Hiroshima Math. J., **54** (2024), 45–59
doi:10.32917/h2022010

Schubert calculus via fermionic variables

Ken Kuwata

(Received February 13, 2022)
(Revised March 9, 2023)

Abstract. Imanishi, Jinzenji and Kuwata provided a recipe for computing Euler number of Grassmann manifold $G(k,N)$ using physical model and its path-integral [S. Imanishi, M. Jinzenji and K. Kuwata, Journal of Geometry and Physics, Volume 180, October 2022, 104623]. They demonstrated that the cohomology ring of $G(k,N)$ is represented by fermionic variables. In this study, using only fermionic variables, we computed an integral of the Chern classes of the dual bundle of the tautological bundle on $G(k,N)$. In other words, the intersection number of the Schubert cycles is obtained using the fermion integral.

1. Introduction

1.1. Background. In this study, we aim to compute the intersection numbers of Schubert cycles. We used fermionic variables and their integrals in [8]. In this section, we explain the background of the study. The complex Grassmann manifold $G(k,N)$ is the space parameterizing all k-dimensional linear subspaces of N-dimensional complex vector space $\mathbb{C}^N$. Because the elements of its cohomology ring are represented by the Poincaré dual of some Schubert cycles of $G(k,N)$, their integral provides the intersection number of Schubert cycles. This research is called Schurbert calculus, and has been studied in combinatorics, representation theory, and other fields [6]. The integral of these cohomology classes can be computed using localization theory or the Landau-Ginzburg formulation. In the localization theory, a fixed-point theorem for a compact manifold with torus action is used. In particular, the formula for the intersection number is provided using the localization theory [5, 9, 11]. However, the Landau-Ginzburg formulation [2, 10] uses a potential function provided by the total Chern class of the tautological bundle of $G(k,N)$ and residue. However, we do not use these theories. We employed the theory of [8]. Imanishi et al. constructed a physical toy model for computing the Euler number of $G(k,N)$. The model was constructed using two types of variables. One is a commutative variable called a bosonic variable, while the other is an anticommutative variable, called a fermionic variable. In [8], it was found that

Key words and phrases. Complex Grassmann manifold, fermionic variables, Schubert calculus.

the cohomology ring of $G(k,N)$ can be represented by fermionic variables, and that the Euler number is provided by their integral. Therefore, the intersection number of Schubert cycles can be obtained using fermion integrals. Generally, it is difficult to perform this calculation. However, in some cases, the number of intersections can be calculated using this method. In this study, we demonstrated the use of the method of [8].

1.2. Organization of the paper. This paper is divided into two sections.

In Section 1, we describe our background and theorem. In addition to the background described above, we introduce the relationship between Chern classes and Schubert cycles, our theorem in this paper, and the theory in [8]. First, we remark on Chern classes and Schubert cycles. Next, we introduce the theorem. Finally, we introduce the relation between the Chern classes and fermionic variables in [8].

In Section 2, we provide the proof of our theorem. We computed the fermion integral to prove the theorem. We also summarize the important results of the fermion integrals.

1.3. Chern classes and Schubert cycles. In this section, we explain the relation between the Chern classes and Schubert cycles and our theorem. In this study, we employed the notation in [8]. First, we introduce the cohomology ring of $G(k,N)$ [1, 8], and then remark on a tautological bundle S and a universal quotient bundle Q. The fiber of S at $\Lambda \in G(k,N)$ is the complex k-dimensional subspace $\Lambda \subset \mathbb{C}^N$ itself ($\mathrm{rk}(S) = k$). Subsequently, a universal quotient bundle Q ($\mathrm{rk}(Q) = N-k$) is defined by the following exact sequence:

$$0 \to S \to \mathbb{C}^N \to Q \to 0, \tag{1.1}$$

where $\mathbb{C}^N$ denotes the trivial bundle $G(k,N) \times \mathbb{C}^N$. We write $c_i(E)$ as the i-th Chern class of the vector bundle E. Let E^* be the dual bundle of E. Subsequently, the cohomology ring $H^*(G(k,N))$ of $G(k,N)$ is

$$H^*(G(k,N)) = \frac{\mathbb{R}[c_1(S^*),\dots,c_k(S^*),c_1(Q^*),\dots,c_{N-k}(Q^*)]}{(c(S^*)c(Q^*)=1)}. \tag{1.2}$$

Consider the problem of representing $H^*(G(k,N))$ using $c_j(S^*)$ $(j = 1,2,\dots,k)$. This was presented in [8]. If we decompose S^* formally by line bundle L_i $(i = 1,2,\dots,k)$:

$$S^* = \bigoplus_{i=1}^{k} L_i, \tag{1.3}$$

$c(S^*)$ and $c_i(S^*)$ are expressed as follows:

$$c(S^*) = \prod_{i=1}^{k}(1 + tx_i) = 1 + \sum_{j=1}^{k} t^j c_j(S^*), \qquad (x_i := c_1(L_i)). \tag{1.4}$$

Hence, $c_j(S^*)$ is written as the degree j elementary symmetric polynomial of $x_1, \dots, x_k$. Then, the relation $c(S^*)c(Q^*) = 1$ can be rewritten as

$$c(Q^*) = \frac{1}{c(S^*)} = \frac{1}{1 + \sum_{j=1}^{k} t^j c_j(S^*)} = \sum_{i=0}^{\infty} a_i t^i. \tag{1.5}$$

We can rewrite a_i in (1.5) as

$$c_i(Q^*) = a_i \quad (i = 1, 2, \dots, N-k), \qquad a_i = 0 \quad (i > N-k). \tag{1.6}$$

Moreover, a_i is the degree i homogeneous polynomial of $c_j(S^*)$'s $(j = 1, 2, \dots, k)$. Thus, $c_i(Q^*)$ in (1.2) can be rewritten as $c_i(S^*)$. Consequently, we obtain another representation of $H^*(G(k,N))$:

$$H^*(G(k,N)) = \frac{\mathbb{R}[c_1(S^*), \dots, c_k(S^*)]}{(a_i = 0 \ \ (i > N-k))}. \tag{1.7}$$

Second, we introduce the Schubert cycle and explain the relationship between Chern classes and Schubert cycles. For a more detailed discussion, please refer to [4]. For any flag $V : 0 \subset V_1 \subset V_2 \subset \cdots \subset V_N = \mathbb{C}^N$, Schubert manifold $\sigma_a(V)$ is defined as follows:

$$\sigma_a(V) := \{\Lambda \in G(k,N) \,|\, \dim(\Lambda \cap V_{N-k+i-a_i}) \geq i \ \ (1 \leq i \leq k)\}, \tag{1.8}$$

where $a = (a_1, \dots, a_k)$ denotes a sequence of natural numbers that satisfies $0 \leq a_k \leq a_{k-1} \leq \cdots \leq a_1 \leq N-k$. $\sigma_a(V)$ is a subvariety of $G(k,N)$ of dimension $\sum_{l=1}^{k} a_l$. The homology class of $\sigma_a(V)$ is independent of the chosen flag. Therefore, let $\sigma_a(V)$ as the homology class be denoted by σ_a. Let $\sigma_a^\vee$ be the Poincaré dual of the cycle σ_a. For simplicity of notation, we omit 0 from a. For example, $\sigma_{a_1, a_2, \dots, a_n}$ denotes $\sigma_{(a_1, a_2, \dots, a_n, 0, \dots, 0)}$. The relationship between i-th Chern class of a vector bundle E and that of its dual bundle E^* is provided by $c_i(E^*) = (-1)^i c_i(E)$. From this formula and the Gauss-Bonnet theorem, we obtain:

$$c_i(S^*) = (-1)^i c_i(S) = \sigma^\vee_{\underbrace{1,\dots,1}_{i}} =: \sigma^\vee_{1^{(i)}}. \tag{1.9}$$

Finally, we introduce.

THEOREM 1.

$$\int_{G(k,N)} (\sigma^{\vee}_{1^{(1)}})^{kN-k^2} = (kN-k^2)!\frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}. \tag{1.10}$$

$$\int_{G(k,N)} (\sigma^{\vee}_{1^{(1)}})^{kN-k^2-2}(\sigma^{\vee}_{1^{(2)}}) = \frac{(kN-k^2-2)!(N-k)(N-k+1)k(k-1)}{2}\frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}. \tag{1.11}$$

$$\begin{aligned}\int_{G(k,N)} (\sigma^{\vee}_{1^{(1)}})^{kN-k^2-4}(\sigma^{\vee}_{1^{(2)}})^2 &= \frac{(kN-k^2-4)!(N-k)(N-k+1)k(k-1)}{4} \\ &\quad\times\frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}[k(k-1)(N-k)(N-k-1) \\ &\quad + 2(k-2)(k-3)(N-k)+4(k-2)(N-k-1)].\end{aligned} \tag{1.12}$$

Here, we assume that N and k in (1.11) *and* (1.12) *satisfy $kN-k^2-2\geq 0$ and $kN-k^2-4\geq 0$, respectively.*

Note that these are the intersection numbers of $\sigma_{1^{(1)}}$ and $\sigma_{1^{(2)}}$. However, the results of (1.10) are already well known [2, 3]. When $k=2$, the intersection numbers of $\sigma^{*}_{1^{(1)}}$ and $\sigma^{*}_{1^{(2)}}$ in $G(2,N)$ are known [2].

$$\int_{G(2,N)} (\sigma^{\vee}_{1^{(1)}})^{2N-4-2l}(\sigma^{\vee}_{1^{(2)}})^{l} = \frac{(2(N-2-l))!}{(N-2-l)!(N-1-l)!}. \tag{1.13}$$

(It is also derived by S. Imanishi's Masters thesis using fermionic variables [7].) In particular, we obtain the following results from (1.10), (1.11), and (1.12):

$$\int_{G(2,N)} (\sigma^{\vee}_{1^{(1)}})^{2N-4} = \frac{(2N-4)!}{(N-2)!(N-1)!}, \tag{1.14}$$

$$\int_{G(2,N)} (\sigma^{\vee}_{1^{(1)}})^{2N-6}(\sigma^{\vee}_{1^{(2)}}) = \frac{(2N-6)!}{(N-3)!(N-2)!}, \tag{1.15}$$

$$\int_{G(2,N)} (\sigma^{\vee}_{1^{(1)}})^{2N-8}(\sigma^{\vee}_{1^{(2)}})^2 = \frac{(2N-8)!}{(N-4)!(N-3)!}. \tag{1.16}$$

1.4. Fermionic variables and Cohomology ring of $G(k,N)$ (Review of [8]). We summarize the representation of the cohomology ring of $G(k,N)$ using fermionic variables [8]. We introduce the fermionic variables ψ_s^j, $\psi_{\bar{s}}^{\bar{j}}$ $(s=1,\dots, N-k,\ j=1,\dots,k)$ and $(k\times k)$ matrix

$$\Phi := \sum_{s=1}^{N-k}\begin{pmatrix}\psi_s^1\psi_{\bar{s}}^{\bar{1}} & \dots & \psi_s^1\psi_{\bar{s}}^{\bar{k}}\\ \vdots & \ddots & \vdots\\ \psi_s^k\psi_{\bar{s}}^{\bar{1}} & \dots & \psi_s^k\psi_{\bar{1}}^{\bar{k}}\end{pmatrix}. \tag{1.17}$$

The fermionic variables ψ_s^j, $\psi_{\bar{s}}^{\bar{j}}$ satisfy the following conditions.

$$\psi_s^j\psi_s^j = \psi_{\bar{s}}^{\bar{j}}\psi_{\bar{s}}^{\bar{j}} = 0, \qquad \psi_s^j\psi_l^i = -\psi_l^i\psi_s^j,$$
$$\psi_{\bar{s}}^{\bar{j}}\psi_{\bar{l}}^{\bar{i}} = -\psi_{\bar{l}}^{\bar{i}}\psi_{\bar{s}}^{\bar{j}}, \qquad \psi_s^j\psi_{\bar{l}}^{\bar{i}} = -\psi_{\bar{l}}^{\bar{i}}\psi_s^j \tag{1.18}$$

$(s,l=1,2,\dots,N-k,\ i,j=1,2,\dots,k)$. The fermion integral is defined as follows:

$$\int D\psi \prod_{s=1}^{N-k}\psi_s^1\psi_{\bar{s}}^{\bar{1}}\cdots\psi_s^k\psi_{\bar{s}}^{\bar{k}} = 1, \tag{1.19}$$

where $D\psi := \prod_{s=1}^{N-k} d\psi_s^1 d\psi_{\bar{s}}^{\bar{1}}\cdots d\psi_s^k d\psi_{\bar{s}}^{\bar{k}}$. We define τ_j $(j=1,2,\dots,k)$ as

$$1+\tau_1 t+\cdots+\tau_k t^k := \det(I_k + t\Phi) = \prod_{j=1}^{k}(1+\lambda_j t). \tag{1.20}$$

Here, λ_j $(j=1,\dots,k)$ are eigenvalues of Φ. Specifically, τ_j is the degree j elementary symmetric polynomial of $\lambda_1,\dots,\lambda_k$. Note that τ_k is identified with $\det(\Phi)$ and τ_1 is identified with $\mathrm{tr}(\Phi)$. In [8], the following theorems were proved:

Theorem 2 ([8]).

$$\frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}\int D\psi(\det(\Phi))^{N-k} = 1. \tag{1.21}$$

Theorem 3 ([8]).

$$H^*(G(k,N)) = \frac{\mathbb{R}[c_1(S^*),\dots,c_k(S^*)]}{(a_i=0\ \ (i>N-k))} \simeq \mathbb{R}[\tau_1,\dots,\tau_k]. \tag{1.22}$$

Theorem 3 is provided by ring homomorphism $f:\mathbb{R}[c_1(S^*),\dots,c_k(S^*)]\to \mathbb{R}[\tau_1,\dots,\tau_k]$, which is defined as

$$f(c_j(S^*)) = \tau_j \qquad (j=1,2,\dots,k). \tag{1.23}$$

From the isomorphism $H^*(G(k,N)) \xrightarrow{\sim} \mathbb{R}[\tau_1, \dots, \tau_k]$, x_j is identified as λ_j. Theorem 2 corresponds to the normalization condition of the integration on $G(k,N)$ given by

$$\int_{G(k,N)} (\sigma^{\vee}_{1^{(k)}})^{N-k} = 1. \tag{1.24}$$

Therefore, we obtain the following formula:

$$\int_{G(k,N)} g(x_1, \dots, x_k) = \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi g(\lambda_1, \dots, \lambda_k), \tag{1.25}$$

where $g(x_1, \dots, x_k)$ is a symmetric polynomial of $x_1, \dots, x_k$ that represents an element of $H^*(G(k,N))$.

2. Proof of our theorem

2.1. Proof of Theorem 1. First, we prove (1.10). From (1.4), (1.20), and (1.25), we have

$$\begin{aligned}\int_{G(k,N)} (\sigma^{\vee}_{1^{(1)}})^{kN-k^2} &= \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi (\mathrm{tr}(\Phi))^{kN-k^2} \\ &= \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi \left(\sum_{s=1}^{N-k} \sum_{j=1}^{k} \psi_s^j \psi_{\bar{s}}^{\bar{j}} \right)^{kN-k^2}. \end{aligned} \tag{2.26}$$

From the multinomial theorem and the conditions of the fermionic variables $\psi_s^j \psi_s^j = \psi_{\bar{s}}^{\bar{j}} \psi_{\bar{s}}^{\bar{j}} = 0$, we obtain:

$$\begin{aligned}\int_{G(k,N)} (\sigma^{\vee}_{1^{(1)}})^{kN-k^2} &= (Nk - k^2)! \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi \prod_{s=1}^{N-k} \prod_{j=1}^{k} \psi_s^j \psi_{\bar{s}}^{\bar{j}} \\ &= (kN - k^2)! \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!}. \end{aligned} \tag{2.27}$$

Second, we show that (1.11) and (1.12). In the same way as in (1.10),

$$\begin{aligned}&\int_{G(k,N)} (\sigma^{\vee}_{1^{(1)}})^{kN-k^2-2l} (\sigma^{\vee}_{1^{(2)}})^{l} \\ &= \frac{\prod_{j=0}^{k-1} j!}{\prod_{j=N-k}^{N-1} j!} \int D\psi (\tau_1)^{kN-k^2-2l} (\tau_2)^{l} \qquad (l = 1, 2). \end{aligned} \tag{2.28}$$

As $\tau_2 = \frac{1}{2}\{(\mathrm{tr}(\Phi))^2 - \mathrm{tr}(\Phi^2)\}$,

$$\int D\psi(\tau_1)^{kN-k^2-2l}(\tau_2)^l$$

$$= \frac{1}{2^l}\int D\psi\{\mathrm{tr}(\Phi)\}^{kN-k^2-2l}\{(\mathrm{tr}(\Phi))^2 - \mathrm{tr}(\Phi^2)\}^l \tag{2.29}$$

$$= \frac{1}{2^l}\sum_{m=0}^{l}\binom{l}{m}(-1)^m\int D\psi(\mathrm{tr}(\Phi))^{kN-k^2-2m}(\mathrm{tr}(\Phi^2))^m. \tag{2.30}$$

Let us define

$$P_m := \int D\psi(\mathrm{tr}(\Phi))^{kN-k^2-2m}(\mathrm{tr}(\Phi^2))^m \qquad (m = 0, 1, 2). \tag{2.31}$$

As can be observed from the calculation in (1.10), $P_0 = (kN - k^2)!$. We can obtain the following result for P_1 and P_2.

Proposition 1.

$$P_1 = (kN - k^2 - 2)!k(N - k)(N - 2k). \tag{2.32}$$

$$P_2 = (kN - k^2 - 4)!k(N - k)$$
$$\times [k(N - k)^3 - 2(N - k)^2(k^2 + 2) + (N - k)(k^3 + 10k) - 4k^2 - 2]. \tag{2.33}$$

We will prove these results later in this paper. From Proposition 1, we have

$$\int D\psi(\tau_1)^{kN-k^2-2}(\tau_2) = \frac{1}{2}(P_0 - P_1)$$
$$= \frac{1}{2}(kN - k^2 - 2)!k(N - k)\{(kN - k^2 - 1) - (N - 2k)\}$$
$$= \frac{1}{2}(kN - k^2 - 2)!(N - k)(N - k + 1)k(k - 1). \tag{2.34}$$

We obtain (1.11). Similarly, we obtain (1.12) from $\int D\psi(\tau_1)^{kN-k^2-4}(\tau_2)^2 = \frac{1}{4}(P_0 - 2P_1 + P_2)$. We have proved Theorem 1.

2.2. Proof of Proposition 1. We compute P_1. Let ω^{ij} be $\sum_{s=1}^{N-k}\psi_s^i\psi_{\bar{s}}^{\bar{j}}$. By definition,

$$\mathrm{tr}(\Phi^2) = \sum_{i,j=1}^{k}\omega^{ij}\omega^{ji} = \sum_{i=1}^{k}(\omega^{ii})^2 + \sum_{i\neq j}\omega^{ij}\omega^{ji}, \tag{2.35}$$

$$P_1 = \int D\psi (\mathrm{tr}(\Phi))^{kN-k^2-2}(\mathrm{tr}(\Phi^2)) \tag{2.36}$$

$$= \sum_{i=1}^{k} \int D\psi \left(\sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-2} (\omega^{ii})^2$$
$$+ \sum_{i \neq j} \int D\psi \left(\sum_{n=1}^{k} \omega^{nn} \right)^{kN-k^2-2} \omega^{ij}\omega^{ji} \tag{2.37}$$

$$= \sum_{i=1}^{k} \sum_{p_n} \frac{(kN-k^2-2)!}{\prod_{n=1}^{k} p_n!} \int D\psi \left(\prod_{n=1}^{k} (\omega^{nn})^{p_n} \right) (\omega^{ii})^2$$
$$+ \sum_{i \neq j} \sum_{p_n} \frac{(kN-k^2-2)!}{\prod_{n=1}^{k} p_n!} \int D\psi \left(\prod_{n=1}^{k} (\omega^{nn})^{p_n} \right) \omega^{ij}\omega^{ji}. \tag{2.38}$$

Here, $\sum_{p_n}$ indicates that the sum includes all combinations from 0 to $kN-k^2-2$ indices p_1 through p_k, such that the sum of all p_n $(n = 1, \ldots, k)$ is $kN-k^2-2$. In the first term, because each ω^{ii} $(i = 1, \ldots, k)$ must be $N-k$ for the fermion integral to be non-zero, $p_n = N-k$ $(n \neq i)$ and $p_i = N-k-2$. In the second term, $p_n = N-k$ $(n \neq i, j)$ and $p_i = p_j = N-k-1$.

$$P_1 = \sum_{i=1}^{k} \frac{(kN-k^2-2)!}{((N-k)!)^{k-1}(N-k-2)!} \int D\psi \left(\prod_{n=1}^{k} (\omega^{nn})^{N-k} \right)$$
$$+ \sum_{i \neq j} \frac{(kN-k^2-2)!}{((N-k)!)^{k-2}((N-k-1)!)^2} \int D\psi \left(\prod_{n \neq i,j} (\omega^{nn})^{N-k} \right)$$
$$\times (\omega^{ii}\omega^{jj})^{N-k-1}\omega^{ij}\omega^{ji}. \tag{2.39}$$

From $\omega^{ii} = \sum_{s=1}^{N-k} \psi_s^i \psi_{\bar{s}}^{\bar{i}}$, the multinomial theorem and conditions of the fermionic variables $\psi_s^j \psi_s^j = \psi_{\bar{s}}^{\bar{j}} \psi_{\bar{s}}^{\bar{j}} = 0$.

$$P_1 = \sum_{i=1}^{k} \frac{(kN-k^2-2)!}{(N-k-2)!} (N-k)!$$
$$+ \sum_{i \neq j} \frac{(kN-k^2-2)!}{((N-k-1)!)^2} \int D\psi \left(\prod_{n \neq i,j} \prod_{l=1}^{N-k} \psi_l^n \psi_{\bar{l}}^{\bar{n}} \right)$$
$$\times (\omega^{ii}\omega^{jj})^{N-k-1} \left(\sum_{s,t=1}^{N-k} \psi_s^i \psi_{\bar{s}}^{\bar{j}} \psi_t^j \psi_{\bar{t}}^{\bar{i}} \right). \tag{2.40}$$

In the second term, $(\omega^{ii}\omega^{jj})^{N-k-1}$ contains $N-k-1$ $\psi_s^i\psi_{\bar{s}}^{\bar{i}}$, and $\psi_t^j\psi_{\bar{t}}^{\bar{i}}$. Therefore, it must be $s=t$ based on the conditions of the fermionic variables.

$$\begin{aligned}
P_1 &= (kN-k^2-2)!k(N-k)(N-k-1) \\
&\quad - \sum_{i\neq j}\sum_{s=1}^{N-k}\frac{(kN-k^2-2)!}{((N-k-1)!)^2}\int D\psi\left(\prod_{n\neq i,j}\prod_{l=1}^{N-k}\psi_l^n\psi_{\bar{l}}^{\bar{n}}\right) \\
&\quad \times (\omega^{ii}\omega^{jj})^{N-k-1}(\psi_s^i\psi_{\bar{s}}^{\bar{i}}\psi_s^j\psi_{\bar{s}}^{\bar{j}}) \qquad (2.41)\\
&= (kN-k^2-2)!k(N-k)(N-k-1) \\
&\quad - \sum_{i\neq j}\sum_{s=1}^{N-k}(kN-k^2-2)!\int D\psi\left(\prod_{n\neq i,j}\prod_{l=1}^{N-k}\psi_l^n\psi_{\bar{l}}^{\bar{n}}\right) \\
&\quad \times \left(\prod_{\substack{q=1\\ q\neq s}}^{N-k}\psi_q^i\psi_{\bar{q}}^{\bar{i}}\psi_q^j\psi_{\bar{q}}^{\bar{j}}\right)(\psi_s^i\psi_{\bar{s}}^{\bar{i}}\psi_s^j\psi_{\bar{s}}^{\bar{j}}) \qquad (2.42)\\
&= (kN-k^2-2)!k(N-k)(N-k-1) - \sum_{i\neq j}\sum_{s=1}^{N-k}(kN-k^2-2)! \qquad (2.43)\\
&= (kN-k^2-2)!\{k(N-k)(N-k-1)-(N-k)k(k-1)\} \qquad (2.44)\\
&= (kN-k^2-2)!k(N-k)(N-2k). \qquad (2.45)
\end{aligned}$$

Therefore, we obtain P_1. We compute P_2.

$$\begin{aligned}
P_2 &= \int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}\left(\sum_{i=1}^{k}(\omega^{ii})^2+\sum_{i\neq j}\omega^{ij}\omega^{ji}\right)^2 \qquad (2.46)\\
&= \int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}\Bigg[\sum_{i,j}(\omega^{ii}\omega^{jj})^2+2\sum_{m=1}^{k}\sum_{i\neq j}(\omega^{mm})^2\omega^{ij}\omega^{ji} \\
&\qquad + \sum_{a\neq b}\sum_{i\neq j}\omega^{ab}\omega^{ba}\omega^{ij}\omega^{ji}\Bigg]. \qquad (2.47)
\end{aligned}$$

We define Q_1, Q_2 and Q_3 as follows.

$$Q_1 := \sum_{i,j}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}(\omega^{ii}\omega^{jj})^2, \qquad (2.48)$$

$$Q_2 := 2\sum_{m=1}^{k}\sum_{i\neq j}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}(\omega^{mm})^2\omega^{ij}\omega^{ji}, \tag{2.49}$$

$$Q_3 := \sum_{a\neq b}\sum_{i\neq j}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}\omega^{ab}\omega^{ba}\omega^{ij}\omega^{ji}. \tag{2.50}$$

Thereafter, $P_2 = Q_1 + Q_2 + Q_3$. We consider Q_1:

$$Q_1 = \sum_{i=1}^{k}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}(\omega^{ii})^4 + \sum_{i\neq j}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}(\omega^{ii}\omega^{jj})^2. \tag{2.51}$$

We can compute the above equation in the same manner as P_1. Consequently,

$$Q_1 = (kN-k^2-4)!k(N-k) \times \{(N-k-1)(N-k-2)(N-k-3) + (k-1)(N-k)(N-k-1)^2\}. \tag{2.52}$$

Subsequently, we calculate Q_2.

$$Q_2 = 2\sum_{i\neq j}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}(\omega^{ii})^2\omega^{ij}\omega^{ji} + 2\sum_{i\neq j}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}(\omega^{jj})^2\omega^{ij}\omega^{ji} + 2\sum_{i\neq j}\sum_{m\neq i,j}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}(\omega^{mm})^2\omega^{ij}\omega^{ji}. \tag{2.53}$$

From $\omega^{ij}\omega^{ji} = \omega^{ji}\omega^{ij}$, if we replace i with j and j with i in the second term, it is the same as in the first term.

$$Q_2 = 4\sum_{i\neq j}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}(\omega^{ii})^2\omega^{ij}\omega^{ji} + 2\sum_{i\neq j}\sum_{m\neq i,j}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}(\omega^{mm})^2\omega^{ij}\omega^{ji} \tag{2.54}$$

$$= 4\sum_{i\neq j}\sum_{p_n}\frac{(kN-k^2-4)!}{\prod_{q=1}^{k}p_q!}\int D\psi\left(\prod_{n=1}^{k}(\omega^{nn})^{p_n}\right)(\omega^{ii})^2\omega^{ij}\omega^{ji}$$

$$+ 2\sum_{i\neq j}\sum_{m\neq i,j}\sum_{p_n}\frac{(kN-k^2-4)!}{\prod_{q=1}^{k}p_q!}\int D\psi\left(\prod_{n=1}^{k}(\omega^{nn})^{p_n}\right)(\omega^{mm})^2\omega^{ij}\omega^{ji}. \quad (2.55)$$

Thereafter, $\sum_{p_n}$ is the sum of all combinations from 0 to $kN-k^2-4$ indices p_1 through p_k such that the sum of p_n $(n=1,\ldots,k)$ is $kN-k^2-4$. From the condition of fermionic integration and the condition of fermionic variables $\psi_s^i\psi_s^i=0$, in the first term, $p_n=N-k$ $(n\neq i,j)$ and $p_i=N-k-3$, $p_j=N-k-1$. In the second term, $p_n=N-k$ $(n\neq i,j,m)$ and $p_i=p_j=N-k-1$, $p_m=N-k-2$. Therefore,

$$Q_2 = 4\sum_{i\neq j}\frac{(kN-k^2-4)!}{(N-k-3)!(N-k-1)!}\int D\psi\left(\prod_{n\neq i,j}\prod_{l=1}^{N-k}\psi_l^n\psi_{\bar{l}}^{\bar{n}}\right)(\omega^{ii}\omega^{jj})^{N-k-1}\omega^{ij}\omega^{ji}$$

$$+ 2\sum_{i\neq j}\sum_{m\neq i,j}\frac{(kN-k^2-4)!(N-k)!}{(N-k-2)!((N-k-1)!)^2}$$

$$\times\int D\psi\left(\prod_{n\neq i,j}\prod_{l=1}^{N-k}\psi_l^n\psi_{\bar{l}}^{\bar{n}}\right)(\omega^{ii}\omega^{jj})^{N-k-1}\omega^{ij}\omega^{ji}. \quad (2.56)$$

Here, we can calculate the fermion integral in the same manner as P_1. We obtain

$$\int D\psi\left(\prod_{n\neq i,j}\prod_{l=1}^{N-k}\psi_l^n\psi_{\bar{l}}^{\bar{n}}\right)(\omega^{ii}\omega^{jj})^{N-k-1}\omega^{ij}\omega^{ji} = -(N-k)((N-k-1)!)^2. \quad (2.57)$$

$$Q_2 = -4\sum_{i\neq j}\frac{(kN-k^2-4)!(N-k)!}{(N-k-3)!}$$

$$- 2\sum_{i\neq j}\sum_{m\neq i,j}\frac{(kN-k^2-4)!(N-k)!(N-k)}{(N-k-2)!} \quad (2.58)$$

$$= -4\frac{(kN-k^2-4)!(N-k)!}{(N-k-3)!}k(k-1)$$

$$- 2\frac{(kN-k^2-4)!(N-k)!(N-k)}{(N-k-2)!}k(k-1)(k-2)$$

$$= (kN-k^2-4)!k(N-k)(k-1)$$

$$\times[-4(N-k-1)(N-k-2)-2(N-k)(N-k-1)(k-2)]. \quad (2.59)$$

Finally, we compute Q_3.

$$Q_3 = \sum_{a\neq b}\sum_{i\neq j}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}\omega^{ab}\omega^{ba}\omega^{ij}\omega^{ji}. \tag{2.60}$$

The sum $\sum_{a\neq b}\sum_{i\neq j}$ can be divided into the following seven cases.

Sum patterns of (i, j) and (a, b)

(1) $i = a,\ j = b$. (2) $i = b,\ j = a$. (3) $i = a,\ j \neq b$. (4) $i = b,\ j \neq a$. (5) $i \neq a,\ j = b$. (6) $i \neq b,\ j = a$. (7) $i \neq a, b\ \ j \neq a, b$.

From the symmetry of a, b and i, j, (1) and (2) have the same form: Similarly, (3), (4), (5), and (6) have the same form: Therefore,

$$\begin{aligned} Q_3 &= 2\sum_{i\neq j}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}(\omega^{ij}\omega^{ji})^2 \\ &\quad + 4\sum_{i\neq j}\sum_{b\neq i,j}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}\omega^{ib}\omega^{bi}\omega^{ij}\omega^{ji} \\ &\quad + \sum_{(i,j,a,b)}{}^{\prime}\int D\psi\left(\sum_{n=1}^{k}\omega^{nn}\right)^{kN-k^2-4}\omega^{ab}\omega^{ba}\omega^{ij}\omega^{ji}. \end{aligned} \tag{2.61}$$

Here, $\sum'_{(i,j,a,b)}$ implies that i, j, a, and b are different from each other in the summation.

$$\begin{aligned} Q_3 &= 2\sum_{i\neq j}\frac{(kN-k^2-4)!}{((N-k)!)^{k-2}((N-k-2)!)^2} \\ &\quad \times\int D\psi\left(\prod_{n\neq i,j}(\omega^{nn})^{N-k}\right)(\omega^{ii}\omega^{jj})^{N-k-2}(\omega^{ij}\omega^{ji})^2 \\ &\quad + 4\sum_{i\neq j}\sum_{b\neq i,j}\frac{(kN-k^2-4)!}{((N-k)!)^{k-3}(N-k-2)!((N-k-1)!)^2} \\ &\quad \times\int D\psi\left(\prod_{n\neq i,j,b}(\omega^{nn})^{N-k}\right)(\omega^{ii})^{N-k-2}(\omega^{bb}\omega^{jj})^{N-k-1}\omega^{ib}\omega^{bi}\omega^{ij}\omega^{ji} \\ &\quad + \sum_{(i,j,a,b)}{}^{\prime}\frac{(kN-k^2-4)!}{((N-k)!)^{k-4}((N-k-1)!)^4} \end{aligned}$$

$$\times \int D\psi \left(\prod_{n \neq a,b,i,j} (\omega^{nn})^{N-k} \right) (\omega^{aa}\omega^{bb}\omega^{ii}\omega^{jj})^{N-k-1} \omega^{ab}\omega^{ba}\omega^{ij}\omega^{ji} \tag{2.62}$$

$$= 2\sum_{i\neq j} \frac{(kN-k^2-4)!}{((N-k-2)!)^2} \int D\psi \left(\prod_{n\neq i,j} \prod_{l=1}^{N-k} \psi_l^n \psi_{\bar{l}}^{\bar{n}} \right) (\omega^{ii}\omega^{jj})^{N-k-2}$$

$$\times \left(\sum_{s_1,s_2,t_1,t_2} \psi_{s_1}^i \psi_{\bar{s}_1}^{\bar{j}} \psi_{t_1}^j \psi_{\bar{t}_1}^{\bar{i}} \psi_{s_2}^i \psi_{\bar{s}_2}^{\bar{j}} \psi_{t_2}^j \psi_{\bar{t}_2}^{\bar{i}} \right)$$

$$+ 4\sum_{i\neq j} \sum_{b\neq i,j} \frac{(kN-k^2-4)!}{(N-k-2)!((N-k-1)!)^2} \int D\psi \left(\prod_{n\neq i,j} \prod_{l=1}^{N-k} \psi_l^n \psi_{\bar{l}}^{\bar{n}} \right) (\omega^{ii})^{N-k-2}$$

$$\times (\omega^{bb}\omega^{jj})^{N-k-1} \left(\sum_{s_1,s_2,t_1,t_2} \psi_{s_1}^i \psi_{\bar{s}_1}^{\bar{b}} \psi_{t_1}^b \psi_{\bar{t}_1}^{\bar{i}} \psi_{s_2}^i \psi_{\bar{s}_2}^{\bar{j}} \psi_{t_2}^j \psi_{\bar{t}_2}^{\bar{i}} \right)$$

$$+ \sum_{(i,j,a,b)}' \frac{(kN-k^2-4)!}{((N-k-1)!)^4} \int D\psi \left(\prod_{n\neq a,b,i,j} \prod_{l=1}^{N-k} \psi_l^n \psi_{\bar{l}}^{\bar{n}} \right) (\omega^{aa}\omega^{bb}\omega^{ii}\omega^{jj})^{N-k-1}$$

$$\times \left(\sum_{s_1,s_2,t_1,t_2} \psi_{s_1}^a \psi_{\bar{s}_1}^{\bar{b}} \psi_{t_1}^b \psi_{\bar{t}_1}^{\bar{a}} \psi_{s_2}^i \psi_{\bar{s}_2}^{\bar{j}} \psi_{t_2}^j \psi_{\bar{t}_2}^{\bar{i}} \right). \tag{2.63}$$

We consider sums of s_1, s_2, t_1 and t_2. In the first term, the sum can be divided into two ways, $(s_1 = t_1,\ s_2 = t_2,\ s_1 \neq s_2)$ and $(s_1 = t_2,\ s_2 = t_1,\ s_1 \neq s_2)$. In the second term, it must be $(s_1 = t_1,\ s_2 = t_2,\ s_1 \neq s_2)$. In the third term, it must be $(s_1 = t_1,\ s_2 = t_2)$. Because the first term is symmetric for s_1 and s_2 and t_1 and t_2,

$$Q_3 = 4\sum_{i\neq j} \sum_{s_1\neq s_2} \frac{(kN-k^2-4)!}{((N-k-2)!)^2} \int D\psi \left(\prod_{n\neq i,j} \prod_{l=1}^{N-k} \psi_l^n \psi_{\bar{l}}^{\bar{n}} \right) (\omega^{ii}\omega^{jj})^{N-k-2}$$

$$\times (\psi_{s_1}^i \psi_{\bar{s}_1}^{\bar{i}} \psi_{s_1}^j \psi_{\bar{s}_1}^{\bar{j}} \psi_{s_2}^i \psi_{\bar{s}_2}^{\bar{i}} \psi_{s_2}^j \psi_{\bar{s}_2}^{\bar{j}})$$

$$+ 4\sum_{i\neq j} \sum_{b\neq i,j} \sum_{s_1\neq s_2} \frac{(kN-k^2-4)!}{(N-k-2)!((N-k-1)!)^2}$$

$$\times \int D\psi \left(\prod_{n\neq i,j} \prod_{l=1}^{N-k} \psi_l^n \psi_{\bar{l}}^{\bar{n}} \right) (\omega^{ii})^{N-k-2} (\omega^{bb}\omega^{jj})^{N-k-1}$$

$$\times (\psi_{s_1}^i \psi_{\bar{s}_1}^{\bar{i}} \psi_{s_1}^b \psi_{\bar{s}_1}^{\bar{b}} \psi_{s_2}^i \psi_{\bar{s}_2}^{\bar{i}} \psi_{s_2}^j \psi_{\bar{s}_2}^{\bar{j}})$$

$$+ \sum_{(i,j,a,b)}' \sum_{s_1,s_2} \frac{(kN-k^2-4)!}{((N-k-1)!)^4} \int D\psi \left(\prod_{n \neq a,b,i,j} \prod_{l=1}^{N-k} \psi_l^n \psi_{\bar{l}}^{\bar{n}} \right) (\omega^{aa} \omega^{bb} \omega^{ii} \omega^{jj})^{N-k-1}$$

$$\times (\psi_{s_1}^a \psi_{\bar{s}_1}^{\bar{a}} \psi_{s_1}^b \psi_{\bar{s}_1}^{\bar{b}} \psi_{s_2}^i \psi_{\bar{s}_2}^{\bar{i}} \psi_{s_2}^j \psi_{\bar{s}_2}^{\bar{j}}) \tag{2.64}$$

$$= 4 \sum_{i \neq j} \sum_{s_1 \neq s_2} (kN-k^2-4)! + 4 \sum_{i \neq j} \sum_{b \neq i,j} \sum_{s_1 \neq s_2} (kN-k^2-4)!$$

$$+ \sum_{(i,j,a,b)}' \sum_{s_1,s_2} (kN-k^2-4)! \tag{2.65}$$

$$= (kN-k^2-4)![4k(k-1)(N-k)(N-k-1)$$

$$+ 4k(k-1)(k-2)(N-k)(N-k-1)$$

$$+ k(k-1)(k-2)(k-3)(N-k)^2] \tag{2.66}$$

$$= (kN-k^2-4)!k(N-k)[4(k-1)^2(N-k-1)$$

$$+ (k-1)(k-2)(k-3)(N-k)]. \tag{2.67}$$

Therefore, we obtain (2.33) from $P_2 = Q_1 + Q_2 + Q_3$ and these results. We complete the proof of proposition 1.

Acknowledgement

We would like to thank Professor Masao Jinzenji for the useful discussions.

References

[1] R. Bott and L. W. Tu. Differential Forms in Algebraic Topology. Graduate Texts in Mathematics. Springer New York, NY, 1982.

[2] N. Chair. Intersection numbers on Grassmannians, and on the space of holomorphic maps from CP^1 into $G_r(C^n)$. J. Geom. Phys., 38(2):170–182, 2001. DOI:10.1016/S0393-0440(00)00059-0, URL:https://doi.org/10.1016/S0393-0440(00)00059-0. arXiv:hep-th/9808170.

[3] W. Fulton. Intersection Theory. Springer-Verlag Berlin Hidelberg, 1998.

[4] P. Griffiths and J. Harris. Principles of algebraic geometry. Pure and Applied Mathematics. Wiley–Interscience, New York, 1978.

[5] D. T. Hiep. Identities involving (doubly) symmetric polynomials and integrals over Grassmannians, 2016. arXiv, DOI: 10.48550/ARXIV.1607.04850 URL: https://arxiv.org/abs/1607.04850.

[6] T. Ikeda and H. Naruse. Modern Schubert calculus, from the special polynomial theory's point of view (in Japanese). Sugaku, 63(3):313–337, 2011. DOI:10.11429/sugaku.0633313, URL:https://doi.org/10.11429/sugaku.0633313.

[7] S. Imanishi. Computation of Euler number for the Grassmann manifold $G(2,N)$ via Mathai-Quillen formalism (in Japanese). Master's thesis, Hokkaido University, 2019.

[8] S. Imanishi, M. Jinzenji, and K. Kuwata. Evaluation of Euler number of complex Grassmann manifold $G(k,N)$ via Mathai-Quillen formalism. Journal of Geometry and Physics, 180:104623, 2022. DOI:https://doi.org/10.1016/j.geomphys.2022.104623.

[9] A. Weber. Equivariant Chern classes and localization theorem. J. Singul., 5:153–176, 2012. DOI:10.5427/jsing.2012.5k, URL: https://doi.org/10.5427/jsing.2012.5k. arXiv:1110.5515.

[10] E. Witten. The Verlinde algebra and the cohomology of the Grassmannian. In Geometry, topology, & physics, Conf. Proc. Lecture Notes Geom. Topology, IV, pages 357–422. Int. Press, Cambridge, MA, 1995. arXiv:hep-th/9312104.

[11] M. Zielenkiewicz. Integration over homogeneous spaces for classical Lie groups using iterated residues at infinity. centr.eur.j.math., 12:574–583, 2014. DOI:10.2478/s11533-013-0372-z, URL: https://doi.org/10.2478/s11533-013-0372-z. arXiv:1212.6623.

Ken Kuwata
Department of General Education
National Institute of Technology, Kagawa College
Chokushi, Takamatsu, 761-8058, *Japan*
E-mail: kuwata-k@t.kagawa-nct.ac.jp