma^\varepsilon)}ds$$

$$+\frac{16G_2^2}{\mu C_W}\int_0^t\int_0^\tau\|\nabla u_\zeta^\varepsilon(s)\|^2_{L^2(\Sigma^\varepsilon)}dsd\tau+2(\mu+3)\|\nabla u_1\|^2_{L^2(\Sigma^\varepsilon)}$$

$$+4g(0)^2\|\nabla u_0\|^2_{L^2(\Sigma^\varepsilon)}+4(\varepsilon\bar{q})^2\left\|\frac{\partial p^\varepsilon(0)}{\partial t}\right\|^2_{L^2(\Sigma^\varepsilon)}$$

$$+\frac{8(\varepsilon\bar{q})^2}{\mu C_W}\left\|\frac{\partial p^\varepsilon(t)}{\partial t}\right\|^2_{L^2(\Sigma^\varepsilon)}+\frac{4(\varepsilon\bar{q})^2}{\mu C_W}\int_0^t\left\|\frac{\partial^2 p^\varepsilon(s)}{\partial t^2}\right\|^2_{L^2(\Sigma^\varepsilon)}ds. \tag{21}$$

By tending ζ to 0 in (21), we find

$$\left\|\frac{\partial^2 u^\varepsilon}{\partial t^2}\right\|^2_{L^2(\Sigma^\varepsilon)}+\frac{\mu C_W}{2}\left\|\nabla\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Sigma^\varepsilon)}$$

$$\leq\left\|\frac{\partial^2 u^\varepsilon}{\partial t^2}(0)\right\|^2_{L^2(\Sigma^\varepsilon)}+\int_0^t\left[\left\|\frac{\partial^2 u^\varepsilon}{\partial t^2}\right\|^2_{L^2(\Sigma^\varepsilon)}+4\mu C_W\left\|\nabla\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Sigma^\varepsilon)}\right]$$

$$+\frac{8g(0)^2}{\mu C_W}\|\nabla u^\varepsilon\|^2_{L^2(\Sigma^\varepsilon)}+\frac{32G_1^2}{\mu C_W}\int_0^t\|\nabla u^\varepsilon(s)\|^2_{L^2(\Sigma^\varepsilon)}ds$$

$$+\frac{16G_2^2}{\mu C_W}\int_0^t\int_0^\tau\|\nabla u^\varepsilon(s)\|^2_{L^2(\Sigma^\varepsilon)}dsd\tau+2(\mu+3)\|\nabla u_1\|^2_{L^2(\Sigma^\varepsilon)}$$

$$+4g(0)^2\|\nabla u_0\|^2_{L^2(\Sigma^\varepsilon)}+4(\varepsilon\bar{q})^2\left\|\frac{\partial p^\varepsilon(0)}{\partial t}\right\|^2_{L^2(\Sigma^\varepsilon)}$$

$$+\frac{8(\varepsilon\bar{q})^2}{\mu C_W}\left\|\frac{\partial p^\varepsilon(t)}{\partial t}\right\|^2_{L^2(\Sigma^\varepsilon)}+\frac{4(\varepsilon\bar{q})^2}{\mu C_W}\int_0^t\left\|\frac{\partial^2 p^\varepsilon(s)}{\partial t^2}\right\|^2_{L^2(\Sigma^\varepsilon)}ds.$$

Now we multiply this inequality by ε to obtain

$$\varepsilon\left[\left\|\frac{\partial^2 u^\varepsilon}{\partial t^2}\right\|^2_{L^2(\Sigma^\varepsilon)}+\frac{\mu C_W}{2}\left\|\nabla\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Sigma^\varepsilon)}\right]$$

$$\leq B+\int_0^t\varepsilon\left[\left\|\frac{\partial^2 u^\varepsilon}{\partial t^2}\right\|^2_{L^2(\Sigma^\varepsilon)}+4\mu C_W\left\|\nabla\frac{\partial u^\varepsilon}{\partial t}\right\|^2_{L^2(\Sigma^\varepsilon)}\right],$$

where B is a constant independent of ε with

$$B = \left(\frac{8g(0)^2}{\mu C_W} + \frac{16G_2^2}{\mu C_W}T^2 + \frac{32G_1^2}{\mu C_W}T\right)A + (C')^2$$

$$+ 2(\mu+3)\|\nabla\hat{u}_1\|^2_{L^2(\Sigma)} + 4g(0)^2\|\nabla\hat{u}_0\|^2_{L^2(\Sigma)} + 4\bar{q}^2\left\|\frac{\partial\hat{p}(0)}{\partial t}\right\|^2_{L^2(\Sigma)}$$

$$+ \frac{8\bar{q}^2}{\mu C_W}\left\|\frac{\partial\hat{p}(t)}{\partial t}\right\|^2_{L^2(\Sigma)} + \frac{4\bar{q}^2}{\mu C_W}\int_0^t\left\|\frac{\partial^2\hat{p}(s)}{\partial t^2}\right\|^2_{L^2(\Sigma)} ds.$$

We employ Gronwall's lemma to obtain the desired estimate (10).

4. Convergence results as $\varepsilon \to 0$ and limit problem

Our convergence result reads as follows.

Theorem 3. *Under the assumptions of Theorem 2, there exist* $u_i^* \in L^2(0,T;P_z) \cap L^\infty(0,T;P_z)$, $i = 1,2$, *such that*

$$\left.\begin{array}{l} \hat{u}_i^\varepsilon \rightharpoonup u_i^* \\ \frac{\partial \hat{u}_i^\varepsilon}{\partial t} \rightharpoonup \frac{\partial u_i^*}{\partial t} \end{array} \quad i=1,2\right\}, \quad \begin{array}{c} \text{weakly in } L^2(0,T;P_z) \\ \text{and weakly } * \text{ in } L^\infty(0,T;P_z), \end{array} \tag{22}$$

$$\left.\begin{array}{l} \varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0 \\ \varepsilon\frac{\partial}{\partial x_j}\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial t}\right) \rightharpoonup 0 \end{array} \quad i,j=1,2\right\}, \quad \begin{array}{c} \text{weakly in } L^2(0,T;L^2(\Sigma)) \\ \text{and weakly } * \text{ in } L^\infty(0,T;L^2(\Sigma)), \end{array} \tag{23}$$

$$\left.\begin{array}{l} \varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial t} \rightharpoonup 0 \\ \varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial t} \rightharpoonup 0, \quad i=1,2 \end{array}\right\}, \quad \begin{array}{c} \text{weakly in } L^2(0,T;L^2(\Sigma)) \\ \text{and weakly } * \text{ in } L^\infty(0,T;L^2(\Sigma)), \end{array} \tag{24}$$

$$\left.\begin{array}{l} \varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0 \\ \varepsilon^2\frac{\partial}{\partial x_i}\left(\frac{\partial \hat{u}_3^\varepsilon}{\partial t}\right) \rightharpoonup 0 \end{array} \quad i=1,2\right\}, \quad \begin{array}{c} \text{weakly in } L^2(0,T;L^2(\Sigma)) \\ \text{and weakly } * \text{ in } L^\infty(0,T;L^2(\Sigma)), \end{array} \tag{25}$$

$$\left.\begin{array}{l} \varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0 \\ \varepsilon^2\frac{\partial}{\partial z}\left(\frac{\partial \hat{u}_3^\varepsilon}{\partial t}\right) \rightharpoonup 0 \end{array}\right\}, \quad \begin{array}{c} \text{weakly in } L^2(0,T;L^2(\Sigma)) \\ \text{and weakly } * \text{ in } L^\infty(0,T;L^2(\Sigma)), \end{array} \tag{26}$$

$$\left.\begin{array}{l} \varepsilon\frac{\partial^2 \hat{u}_i^\varepsilon}{\partial t^2} \rightharpoonup 0, \quad i=1,2 \\ \varepsilon^2\frac{\partial^2 \hat{u}_3^\varepsilon}{\partial t^2} \rightharpoonup 0 \end{array}\right\}, \quad \begin{array}{c} \text{weakly in } L^2(0,T;L^2(\Sigma)) \\ \text{and weakly } * \text{ in } L^\infty(0,T;L^2(\Sigma)). \end{array} \tag{27}$$

Proof. According to Theorem 2, there exists a constant C independent of ε such that

$$\left\|\frac{\partial \hat{u}_i^\varepsilon}{\partial z}\right\|^2_{L^2(\Sigma)} \leq C, \qquad i=1,2.$$

By using this estimate with the Poincaré inequality in the domain Σ

$$\|\hat{u}^\varepsilon\|_{P_z}^2 \leq \bar{q}^2 \left\| \frac{\partial \hat{u}^\varepsilon}{\partial z} \right\|_{L^2(\Sigma)}^2,$$

we obtain (22). The convergence properties (23)–(27) follow from (9), (10) and (22).

Thus, we have the following estimates.

Theorem 4. *With the same assumptions of Theorem* 3, u^* *satisfies*

$$\begin{aligned} \mu \sum_{i=1}^2 \int_\Sigma \frac{\partial u_i^*}{\partial z} \frac{\partial}{\partial z}\left(\hat{\varphi}_i - \frac{\partial u_i^*}{\partial t}\right) dx' dz & \\ + \sum_{i=1}^2 \int_\Sigma \left(\int_0^t g(t-s) \frac{\partial}{\partial z}\left(\frac{\partial u_i^*}{\partial t}(s)\right) ds \right) \frac{\partial}{\partial z}\left(\hat{\varphi}_i - \frac{\partial u_i^*}{\partial t}\right) dx' dz & \\ + \hat{J}(\hat{\varphi}) - \hat{J}\left(\frac{\partial u^*}{\partial t}\right) & \\ \geq \sum_{i=1}^2 \int_\Sigma \hat{p}_i \left(\hat{\varphi}_i - \frac{\partial u_i^*}{\partial t}\right) dx' dz, \qquad \forall \hat{\varphi} \in \Pi(W), & \end{aligned} \tag{28}$$

$$-\frac{\partial^2}{\partial z^2}\left[\mu u_i^*(t) - \int_0^t g(t-s) u_i^*(s) ds\right] = \hat{p}_i(t), \qquad for\ \ i = 1, 2\ \ in\ \ L^2(\Sigma), \tag{29}$$

$$u_i^*(x', z, 0) = u_{0,i}, \qquad i = 1, 2.$$

Proof. As $\hat{J}(.)$ is convex and lower-semicontinuous

$$\lim_{\varepsilon \to 0} \inf \int_\Lambda \hat{\kappa} \left| \frac{\partial \hat{u}^\varepsilon}{\partial t} \right| dx' \geq \int_\Lambda \hat{\kappa} \left| \frac{\partial u^*}{\partial t} \right| dx',$$

thus we pass to the limit when ε tends to 0 in (8). By using the convergence results of Theorem 3 we obtain

$$\begin{aligned} \mu \sum_{i=1}^2 \int_\Sigma \frac{\partial u_i^*}{\partial z} \frac{\partial}{\partial z}\left(\hat{\varphi}_i - \frac{\partial u_i^*}{\partial t}\right) dx' dz & \\ + \sum_{i=1}^2 \int_\Sigma \left(\int_0^t g(t-s) \frac{\partial}{\partial z}\left(\frac{\partial u_i^*}{\partial t}(s)\right) ds \right) \frac{\partial}{\partial z}\left(\hat{\varphi}_i - \frac{\partial u_i^*}{\partial t}\right) dx' dz & \\ + \hat{J}(\hat{\varphi}) - \hat{J}\left(\frac{\partial u^*}{\partial t}\right) & \\ \geq \sum_{i=1}^2 \int_\Sigma \hat{p}_i \left(\hat{\varphi}_i - \frac{\partial u_i^*}{\partial t}\right) dx' dz, \qquad \forall \hat{\varphi} \in \Pi(W). & \end{aligned} \tag{30}$$

Choosing in the variational inequation (30)

$$\hat{\varphi}_i = \frac{\partial u_i^*}{\partial t} \pm \psi_i, \qquad \psi_i \in H_0^1(\Sigma), \qquad i = 1, 2,$$

we find

$$\mu \sum_{i=1}^{2} \int_{\Sigma} \frac{\partial u_i^*}{\partial z} \frac{\partial \psi_i}{\partial z}\, dx'dz + \sum_{i=1}^{2} \int_{\Sigma} \int_0^t g(t-s) \frac{\partial u_i^*}{\partial z}(s) ds \frac{\partial \psi_i}{\partial z}\, dx'dz$$
$$= \sum_{i=1}^{2} \int_{\Sigma} \hat{p}_i \psi_i\, dx'dz,$$

by using Green's formula. Choosing, $\psi_1 = 0$ and $\psi_2 \in H_0^1(\Sigma)$, then $\psi_2 = 0$ and $\psi_1 \in H_0^1(\Sigma)$, we obtain

$$-\int_{\Sigma} \mu \frac{\partial^2 u_i^*}{\partial z^2} \psi_i\, dx'dz - \int_{\Sigma} \int_0^t g(t-s) \frac{\partial^2 u_i^*}{\partial z^2}(s) ds \psi_i\, dx'dz = \int_{\Sigma} \hat{p}_i \psi_i\, dx'dz.$$

Thus

$$-\frac{\partial^2}{\partial z^2}\left[\mu u_i^* - \int_0^t g(t-s) u_i^*(s) ds\right] = \hat{p}_i, \qquad \text{for } i = 1, 2 \text{ in } H^{-1}(\Sigma). \quad (31)$$

Since $\hat{p}_i \in L^2(\Sigma)$ $i = 1, 2$, (31) is valid in $L^2(\Sigma)$.

Hence, we have the following theorem.

Theorem 5. *The traces* $\tau^*(x', t) = \frac{\partial u^*}{\partial z}(x', 0, t)$, *and* $s^*(x', t) = u^*(x', 0, t)$ *satisfy the following inequality*

$$\int_{\Lambda} \hat{\kappa}\left(\left|\psi + \frac{\partial s^*}{\partial t}\right| - \left|\frac{\partial s^*}{\partial t}\right|\right) dx' - \int_{\Sigma}\left(\mu \tau^* + \int_0^t g(t-\chi) \tau^*(\chi) d\chi\right) \psi\, dx'$$
$$\geq 0, \qquad \forall \psi \in L^2(\Sigma)^2, \quad (32)$$

and the following limit form of the Tresca boundary condition in $\Lambda \times]0, T[$

$$\left|\mu \tau^* + \int_0^t g(t-\chi) \tau^*(\chi) d\chi\right| < \hat{\kappa} \Rightarrow \frac{\partial s^*}{\partial t} = 0, \quad (33)$$

$$\left|\mu \tau^* + \int_0^t g(t-\chi) \tau^*(\chi) d\chi\right| = \hat{\kappa}$$
$$\Rightarrow \exists \beta > 0 : \frac{\partial s^*}{\partial t} = \beta\left(\tau^* + \int_0^t g(t-\chi) \tau^*(\chi) d\chi\right). \quad (34)$$

Moreover, u^* and s^* satisfy the following weak form of the Reynolds equation

$$\int_\Lambda \left(\tilde{F} - \frac{h}{2}\left(\mu s^* + \int_0^t g(t-\chi)s^*(\chi)d\sigma\right)\right)\nabla\psi(x')dx'$$
$$+ \int_\Lambda \left(\int_0^q \left(\mu u^*(x',z,t) + \int_0^t g(t-\chi)u^*(x',z,\chi)d\chi\right)dz\right)\nabla\psi(x')dx'$$
$$= 0, \qquad \forall\psi \in H^1(\Lambda), \tag{35}$$

where

$$\tilde{F}(x',q,t) = \int_0^q F(x',z,t)dz - \frac{q}{2}F(x',q,t),$$
$$F(x',q,t) = \int_0^z \int_0^\zeta \hat{p}(x',\eta,t)d\eta d\zeta.$$

Proof. In order to prove (32)–(34), we use the same techniques used in [1]. To prove (35) we integrate (29) twice from 0 to z to obtain

$$\mu u_i^*(x',z,t) + \int_0^t g(t-\chi)u_i^*(x',z,\chi)d\chi$$
$$= \mu(s_i^* + z\tau_i^*) + \int_0^t g(t-\chi)(s_i^*(\chi) + z\tau_i^*(\chi))d\chi$$
$$- \int_0^z \int_0^\zeta \hat{p}_i(x',\eta,t)d\eta d\zeta. \tag{36}$$

As $u_i^*(x',q(x'),t) = 0$, we obtain

$$\mu(s_i^* + q\hat{\tau}_i^*) + \int_0^t g(t-\chi)(s_i^*(\chi) + q\tau_i^*(\chi))d\chi$$
$$= \int_0^q \int_0^\zeta p_i(x',\eta,t)d\eta d\zeta. \tag{37}$$

By integrating (36) from 0 to q, we get

$$\int_0^q \left(\mu u_i^*(x',z,t) + \int_0^t g(t-\chi)u_i^*(x',z,\chi)d\chi\right)dz$$
$$= \mu\left(qs_i^* + \frac{1}{2}q^2\hat{\tau}_i^*\right) + \int_0^t g(t-\chi)\left(qs_i^*(\chi) + \frac{1}{2}q^2\hat{\tau}_i^*(\chi)\right)d\chi$$
$$- \int_0^q \int_0^z \int_0^\zeta \hat{p}_i(x',\eta,t)d\eta d\zeta dz. \tag{38}$$

From (37)–(38), we deduce

$$\int_0^q \left(\mu u_i^*(x', z, t) + \int_0^t g(t-\chi) u_i^*(x', z, \chi) d\chi\right) dz - \frac{q}{2}\left(\mu s_i^* + \int_0^t g(t-\chi) s_i^*(\chi) d\chi\right) + \tilde{F}_i = 0,$$

with

$$\tilde{F}_i(x', q, t) = \int_0^q F_i(x', z, t) dz - \frac{q}{2} F_i(x', q, t),$$

$$F_i(x', z, t) = \int_0^z \int_0^\zeta \hat{p}_i(x', \eta, t) d\eta d\zeta.$$

We next prove the uniqueness of a solution of variational inequalities given in Theorem 4.

Theorem 6. *The solution u^* of the limit problem* (28) *and* (29) *is unique in* $L^\infty(0, T; P_z)$.

Proof. Let u^{*1} and u^{*2} be two solutions of the variational inequality (28). Then we have

$$\mu \sum_{i=1}^2 \int_\Sigma \frac{\partial u_i^{*1}}{\partial z} \frac{\partial}{\partial z}\left(\hat{\varphi}_i - \frac{\partial u_i^{*1}}{\partial t}\right) dx' dz + \sum_{i=1}^2 \int_\Sigma \int_0^t g(t-s) \frac{\partial u_i^{*1}}{\partial z}(s) ds \frac{\partial}{\partial z}\left(\hat{\varphi}_i - \frac{\partial u_i^{*1}}{\partial t}\right) dx' dz + \hat{J}(\hat{\varphi}) - \hat{J}\left(\frac{\partial u^{*1}}{\partial t}\right) \geq \sum_{i=1}^2 \int_\Sigma \hat{p}_i \left(\hat{\varphi}_i - \frac{\partial u_i^{*1}}{\partial t}\right) dx' dz, \tag{39}$$

and

$$\mu \sum_{i=1}^2 \int_\Sigma \frac{\partial u_i^{*2}}{\partial z} \frac{\partial}{\partial z}\left(\hat{\varphi}_i - \frac{\partial u_i^{*2}}{\partial t}\right) dx' dz + \sum_{i=1}^2 \int_\Sigma \int_0^t g(t-s) \frac{\partial u_i^{*2}}{\partial z}(s) ds \frac{\partial}{\partial z}\left(\hat{\varphi}_i - \frac{\partial u_i^{*2}}{\partial t}\right) dx' dz + \hat{J}(\hat{\varphi}) - \hat{J}\left(\frac{\partial u^{*2}}{\partial t}\right) \geq \sum_{i=1}^2 \int_\Sigma \hat{p}_i \left(\hat{\varphi}_i - \frac{\partial u_i^{*2}}{\partial t}\right) dx' dz. \tag{40}$$

By taking $\hat{\varphi} = \frac{\partial u^{*2}}{\partial t}$ in (39) and $\hat{\varphi} = \frac{\partial u^{*1}}{\partial t}$ in (40) respectively, we get

$$\mu \sum_{i=1}^{2} \int_{\Sigma} \frac{\partial}{\partial z}(u_i^{*1} - u_i^{*2}) \frac{\partial}{\partial z}\left(\frac{\partial u_i^{*1}}{\partial t} - \frac{\partial u_i^{*2}}{\partial t}\right) dx' dz$$
$$+ \sum_{i=1}^{2} \int_{\Sigma} \int_0^t g(t-s)\left(\frac{\partial u_i^{*1}}{\partial z}(s) - \frac{\partial u_i^{*2}}{\partial z}(s)\right) ds \frac{\partial}{\partial z}\left(\frac{\partial u_i^{*1}}{\partial t} - \frac{\partial u_i^{*2}}{\partial t}\right) dx' dz$$
$$\leq 0.$$

We put $\overline{Y}(t) = u^{*1}(t) - u^{*2}(t)$. We then see from above inequality that $\overline{Y}$ satisfies

$$\frac{1}{2}\frac{d}{dt}\left[\left(\mu + \int_0^t g(s)ds\right)\left\|\frac{\partial \overline{Y}}{\partial z}\right\|^2_{L^2(\Sigma)} - \left(g \star \frac{\partial \overline{Y}}{\partial z}\right)(t)\right]$$
$$+ \left(g' \star \frac{\partial \overline{Y}}{\partial z}\right)(t) - \frac{1}{2} g(t)\left\|\frac{\partial \overline{Y}}{\partial z}\right\|^2_{L^2(\Sigma)} \leq 0.$$

We finally integrate this inequality over $(0, t)$ and use Poincaré's inequality to find

$$\|\overline{Y}\|_{L^\infty(0,T;P_z)} = 0.$$

This implies $\overline{Y} = 0$ to show the uniqueness.

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