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Left-invariant symplectic structures on diagonal almost abelian Lie groups*

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ABSTRACT. We are interested in the classification or finding conditions for the existence of left-invariant symplectic structures on Lie groups. Some classifications are known, especially in low dimensions. We approach this problem by studying the "moduli space of left-invariant nondegenerate 2-forms", which is a certain orbit space in the set of all nondegenerate 2-forms on a Lie algebra. In this paper, using this approach, we give a classification of left-invariant symplectic structures on all almost abelian Lie algebras determined by diagonal matrices.

1. Introduction

The problem of determining whether a given manifold admits a symplectic structure is a classical and hard problem. In the setting of Lie groups, it is natural to ask about the existence of left-invariant structures. A *symplectic Lie group* is a Lie group G endowed with a left-invariant symplectic form ω (that is, a nondegenerate closed 2-form). The study of symplectic Lie groups reduces to the study of *symplectic Lie algebras* $(\mathfrak{g}, \omega)$, that is, Lie algebras $\mathfrak{g}$ endowed with nondegenerate closed 2-forms (or equivalently two-cocycles $\omega \in Z^2(\mathfrak{g})$). Still the problem of determining if a given Lie algebra admits a symplectic structure remains difficult in general and the picture seems far from complete. Only some classifications in low dimensions and some special cases in higher dimensions are known. Some of the known classification in low dimensions include: complete classification for the 4-dimensional case ([17]), filiform Lie algebras up to dimension 10 ([8]), most of solvable Lie algebras up to dimension 6 ([5], [14]). Some other special higher dimensional cases can be found, for example, in [15], [18].

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In [9], we can find a novel method to find nice (e.g., Einstein or Ricci soliton) left-invariant Riemannian metrics. The method is based on the moduli space of left-invariant Riemannian metrics on a Lie group G (the orbit of space of certain group action). In [12] and [13], the authors adapted the same ideas in the pseudo-Riemannian case successfully. It was natural then, to try to use the same ideas for symplectic Lie groups. Inspired by those previous studies in [6] we developed a similar approach for the study of symplectic Lie groups. We study the moduli space of left-invariant nondegenerate 2-forms on Lie algebras $\mathfrak{g}$, which is the orbit space of the action of $\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g})$ on the space $\Omega(\mathfrak{g})$ of nondegenerate 2-forms on $\mathfrak{g}$. As a first application of these ideas also in [6] we studied two particular Lie algebras: the Lie algebra of the real hyperbolic space $\mathfrak{g}_{\mathbb{R}\mathrm{H}^{2n}}$, and the direct sum of the 3-dimensional Heisenberg Lie algebra and the abelian Lie algebra $\mathfrak{h}^3 \oplus \mathbb{R}^{2n-3}$. We obtained a classification of symplectic structures on both of them.

These two Lie algebras belong to a special family of Lie algebras: they are both almost abelian Lie algebras. An almost abelian Lie algebra is a Lie algebra that contains a codimension one abelian subalgebra. The structure of almost abelian Lie algebras has been studied in [1]. An interesting result is that isomorphism classes of almost abelian Lie algebras are related to similarity classes of linear operators. Almost abelian Lie algebras represent a good candidate for applying our method. In this paper we study a particular family of almost abelian Lie algebras whose adjoint homomorphism is diagonalizable (see Section 4). In fact, we obtain

THEOREM 1.1. *Let $\mathfrak{g}$ be an almost abelian Lie algebra determined by a diagonal matrix* $\operatorname{diag}(\lambda_2, \dots, \lambda_{2n})$.

(1) *There exists a symplectic form ω on $\mathfrak{g}$ if and only if there exists a permutation σ of $\{2, \dots, 2n\}$ such that*
$$\lambda_{\sigma(i)} + \lambda_{\sigma(i+n)} = 0 \qquad \text{for } i = 2, \dots, n.$$
(2) *If there exists a symplectic form ω on $\mathfrak{g}$, then it is unique up to automorphism and scale.*

Condition (1) is an easy to check condition for the existence in terms of the structure constants of the Lie algebra. If a symplectic form does exist, (2) states it is actually unique.

One of the important problems in the context of symplectic Lie algebras $(\mathfrak{g}, \omega)$ is to determine the existence of certain special subspaces (see [2] for more details). In particular, a subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is said to be *Lagrangian* if
$$\mathfrak{l} = \mathfrak{l}^\perp := \{v \in \mathfrak{g} \mid \omega(v, w) = 0 \text{ for all } w \in \mathfrak{l}\}.$$
From Theorem 1.1, we get the following.

Corollary 1.2. *Any diagonal almost abelian Lie algebra* $\mathfrak{g}$ *with a symplectic structure* $\omega \in \Omega(\mathfrak{g})$ *contains a Lagrangian ideal.*

Useful tools for studying the moduli space of left-invariant nondegenerate 2-forms are decompositions of matrices in terms of symplectic matrices. In [6], we obtained a slight modification of a decomposition theorem of symplectic matrices called symplectic QR decomposition. In this paper again we obtain an improvement on another decomposition theorem of symplectic matrices called symplectic SR decomposition: we showed that up to permutation every nonsingular matrix has an SR decomposition.

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2. Preliminaries

In this section, we recall some basic notions on left-invariant symplectic forms on Lie groups.

2.1. Left-invariant symplectic 2-forms. Let G be a simply connected Lie group with dimension $2n$ and $\mathfrak{g}$ its corresponding Lie algebra. We are interested in the set of all nondegenerate left-invariant 2-forms on G, denoted by

$$\Omega(G) := \{\omega(\cdot\,,\cdot) \in \textstyle\bigwedge^2 T^*G \mid \omega^n \neq 0, \text{ left-invariant}\}.$$

We want to find closed 2-forms in this set. For this set, we have the following natural equivalence relation.

Definition 2.1. *Let* $\omega_1, \omega_2 \in \Omega(G)$. *Then,* (G, ω_1) *and* (G, ω_2) *are said to be equivalent up to automorphism (resp. equivalent up to automorphism and scale) if there exists* $\phi \in \mathrm{Aut(G)}$ *such that* $\phi^*\omega_1 = \omega_2$ *(resp. if there exist* $\phi \in \mathrm{Aut(G)}$ *and a constant* $c \neq 0$ *such that* $c \cdot (\phi)^*\omega_1 = \omega_2$*).*

It is well known that the space $\Omega(G)$ can be identified with the space of nondegenerate 2-forms on $\mathfrak{g}$, denoted by

$$\Omega(\mathfrak{g}) := \{\omega(\cdot\,,\cdot) \in \textstyle\bigwedge^2 \mathfrak{g}^* \mid \omega^n \neq 0\}.$$

For this set, we have the following natural equivalence relation.

Definition 2.2. *Let* $\omega_1, \omega_2 \in \Omega(\mathfrak{g})$. *Then,* $(\mathfrak{g}, \omega_1)$ *and* $(\mathfrak{g}, \omega_2)$ *are said to be equivalent up to automorphism (resp. equivalent up to automorphism and scale) if there exists* $\phi \in \mathrm{Aut}(\mathfrak{g})$ *such that* $\phi^*\omega_1 = \omega_2$ *(resp. if there exist* $\phi \in \mathrm{Aut}(\mathfrak{g})$ *and a constant* $c \neq 0$ *such that* $c \cdot (\phi)^*\omega_1 = \omega_2$*).*

When the Lie group is simply connected, which we always assume, both notions Definitions 2.1 and 2.2 of equivalence coincide. This fact allows us to work at the Lie algebra level.

REMARK 2.3. *If* (S, ω_1) *and* (S, ω_2) *are symplectic manifolds and there exists* $\phi \in \mathrm{Diff}(S)$ *such that* $\phi^* \omega_1 = \omega_2$, *then* (S, ω_1) *and* (S, ω_2) *are said to be symplectomorphically equivalent and* ϕ *is called a symplectomorphism. Notice that the equivalence relation in Definition* 2.1 *(and the corresponding notion in Definition* 2.2*) is stronger, but this would be the usual notion of equivalence in symplectic Lie groups. In fact, in the context of symplectic Lie groups, the map in Definition* 2.1 *or Definition* 2.2 *is also sometimes called a symplectomorphism.*

Remember that a symplectic vector space is a pair (V, ω), where V is a vector space and ω is a nondegenerate 2-form. For every $\omega_{\mathfrak{g}} \in \Omega(\mathfrak{g})$, the pair $(\mathfrak{g}, \omega_{\mathfrak{g}})$ is a symplectic vector space. The next is a well known fact.

PROPOSITION 2.4 (cf. [2], Chapter 0). *Let* $\omega_{\mathfrak{g}} \in \Omega(\mathfrak{g})$, *and* $\omega_G \in \Omega(G)$ *be the corresponding* 2*-form on the Lie group. Then* ω_G *is closed if and only if* $\omega_{\mathfrak{g}}$ *satisfies, for all* $x, y, z \in \mathfrak{g}$

$$d\omega_{\mathfrak{g}}(x, y, z) := \omega_{\mathfrak{g}}(x, [y, z]) + \omega_{\mathfrak{g}}(z, [x, y]) + \omega_{\mathfrak{g}}(y, [z, x]) = 0.$$

A 2-form $\omega_{\mathfrak{g}} \in \Omega(\mathfrak{g})$ that satisfies the previous property is called a *closed* 2*-form* or *symplectic form* on the Lie algebra $\mathfrak{g}$.

REMARK 2.5. *The previous condition can be expressed in terms of the cohomology of Lie algebras. One knows that* $\omega_{\mathfrak{g}}$ *is closed if and only if* $\omega \in Z^2(\mathfrak{g})$, *where* $Z^2(\mathfrak{g})$ *is the set of* 2*-cocyles in the trivial representation over* $\mathbb{R}$.

From the theory of homogeneous spaces we have the identification

$$\Omega(\mathfrak{g}) \cong \mathrm{GL}(2n, \mathbb{R}) / \mathrm{Sp}_n(\mathbb{R}).$$

Here we identify $\mathfrak{g} \cong \mathbb{R}^{2n}$, and then the general linear group $\mathrm{GL}(2n, \mathbb{R})$ acts transitively on $\Omega(\mathfrak{g})$ by

$$g.\omega(\cdot\,, \cdot) = \omega(g^{-1}(\cdot), g^{-1}(\cdot)) \qquad \forall g \in \mathrm{GL}(2n, \mathbb{R}).$$

We also recall that $\mathrm{Sp}_n(\mathbb{R})$ is the symplectic group, that is, the group of linear maps which preserve the canonical symplectic 2-form ω_0 in $\mathbb{R}^{2n}$. If $\{e_1, \dots, e_{2n}\}$ is the canonical basis in $\mathbb{R}^{2n}$ and $\{\varepsilon^1, \dots, \varepsilon^{2n}\}$ is the corresponding dual basis, then the canonical symplectic 2-form is given by

$$\omega_0 := \varepsilon^1 \wedge \varepsilon^{n+1} + \cdots + \varepsilon^n \wedge \varepsilon^{2n}. \tag{1}$$

Then the group $\mathrm{Sp}_n(\mathbb{R})$ can be described as

$$\mathrm{Sp}_n(\mathbb{R}) := \{A \in \mathrm{GL}(2n, \mathbb{R}) \mid (A^t)JA = J\}, \tag{2}$$

where $J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, with I_n the identity matrix. Note that the symplectic group $\mathrm{Sp}_n(\mathbb{R})$ is closed under the transposition of matrices, which can be checked just by taking the inverse of $(A^t)JA = J$.

We want to use Proposition 2.4 to search for 2-forms that are closed in the set $\Omega(\mathfrak{g})$, but this set can be rather big so next we introduce the concept of the *moduli space*.

2.2. The definition. Consider the automorphism group of $\mathfrak{g}$ defined by

$$\mathrm{Aut}(\mathfrak{g}) := \{\phi \in \mathrm{GL}(2n, \mathbb{R}) \mid \phi[\cdot\,,\cdot] = [\phi(\cdot), \phi(\cdot)]\}.$$

Also define $\mathbb{R}^\times := \mathbb{R}\backslash 0$. Then we can consider the set

$$\mathbb{R}^\times \,\mathrm{Aut}(\mathfrak{g}) := \{\phi \in \mathrm{GL}(2n, \mathbb{R}) \mid \phi \in \mathrm{Aut}(\mathfrak{g}),\ c \in \mathbb{R}^\times\},$$

which is a subgroup of $\mathrm{GL}(2n, \mathbb{R})$. Hence it naturally acts on $\Omega(\mathfrak{g})$. Note that $\mathbb{R}^\times$ does not act as the usual scaling. In fact, for $cI_{2n} \in \mathbb{R}^\times \,\mathrm{Aut}(\mathfrak{g})$, we have

$$(cI_{2n}).\omega(\cdot\,,\cdot) = \omega((cI_{2n})^{-1}(\cdot), (cI_{2n})^{-1}(\cdot)) = c^{-2}\omega.$$

We can then consider the orbit space of this action.

Definition 2.6. *The orbit space of the action of* $\mathbb{R}^\times \,\mathrm{Aut}(\mathfrak{g})$ *on* $\Omega(\mathfrak{g})$ *is called the moduli space of left-invariant nondegenerate* 2*-forms and is denoted by*

$$\mathfrak{P}\Omega(\mathfrak{g}) := \mathbb{R}^\times \,\mathrm{Aut}(\mathfrak{g})\backslash\Omega(\mathfrak{g}) := \{\mathbb{R}^\times \,\mathrm{Aut}(\mathfrak{g}).\omega \mid \omega \in \Omega(\mathfrak{g})\}.$$

One can easily see that, if $\omega_1, \omega_2 \in \Omega(\mathfrak{g})$ are in the same $\mathbb{R}^\times \,\mathrm{Aut}(\mathfrak{g})$-orbit, then they are equivalent up to automorphism and scale. Therefore there is a surjection from the moduli space $\mathfrak{P}\Omega(\mathfrak{g})$ onto the quotient space

$$\Omega(\mathfrak{g})/\text{“up to automorphism and scale”}.$$

This would be not bijective, since ω and $-\omega$ are possibly not in the same $\mathbb{R}^\times \,\mathrm{Aut}(\mathfrak{g})$-orbit. In fact, for the canonical form ω_0 we have

$$I_{n,n}\omega_0 = -\omega_0, \qquad I_{n,n} = \left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & -I_n \end{array}\right).$$

We could consider instead the action of $(\mathbb{Z}_2 I_{n,n})\mathbb{R}^\times \,\mathrm{Aut}(\mathfrak{g})$ to avoid this, but in many cases this is not necessary.

In this paper, we just consider the moduli space $\mathfrak{P}\Omega(\mathfrak{g})$. Notice that the action of $\mathbb{R}^{\times}\operatorname{Aut}(\mathfrak{g})$ preserves the closedness of 2-forms. In the latter sections, instead of studying $\Omega(\mathfrak{g})$ directly we will focus on studying $\mathfrak{P}\Omega(\mathfrak{g})$: we want to find orbits that correspond to closed 2-forms.

2.3. Milnor frames procedure. Remember that given a symplectic vector space (V,ω) with $\dim V = 2n$, we can always choose a basis $\{x_1,\dots,x_{2n}\}$ of V such that for $i<j$

$$\omega(x_i,x_j) = \begin{cases} 1 & (\text{if } j = i+n), \\ 0 & (\text{all other cases}). \end{cases}$$

Such a basis is called a *symplectic basis*.

Remember that $\mathfrak{g}$ is a Lie algebra of dimension $2n$, and we identify $\mathfrak{g} \cong \mathbb{R}^{2n}$ with the canonical basis $\{e_1,\dots,e_{2n}\}$. Let ω_0 be the canonical 2-form as in (1). Then $\{e_1,\dots,e_{2n}\}$ is a symplectic basis with respect to ω_0. To simplify the notation let us denote the orbit of $\mathbb{R}^{\times}\operatorname{Aut}(\mathfrak{g})$ through $\omega \in \Omega(\mathfrak{g})$ by

$$[\omega] := (\mathbb{R}^{\times}\operatorname{Aut}(\mathfrak{g})).\omega := \{\phi.\omega \mid \phi \in \mathbb{R}^{\times}\operatorname{Aut}(\mathfrak{g})\}.$$

Definition 2.7. *A subset $U \subset \mathrm{GL}(2n,\mathbb{R})$ is called a set of representatives of $\mathfrak{P}\Omega(\mathfrak{g})$ if it satisfies*

$$\mathfrak{P}\Omega(\mathfrak{g}) = \{[h.\omega_0] \mid h \in U\}.$$

Let $[[g]]$ denote the double coset of $g \in \mathrm{GL}(2n,\mathbb{R})$ defined by

$$[[g]] := \mathbb{R}^{\times}\operatorname{Aut}(\mathfrak{g})g\,\mathrm{Sp}(2n,\mathbb{R}) := \{\phi g s \mid \phi \in \mathbb{R}^{\times}\operatorname{Aut}(\mathfrak{g}),\ s \in \mathrm{Sp}_n(\mathbb{R})\}.$$

By standard theory of double coset spaces, we have a criterion for a set U to be a set of representatives (we refer to [9]).

Lemma 2.8. *Let $U \subset \mathrm{GL}(2n,\mathbb{R})$, and assume that for every $g \in \mathrm{GL}(2n,\mathbb{R})$ there exists $h \in U$ such that $h \in [[g]]$. Then U is a set of representatives of $\mathfrak{P}\Omega(\mathfrak{g})$.*

Now we state a theorem for obtaining Milnor type frames in the symplectic case.

Theorem 2.9 ([6]). *Let U be a set of representatives of $\mathfrak{P}\Omega(\mathfrak{g})$. Then for every $\omega \in \Omega(\mathfrak{g})$ there exist $k>0$, $\phi \in \operatorname{Aut}(\mathfrak{g})$ and $h \in U$ such that $\{\phi h e_1,\dots,\phi h e_{2n}\}$ is a symplectic basis with respect to $k\omega$.*

The basis obtained in this theorem will be called *Milnor frames*. Notice that if U has a nice form, the bracket relations of the Milnor frames will also be given in terms of a nice set of parameters. In such cases, it becomes much easier to search for closed 2-forms inside of $\Omega(\mathfrak{g})$. In the next section, we introduce a tool that will be useful to calculate a nice set of representatives and to obtain nice Milnor frames.

3. SR decomposition

To obtain a nice set of representatives, it is useful to have general results for decomposing matrices using symplectic matrices. In this section, we give an improvement of the so called *SR* decomposition. Some of the known results can be seen in [3] or [4]. First we define some notations.

$\mathrm{M}(n,\mathbb{R})$ denotes the set of all $n \times n$ real matrices. As before, $\mathrm{GL}(n,\mathbb{R})$ denotes the set of all $n \times n$ nonsingular real matrices and I_k the $k \times k$ identity matrix. E_{ij} denotes the matrix with 1 at position (i,j) and zeros everywhere else. For a matrix M, M^t denotes its transpose.

Definition 3.1. *A $2n \times 2n$ block square matrix*

$$R = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

is called upper J-triangular if A, B, D are upper triangular and C is strictly upper triangular. If in addition B is strictly upper triangular it is called strictly upper J-triangular. A matrix L is called lower J-triangular (resp. strictly lower J triangular) if L^t is upper J-triangular (resp. strictly upper J-triangular).

Recall that $P \in \mathrm{GL}(n,\mathbb{R})$ is called a *permutation matrix* if it induces a permutation among the elements in the standard basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$. Denote by $\mathrm{Per}(n,\mathbb{R}) < \mathrm{GL}(n,\mathbb{R})$ the group of all permutation matrices. In particular, we have the stabilizer of e_1

$$\mathrm{Per}(n,\mathbb{R})_{e_1} = \left\{ \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & P' \end{array}\right) \,\middle|\, P' \in \mathrm{Per}(n-1,\mathbb{R}) \right\}.$$

We also define the particular permutation matrix $\hat{P} \in \mathrm{Per}(2n,\mathbb{R})_{e_1}$ by

$$\hat{P}e_l = \begin{cases} e_{2l-1} & (1 \le l \le n), \\ e_{2(l-n)} & (n+1 \le l \le 2n). \end{cases} \tag{3}$$

The next proposition is well known and can also be used as the definition of upper J-triangular matrices.

PROPOSITION 3.2 (See [4]). *A matrix R is upper J-triangular if and only if there exists an upper triangular matrix T such that*

$$(\hat{P}^t)T\hat{P} = R = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right).$$

In particular, $\mathrm{Det}(R) = \mathrm{Det}(A)\,\mathrm{Det}(D)$.

In [4], the decomposition of $2n \times 2n$ matrices into the product of a symplectic matrix and a J-triangular matrix is considered. They gave a condition for a matrix to have such a decomposition as follows. $M[k,k]$ will denote the leading principal submatrix of dimension k, consisting of the first k rows and columns. Also recall J as defined in (2).

THEOREM 3.3 (SR decomposition). *Let $M \in \mathrm{GL}(2n, \mathbb{R})$. Then there exists a decomposition of the form*

$$M = SR,$$

with $S \in Sp_n(\mathbb{R})$ and R a strictly upper J-triangular matrix if and only if

$$\det(\hat{P}M^tJM\hat{P}^t)[2k, 2k] \neq 0 \tag{4}$$

for all $k = 1, \dots, n$.

PROOF. See Theorem 3.8 and Remark 3.9 in [4]. □

In the next section of this paper, it will be more convenient to use the "transpose" version of Theorem 3.3.

LEMMA 3.4 (LS decomposition). *Let $M \in \mathrm{GL}(2n, \mathbb{R})$. Then there exists a decomposition of the form*

$$M = LS,$$

with $S \in Sp_n(\mathbb{R})$ and L a strictly lower J-triangular matrix if and only if M^t has an SR decomposition.

A well known decomposition result is the so called *LU-decomposition*: a decomposition in terms of a lower triangular matrix L and an upper triangular matrix U. In fact, this result is related to the proof of Theorem 3.3. It is also well known that up to permutation any matrix has a LU-decomposition ([10] Theorem 3.5.8). We shall prove a similar result for the SR decomposition: up to permutation every nonsingular matrix has an SR decomposition (cf. Theorem 3.8).

The matrix M^tJM that appears in (4) is a skew-symmetric matrix. We recall some basic facts about skew-symmetric matrices that will be useful. Let

$\mathrm{Skew}(n, \mathbb{R})$ denote the set of all $n \times n$ skew-symmetric matrices. The determinant of a skew-symmetric matrix can always be written as the square of a polynomial in the matrix entries. The value of this polynomial evaluated in the coefficients of a matrix A is called the *Pfaffian* of A and is denoted by $\mathrm{pf}(A)$. The Pfaffian can be defined in several ways, here we present a recursive definition.

Definition 3.5 (Pfaffian). *Let* $A = (a_{ij}) \in \mathrm{Skew}(2n, \mathbb{R})$, *then its Pfaffian is defined inductively by*

$$\mathrm{pf}(A) = \sum_{j \neq i} (-1)^{i+j+1+\theta(i-j)} a_{ij}\, \mathrm{pf}(A_{\hat{\imath}\hat{\jmath}}),$$

where θ *is the Heaviside step function,* i *can be chosen freely,* $A_{\hat{\imath}\hat{\jmath}} \in \mathrm{Skew}(2n-2, \mathbb{R})$ *is the matrix obtained by removing both the* i*-th and* j*-th row and columns from A. The Pffafian of a* 0×0 *matrix is defined as* 1.

Remember the Heaviside step function θ is defined by

$$\theta(x) = \begin{cases} 1 & (x > 0), \\ 0 & (x \leq 0). \end{cases} \tag{5}$$

Theorem 3.6 (Cayley 1842). *If* $A \in \mathrm{Skew}(2n, \mathbb{R})$, *then the determinant of* A *is given by* $\det(A) = (\mathrm{pf}(A))^2$.

The proof of the second part of the next lemma might follow easily from a known formula for minors of skew symmetric matrices, but for the sake of completeness we present the proof.

Lemma 3.7. *Let* $A \in \mathrm{Skew}(2n, \mathbb{R}) \cap \mathrm{GL}(2n, \mathbb{R})$, $n > 1$. *Then there exists a permutation matrix* P *such that* $\det((P^t)AP[2n-2, 2n-2]) \neq 0$. *Furthermore,* P *can be an element of* $\mathrm{Per}(2n, \mathbb{R})_{e_1}$.

Proof. We have $\det(A) = (\mathrm{pf}(A))^2 \neq 0$. From the definition of $\mathrm{pf}(A)$, there exist i, j such that $\mathrm{pf}(A_{\hat{i}\hat{j}}) \neq 0$. Since $A_{\hat{i}\hat{j}} \in \mathrm{Skew}(2n-2, \mathbb{R})$, we have $\det(A_{\hat{i}\hat{j}}) = (\mathrm{pf}(A_{\hat{i}\hat{j}}))^2 \neq 0$. Now it is easy to see that there is a permutation P such that

$$(P^t)AP = \left(\begin{array}{c|c} A_{\hat{i}\hat{j}} & * \\ \hline * & * \end{array}\right).$$

For the second part, it is enough to show that there exist $i, j \neq 1$ such that $\det(A_{\hat{i}\hat{j}}) \neq 0$. If this is the case, then it is easy to see that the permutation

P defined previously can be of the desired form. Suppose that $\det(A_{\hat{i}\hat{j}}) = (\mathrm{pf}(A_{\hat{i}\hat{j}}))^2 = 0$ for all $i, j \neq 1$, then for all $i \neq 1$,

$$\mathrm{pf}(A) = \sum_{j \neq i} (-1)^{i+1+j+\theta(i-j)} a_{ij} \,\mathrm{pf}\, A_{\hat{i}\hat{j}} = (-1)^{i+1+1+\theta(i-1)} a_{i1} \,\mathrm{pf}\, A_{\hat{i}1}$$

$$= -(-1)^i a_{i1} \,\mathrm{pf}\, A_{\hat{i}1} = (-1)^i a_{1i} \,\mathrm{pf}\, A_{1\hat{i}}.$$

By summing over i, we get

$$(2n-1)\,\mathrm{pf}(A) = \sum_{i=2}^{2n} (-1)^i a_{1i} \,\mathrm{pf}\, A_{1\hat{i}} = \mathrm{pf}(A),$$

then $\mathrm{pf}(A) = 0$ which is not possible. □

Theorem 3.8. *Let $M \in \mathrm{GL}(2n, \mathbb{R})$. Then there exists a permutation P such that MP has an SR decomposition. Furthermore, P can be an element of* $\mathrm{Per}(2n, \mathbb{R})_{e_1}$.

Proof. From Theorem 3.3, we just need to prove that there is a permutation matrix P such that for all $k = 1, \dots, n$,

$$\det(\hat{P}P^t M^t JMP\hat{P}^t)[2k, 2k] \neq 0. \tag{6}$$

In fact, because $\hat{P}$ is itself a permutation matrix, it will be enough to show that there is a permutation matrix P such that for all $k = 1, \dots, n$,

$$\det(P^t M^t JMP)[2k, 2k] \neq 0. \tag{7}$$

Note that if such matrix P exists, then the permutation matrix $P\hat{P}$ is sufficient for satisfying (6).

To prove (7), we just need to apply Lemma 3.7 repeatedly. Note that $M^t JM \in \mathrm{Skew}(2n, \mathbb{R})$. If $n = 1$ there is nothing to prove, because $M^t JM$ is nonsingular. If $n \neq 1$, from Lemma 3.7, there exists a permutation P_1 such that

$$\det(P_1^t M^t JMP_1)[2n-2, 2n-2] \neq 0.$$

We again have

$$B := P_1^t M^t JMP_1[2n-2, 2n-2] \in \mathrm{Skew}(2n-2, \mathbb{R}) \cap \mathrm{GL}(2n-2, \mathbb{R}),$$

so by Lemma 3.7, there exists a matrix P_2' such that

$$\det(P_2'^t BP_2')[2n-4, 2n-4] \neq 0.$$

If we define

$$P_2 := \left(\begin{array}{c|c} P_2' & 0 \\ \hline 0 & I_2 \end{array}\right),$$

then we have that for $k = n, n-1, n-2$

$$\det(P_2^t P_1^t M^t J M P_1 P_2)[2k, 2k] \neq 0.$$

It is easy to see that we can continue this procedure until we have a matrix P such that (7) is satisfied.

For the second part just notice that also from Lemma 3.7 all the permutation matrices used can be selected to have the desired form. $\square$

Hence it is clear that the same result extends to Lemma 3.4. In the latter sections, we use the next statement.

Corollary 3.9. *Let $M \in \mathrm{GL}(2n, \mathbb{R})$. Then there exist a permutation matrix $P \in \mathrm{Per}(2n, \mathbb{R})_{e_1}$ and $S \in Sp_n(\mathbb{R})$ such that*

$$PMS = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

is strictly lower J-triangular (with A and D nonsingular lower triangular, and B and C strictly lower triangular).

4. Almost abelian Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ of dimension n (not necessarily even).

Definition 4.1. *A non-abelian Lie algebra $\mathfrak{g}$ is called almost abelian if it contains a codimension 1 abelian subalgebra.*

Remark 4.2. *In our definition we exclude abelian Lie algebras from almost abelian Lie algebras. Some authors prefer to include abelian Lie algebras in the class of almost abelian Lie algebras. We follow the convention in* [1].

Proposition 4.3 ([1]). *An almost abelian Lie algebra $\mathfrak{g}$ has a codimension 1 abelian ideal $\mathfrak{I}$, and is therefore isomorphic to the semidirect product*

$$\mathfrak{g} = \mathbb{R}e \ltimes \mathfrak{I}$$

for some $e \in \mathfrak{g} \backslash \mathfrak{I}$.

The Lie algebra structure of an almost abelian Lie algebra $\mathfrak{g} = \mathbb{R}e \ltimes \mathfrak{I}$, as in the previous proposition, is completely determined by $\mathrm{ad}_e|_{\mathfrak{I}} \in \mathrm{End}(\mathfrak{I})$:

$$[e, v] = \mathrm{ad}_e|_{\mathfrak{I}} v, \qquad v \in \mathfrak{I}.$$

Any pair (V, T), where V is a vector space and T a nonzero linear map, determines an almost abelian Lie algebra $\mathfrak{g}$ and vice versa, but different maps can yield isomorphic Lie algebras.

DEFINITION 4.4. *Two pairs (V_1, T_1) and (V_2, T_2) are said to be similar if there exist $\lambda \in \mathbb{R}^\times$ and an invertible linear map ϕ such that $T_2 = \lambda(\phi T_1 \phi^{-1})$. In this case we write $(V_1, T_1) \sim (V_2, T_2)$.*

Isomorphism classes of almost abelian Lie algebras correspond to the similarity classes of linear operators on vector spaces.

THEOREM 4.5 ([1]). *Two almost abelian Lie algebras $\mathfrak{g} = \mathbb{R}e \ltimes \mathfrak{I}$ and $\mathfrak{g}' = \mathbb{R}e' \ltimes \mathfrak{I}'$ are isomorphic if and only if $(\mathfrak{I}, \mathrm{ad}_e|_{\mathfrak{I}}) \sim (\mathfrak{I}', \mathrm{ad}_{e'}|_{\mathfrak{I}'})$.*

For the proof of our main result in the next section, we need a description of the automorphism group of an almost abelian Lie algebra $\mathfrak{g}$. A complete description can be found in [1]. In fact, we will not need the complete description, and only use the following weaker result. For a given almost abelian Lie algebra $\mathfrak{g} = \mathbb{R}y_1 \ltimes \mathfrak{I}$, we fix a basis $\{y_1, \dots, y_n\}$ of $\mathfrak{g}$ such that $\mathfrak{I} = \mathrm{Span}\{y_2, \dots, y_n\}$.

THEOREM 4.6. *For the automorphism group of an almost abelian Lie algebra $\mathfrak{g} = \mathbb{R}y_1 \ltimes \mathfrak{I}$, we have*

$$\mathrm{Aut}(\mathfrak{g}) \supset \left\{ \begin{pmatrix} \alpha & 0 \\ \beta & A \end{pmatrix} \middle|\, \alpha \in \mathbb{R}^\times,\, \beta \in \mathbb{R}^{n-1},\, A(\mathrm{ad}_{y_1}|_{\mathfrak{I}}) - \alpha(\mathrm{ad}_{y_1}|_{\mathfrak{I}})A = 0 \right\}.$$

PROOF. By a direct calculation. Compare with [1], Proposition 8. □

In this paper we focus on a special family of almost abelian Lie algebras: those that are diagonalizable.

DEFINITION 4.7. *Let $\mathfrak{g} = \mathbb{R}y_1 \ltimes \mathfrak{I}$ be an almost abelian Lie algebra of dimension n such that*

$$\mathrm{ad}_{y_1} = \begin{pmatrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

We will call $\mathfrak{g}$ a diagonal almost abelian Lie algebra and will usually just say $\mathfrak{g}$ is an almost abelian Lie algebra determined by $\mathrm{diag}(\lambda_2, \dots, \lambda_{2n})$.

In particular, for the automorphism group of a diagonal almost abelian Lie algebra, we have the following elements.

Proposition 4.8. *For a diagonal almost abelian Lie algebra* $\mathfrak{g} = \mathbb{R}y_1 \ltimes \mathfrak{I}$, *any nonsingular matrix of the following form is an element of* $\mathbb{R}^\times \operatorname{Aut}(\mathfrak{g})$*:*

$$\left(\begin{array}{c|ccc} * & & & \\ \hline * & * & & \\ \vdots & & \ddots & \\ * & & & * \end{array}\right).$$

Proof. From Theorem 4.6, we have that

$$\left(\begin{array}{c|ccc} 1 & & & \\ \hline * & * & & \\ \vdots & & \ddots & \\ * & & & * \end{array}\right) \in \operatorname{Aut}(\mathfrak{g}).$$

Then the proposition follows immediately. □

Proposition 4.9. *Let* $\mathfrak{g} = \mathbb{R}y_1 \ltimes \mathfrak{I}$ *be the diagonal almost abelian Lie algebra determined by* $\operatorname{diag}(\lambda_2, \dots, \lambda_n)$. *If* $\lambda_k = \lambda_l$, *then a map h satisfying the following is an automorphism:*

$$h(\operatorname{Span}\{y_k, y_l\}) = \operatorname{Span}\{y_k, y_l\}, \qquad h(y_i) = (y_i) \quad (i \neq k, l).$$

Proof. It follows directly from Theorem 4.6. □

5. Proof of the main theorem

In this section, we prove Theorem 1.1, that is, for diagonal almost abelian Lie algebras, we give a condition for the existence of symplectic structures and show the uniqueness.

First we prove some results that are true for particular almost abelian Lie algebras, not necessarily diagonal. Let $\mathfrak{g} = \operatorname{Span}\{x_1, \dots, x_{2n}\} = \mathbb{R}x_1 \ltimes \mathfrak{I}$ with $\mathfrak{I} = \operatorname{Span}\{x_2, \dots, x_{2n}\}$ be an almost abelian Lie algebra of dimension $2n$. The structure constants of $\mathfrak{g}$ can be described by the matrix

$$C_{\mathfrak{g}} = \left(\begin{array}{c|c} 0 & \\ \hline & \operatorname{ad}_{x_1}|_{\mathfrak{I}} \end{array}\right) = (c_{ij}),$$

so that

$$[x_1, x_i] = \sum_j c_{ji} x_j.$$

First of all we show that if $\{x_i\}$ is a symplectic basis and the corresponding matrix $C_{\mathfrak{g}}$ is J-triangular, then the closed condition (Proposition 2.4) has a very simple form. Recall that $\Omega(\mathfrak{g})$ is the set of nondegenerate 2-forms on $\mathfrak{g}$.

LEMMA 5.1. *Let $\mathfrak{g}$ be an almost abelian Lie algebra and $\omega \in \Omega(\mathfrak{g})$. Let $\{x_1, \dots, x_{2n}\}$ be a symplectic basis of $\mathfrak{g}$ such that $\mathfrak{g} = \mathbb{R}x_1 \ltimes \mathrm{Span}\{x_2, \dots, x_{2n}\}$ and the matrix $C_{\mathfrak{g}} = (c_{ij})$ of structure constant is strictly lower J-triangular. Then $d\omega = 0$ if and only if $C_{\mathfrak{g}}$ is diagonal and*

$$c_{ii} = -c_{i+n\, i+n} \qquad (i \in \{2, \dots, n\}).$$

Notice that $c_{11} = 0$ and there is no condition on the element $c_{n+1\, n+1}$.

PROOF. It is easy to see that $d\omega = 0$ if and only if $d\omega(x_1, x_i, x_j) = 0$ for $1 < i < j$. By a direct calculation, we get

$$\begin{aligned} d\omega(x_1, x_i, x_j) &= \omega(x_j, [x_1, x_i]) + \omega(x_i, [x_j, x_1]) \\ &= \sum_k c_{ki}\omega(x_j, x_k) - \sum_l c_{lj}\omega(x_i, x_l). \end{aligned} \tag{8}$$

We can now consider different cases. Let us consider the square block decomposition

$$C_{\mathfrak{g}} = \left(\begin{array}{c|c} T_1 & T_2 \\ \hline T_3 & T_4 \end{array}\right).$$

Case 1: $1 < i < j \le n$. In this case, by Equation (8), we have

$$d\omega(x_1, x_i, x_j) = c_{j+n\, i}\omega(x_j, x_{j+n}) - c_{i+n\, j}\omega(x_i, x_{i+n}) = c_{j+n\, i} - c_{i+n\, j} = 0. \tag{9}$$

This is a condition on block T_3. Recall that this block can be written as

$$T_3 = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & T_3' \end{array}\right),$$

where $T_3' \in \mathrm{M}(n-1, \mathbb{R})$ is strictly lower triangular. Then Equation (9) is equivalent to $T_3' = (T_3')^t$, but T_3' is strictly lower triangular so $T_3' = 0$. Then we get $T_3 = 0$.

Case 2: $n < i < j$. In this case, by Equation (8), we have

$$d\omega(x_1, x_i, x_j) = c_{j-n\, i}\omega(x_j, x_{j-n}) - c_{i-n\, j}\omega(x_i, x_{i-n}) = -c_{j-n\, i} + c_{i-n\, j} = 0. \tag{10}$$

This is a condition on block T_2, and is equivalent to $T_2 = T_2^t$. Since T_2 is also strictly lower triangular, we get $T_2 = 0$.

Case 3: $1 < i \le n < j$. In this case, by Equation (8), we have

$$d\omega(x_1, x_i, x_j) = c_{j-n\, i}\omega(x_j, x_{j-n}) - c_{i+n\, j}\omega(x_i, x_{i+n}) = -c_{j-n\, i} - c_{i+n\, j} = 0. \tag{11}$$

This condition relates blocks T_1 and T_4. These blocks can be written as

$$T_1 = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & T_1' \end{array}\right), \qquad T_4 = \left(\begin{array}{c|c} * & 0 \\ \hline \alpha & T_4' \end{array}\right),$$

where $T_1', T_4' \in \mathrm{M}(n-1, \mathbb{R})$ are both lower triangular. When $j = n+1$, Equation (11) implies $\alpha = 0$. When $j \neq n+1$, Equation (11) implies $T_1' = -(T_4')^t$. Since T_1' and T_4' are lower triangular, only the diagonal elements are nonzero and we get the condition of this lemma. $\square$

From here onward, we will restrict our attention to diagonal almost abelian Lie algebras. Let $\mathfrak{g}_A \cong \mathbb{R}^{2n} = \mathrm{Span}\{e_1, \dots, e_{2n}\} = \mathbb{R}e_1 \ltimes \mathrm{Span}\{e_2, \dots, e_{2n}\}$ denote the diagonal almost abelian Lie algebra determined by $A = \mathrm{diag}(\lambda_2, \dots, \lambda_{2n})$, so the matrix of structure constants with respect to the canonical basis is given by

$$C_{\mathfrak{g}_A} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & A \end{array}\right) = (a_{ij}). \tag{12}$$

Lemma 5.2. *Let $\mathfrak{g}_A$ be a diagonal almost abelian Lie algebra as above. A set of representatives U for the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}_A)$ on $\Omega(\mathfrak{g}_A)$ is given by*

$$U = \left\{ PL \;\middle|\; \begin{array}{l} L = (l_{ij}) \text{ is strictly lower } J\text{-triangular} \\ l_{ii} = 1 \text{ for all } i \text{ and } l_{i1} = 0 \text{ for } i > 1 \\ P \in \mathrm{Per}(2n, \mathbb{R})_{e_1} \end{array} \right\}.$$

Proof. Take $g \in \mathrm{GL}(2n, \mathbb{R})$. By Corollary 3.9, there exists $S \in \mathrm{Sp}_n(\mathbb{R})$, a permutation matrix $P \in \mathrm{Per}(2n, \mathbb{R})_{e_1}$ and a strictly lower J-triangular matrix L with all the diagonal elements nonzero such that

$$PgS = L.$$

Therefore,

$$[[g]] \ni gS = P^{-1}L.$$

We can take a matrix

$$h_1 = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix},$$

such that $h_1 L =: L' = (l_{ij}')$ with $l_{ii}' = 1$ for all i. From Proposition 4.8, we have $P^{-1}h_1 P \in \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}_A)$, since it is still diagonal. Hence,

$$[[g]] \ni (P^{-1}h_1 P)(P^{-1}L) = P^{-1}L'.$$

We can take a matrix

$$h_2 = \left(\begin{array}{c|ccc} 1 & & & \\ \hline * & 1 & & \\ \vdots & & \ddots & \\ * & & & 1 \end{array}\right),$$

such that $h_2 L' =: L'' = (l''_{ij})$ with $l''_{ii} = 1$ for all i and $l''_{j1} = 0$ for $j > 1$. Since $P^{-1} \in \mathrm{Per}(2n, \mathbb{R})_{e_1}$, $P^{-1} h_2 P$ remains of the same shape as h_2. Therefore, from Proposition 4.8, $P^{-1} h_2 P \in \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}_A)$. Hence,

$$[[g]] \ni (P^{-1} h_2 P)(P^{-1} L') = P^{-1} L''.$$

Note that L'' remains strictly lower triangular. This finishes the proof. □

Now we can state a Milnor-type theorem using the set of representatives obtained in the previous lemma.

THEOREM 5.3 (Milnor-type). *Let $\mathfrak{g}_A$ be a diagonal almost abelian Lie algebra as before. For all $\omega \in \Omega(\mathfrak{g}_A)$, there exist $t > 0$, $u \in U$ (as in Lemma 5.2) and a symplectic basis $\{x_1, \ldots, x_{2n}\} \subset \mathfrak{g}_A$ with respect to $t\omega$ such that the only nonzero brackets are given by*

$$[x_1, x_k] = \sum_l b_{lk} x_l \qquad \textit{for } k = 2, \ldots, 2n,$$

where $B = u^{-1} C_{\mathfrak{g}_A} u = (b_{ij})$.

PROOF. Let $\{e_1, \ldots, e_{2n}\}$ be the canonical basis of $\mathfrak{g}_A$, whose bracket relation is given by $[e_1, e_i] = \sum_l a_{li} e_l$ for $i > 1$. In Lemma 5.2, we obtained a set of representatives U for the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}_A)$ on $\Omega(\mathfrak{g}_A)$. Take any $\omega \in \Omega(\mathfrak{g}_A)$. Then it follows from Theorem 2.9 that there exist $u = (u_{ij}) \in U, t > 0$ and $\phi \in \mathrm{Aut}(\mathfrak{g}_A)$ such that $\{x_1 := \phi u e_1, \ldots, x_n := \phi u e_n\}$ is symplectic with respect to $t\omega$. Hence we only have to check the bracket relations among them.

For any $u \in U$, we have $ue_1 = e_1$ and $ue_i \in \mathrm{Span}\{e_2, \ldots, e_{2n}\}$ for $i \neq 1$. Among $\{ue_1, \ldots, ue_{2n}\}$, the only possible nonzero bracket relations are

$$[ue_1, ue_i] = [e_1, ue_i], \qquad i \neq 1.$$

By the usual argument of change-of-basis matrix, we get

$$[e_1, ue_i] = \sum_{j,k,t} u_{ji} (u^{-1})_{tk} a_{kj} u e_t.$$

Finally, apply ϕ to both sides

$$[x_1, x_i] = \sum_{j,k,t} u_{ji}(u^{-1})_{tk} a_{kj} \phi u e_t = \sum_{j,k,t} u_{ji}(u^{-1})_{tk} a_{kj} x_t.$$

This finishes the proof. □

We introduce a notation. For each permutation $P \in \mathrm{Per}(2n, \mathbb{R})$, we denote by σ_P the permutation on $\{1, \dots, 2n\}$ corresponding to P. That is, $Pe_i = e_{\sigma_P(i)}$ for $i = 1, \dots, 2n$.

Lemma 5.4. *Let $\mathfrak{g}_A$ be a diagonal almost abelian Lie algebra as before. For $PL \in U$ (as in Lemma 5.2), consider $PL.\omega_0 \in \Omega(\mathfrak{g}_A)$. Then we have $dPL.\omega_0 = 0$ if and only if*

$$\lambda_{\sigma_P(i)} + \lambda_{\sigma_P(i+n)} = 0 \qquad (i = 2, \dots, n), \tag{13}$$

$$l_{ij}(\lambda_{\sigma_P(i)} - \lambda_{\sigma_P(j)}) = 0 \qquad (i \neq j). \tag{14}$$

Proof. By Theorem 5.3, there exist $t > 0$ and a symplectic basis with respect to $tPL.\omega_0$ such that the structure constants are given by the matrix $B = L^{-1}P^{-1}C_{\mathfrak{g}_A}PL = (b_{ij})$. Notice that $d(tPL).\omega_0 = 0$ if and only if $d(PL).\omega_0 = 0$. Remember that

$$C_{\mathfrak{g}_A} = \mathrm{diag}(0, \lambda_2, \dots, \lambda_{2n}),$$

so that

$$P^{-1}C_{\mathfrak{g}_A}P = \mathrm{diag}(0, \lambda_{\sigma_P(2)}, \dots, \lambda_{\sigma_P(2n)}) =: D.$$

Note that both L and L^{-1} are strictly lower J-triangular with diagonal elements equal to 1. Then B can be calculated as follows

$$B = L^{-1}DL = \left(\begin{array}{c|ccc|ccc} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline 0 & \lambda_{\sigma_P(2)} & & 0 & 0 & & 0 \\ \vdots & & \ddots & & & \ddots & \\ 0 & * & & \lambda_{\sigma_P(n)} & * & & 0 \\ \hline 0 & 0 & & 0 & \lambda_{\sigma_P(n+1)} & & 0 \\ \vdots & & \ddots & & & \ddots & \\ 0 & * & & 0 & * & & \lambda_{\sigma_P(2n)} \end{array}\right),$$

so that B is again a strictly lower J-triangular matrix. Therefore we can use Lemma 5.1 to obtain $dPL.\omega_0 = 0$ if and only if B is diagonal and

$$b_{ii} = -b_{i+n\,i+n} \qquad (i \in \{2, \dots, n\}).$$

For the elements of the diagonal, we obtain immediately Equation (13). Non-diagonal elements are all equal to 0, this is equivalent to the condition

$$B = D.$$

From the definition of B, this is equivalent to the equation

$$DL = LD.$$

Finally, if we write this equation in terms of the parameters of the matrices L and D we get Equation (14). $\square$

Equation (13) proves the first part of the main result in Theorem 1.1 as follows.

PROPOSITION 5.5. *Let $\mathfrak{g}$ be a diagonal almost abelian Lie algebra determined* $\mathrm{diag}(\lambda_2, \dots, \lambda_{2n})$. *There exists a symplectic form $\omega \in \Omega(\mathfrak{g})$ if and only if there exists a permutation σ of $\{2, \dots, 2n\}$ such that*

$$\lambda_{\sigma(i)} + \lambda_{\sigma(i+n)} = 0 \qquad \textit{for } i = 2, \dots, n.$$

PROOF. First suppose there exists a symplectic form $\omega \in \Omega(\mathfrak{g})$. Then we have $[\omega] = [PL.\omega_0]$, where PL is as in Lemma 5.2. Then from (13) in Lemma 5.4, we get the desired condition.

Now suppose that there exists σ such that $\lambda_{\sigma(i)} + \lambda_{\sigma(i+n)} = 0$ for $i = 2, \dots, n$. We can rearrange the basis to the following order

$$\{e_1, e_{\sigma(2)}, \dots, e_{\sigma(2n)}\}.$$

Then from Lemma 5.4 with $L = I_{2n}$, we can check that the 2-form ω associated with this basis is symplectic. $\square$

REMARK 5.6. *In fact, the first part of our main result can also be obtained relatively easily by direct calculations. Our method will be particularly useful for the second part of our Main result. Our method will also provide ideas to study other examples.*

From now on we prove the second part of our main theorem. Let $\mathfrak{g} \cong \mathbb{R}^{2n} = \mathrm{Span}\{e_1, \dots, e_{2n}\} = \mathbb{R}e_1 \ltimes \mathrm{Span}\{e_2, \dots, e_{2n}\}$ denote the diagonal almost abelian Lie algebra determined by $\mathrm{diag}(\lambda_2, \dots, \lambda_{2n})$. Assume that there exist a symplectic form $\omega \in \Omega(\mathfrak{g})$. Then (13) must be satisfied and similarly as in the proof of Proposition 5.5 we can assume without loss of generality that $\mathfrak{g} = \mathfrak{g}_A$ is a diagonal almost abelian Lie algebra determined by

$$A = \mathrm{diag}(\lambda_2, \dots, \lambda_n, \lambda_{n+1}, -\lambda_2, \dots, -\lambda_n) \qquad \text{with } \lambda_2 \geq \cdots \geq \lambda_n \geq 0. \tag{15}$$

As before ω_0 denotes the canonical 2-form.

Lemma 5.7. *Let* $\mathfrak{g} = \mathfrak{g}_A$ *be the diagonal almost abelian Lie algebra determined by* A *as in* (15). *Let* $PL \in U$ *(as in Lemma 5.2). If* $dPL.\omega_0 = 0$*, then* $[PL.\omega_0] = [P.\omega_0]$.

Proof. Write the matrix $L = (l_{ij})$ as

$$L = (L_1, \dots, L_{2n}),$$

that is, $L_i = (l_{1i}, \dots, l_{i-1i}, 1, l_{i+1i}, \dots, l_{2ni})^t$. Define the matrix

$$Q_i = (e_1, \dots, L_i', \dots, e_{2n}),$$

where $L_i' := (-l_{1i}, \dots, -l_{i-1i}, 1, -l_{i+1i}, \dots, -l_{2ni})^t$. Now suppose that j is the largest integer such that $L = (e_1, \dots, e_{j-1}, L_j, \dots, L_{2n})$ and $L_j \neq e_j$. By the definition of L, we know $j \geq 2$. Notice that

$$Q_j L = (e_1, \dots, e_j, *, \dots, *).$$

This operation eliminates the parameters in the j-th column of L. It leaves the previous columns without change. It changes the values of the latter columns, but L remains a J-triangular matrix with diagonal elements equal to 1. We can repeat this procedure until L is transformed into the identity matrix. Therefore, we just have to show that if $dPL.\omega_0 = 0$, then

$$[PL.\omega_0] = [PQ_j L.\omega_0].$$

Since $PQ_j L = (PQ_j P^{-1})PL$, it is enough to prove that

$$PQ_j P^{-1} \in \mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}_A). \tag{16}$$

Q_j can be written as

$$Q_j = I - \sum_{i \neq j} l_{ij} E_{ij} = \prod_{i \neq j} (I - l_{ij} E_{ij}).$$

Then we have

$$\begin{aligned} PQ_j P^{-1} = P\left(\prod_{i \neq j} (I - l_{ij} E_{ij})\right) P^{-1} &= \prod_{i \neq j} (I - l_{ij} P E_{ij} P^{-1}) \\ &= \prod_{i \neq j} (I - l_{ij} E_{\sigma_P(i)\sigma_P(j)}). \end{aligned} \tag{17}$$

If $dPL.\omega_0 = 0$, P and L must satisfy (13) and (14). From (14), we get $l_{ij} = 0$ or $\lambda_{\sigma_P(i)} = \lambda_{\sigma_P(j)}$. In both cases, it is easy to see from Proposition 4.9 that

$$I - l_{ij} E_{\sigma_P(i)\sigma_P(j)} \in \mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}_A).$$

This and (17) imply (16). This finishes the proof. □

LEMMA 5.8. *Let $\mathfrak{g} = \mathfrak{g}_A$ be the diagonal almost abelian Lie algebra determined by A as in* (15). *Let $P \in \mathrm{Per}(2n, \mathbb{R})_{e_1}$. If $dP.\omega_0 = 0$, then $[P.\omega_0] = [\omega_0]$.*

PROOF. Remember that a permutation σ is called a transposition when it is a cycle with only two elements. In the usual cycle notation, if σ permutes the i-th and j-th element, then the permutation is denoted by $\sigma = (ij)$. Now let $P \in \mathrm{Per}(2n, \mathbb{R})_{e_1}$ and assume that $dP.\omega_0 = 0$.

Claim 1: there exists P' such that

$$[P.\omega_0] = [P'.\omega_0], \qquad |\lambda_{\sigma_{P'}(2)}| \geq \cdots \geq |\lambda_{\sigma_{P'}(n)}|.$$

We can choose a matrix

$$P_3 = \left(\begin{array}{c|c} P_3' & \\ \hline & P_3' \end{array}\right), \qquad P_3' \in \mathrm{Per}(n, \mathbb{R})_{e_1}$$

such that $|\lambda_{\sigma_{PP_3'}(2)}| \geq \cdots \geq |\lambda_{\sigma_{PP_3'}(n)}|$. Set $P' := PP_3$. Since $P_3 \in \mathrm{Sp}_n(\mathbb{R})$, we have

$$[P.\omega_0] = [PP_3\omega_0] = [P'.\omega_0].$$

Claim 2: there exists P'' such that

$$[P'.\omega_0] = [P''.\omega_0], \qquad \lambda_{\sigma_{P''}(2)} \geq \cdots \geq \lambda_{\sigma_{P''}(n)} \geq 0.$$

Suppose $\lambda_{\sigma_{P'}(i)} < 0$ for some $0 < i \leq n$. Consider the permutation P_2 such that $\sigma_{P_2} = (i\; i+n)$. The effect of this permutation is

$$\lambda_{\sigma_{P'P_2}(k)} = \begin{cases} \lambda_{\sigma_{P'(k)}} & (k \neq i, i+n), \\ -\lambda_{\sigma_{P'(k)}} & (k = i, i+n). \end{cases} \tag{18}$$

We can repeat this operation to obtain the desired order. Then we have only to show that

$$[P'.\omega_0] = [P'P_2.\omega_0].$$

First notice that

$$(I - 2E_{ii})P_2 \in \mathrm{Sp}_n(\mathbb{R}).$$

Also from Proposition 4.8, we have

$$P'(I - 2E_{ii})(P')^{-1} = (I - 2E_{\sigma_{P'}(i)\sigma_{P'}(i)}) \in \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}_A).$$

Therefore we have

$$\begin{aligned} [P'.\omega_0] &= [P'((I - 2E_{ii})P_2).\omega_0] = [(P'(I - 2E_{ii})(P')^{-1})P'P_2.\omega_0] \\ &= [(I - 2E_{\sigma_{P'}(i)\sigma_{P'}i})P'P_2.\omega_0] = [P'P_2.\omega_0]. \end{aligned}$$

Claim 3: $P'' \in \mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}_A)$. Therefore

$$[P''.\omega_0] = [\omega_0].$$

For each pair i, j $(i \neq j)$ such that $\lambda_i = \lambda_j$, consider a permutation P_3 such that $\sigma_{P_3} = (ij)$. From Proposition 4.9, we have that

$$P_3 \in \mathbb{R}^\times \operatorname{Aut}(\mathfrak{g}_A). \tag{19}$$

Since P'' is just a composition of this type of permutations, Claim 3 follows. This completes the proof of this lemma. □

This lemma proves the second part of the main result Theorem 1.1 as follows.

Proposition 5.9. *Let $\mathfrak{g}$ be a diagonal almost abelian Lie algebra. If there exists a symplectic form on $\mathfrak{g}$, then it is unique up to symplectomorphism and scale.*

Proof. We can assume without loss of generality that $\mathfrak{g} = \mathfrak{g}_A$, where A is as in (15). Let ω_0 denote the canonical 2-form. Note that in this case, ω_0 is closed by Lemma 5.4. Take a symplectic form ω. Then we have $[\omega] = [PL.\omega_0]$ where PL is as in Lemma 5.2. Finally by Lemmas 5.7 and 5.8, we have

$$[\omega] = [PL.\omega_0] = [P.\omega_0] = [\omega_0],$$

which finishes the proof. □

Finally, we can give a proof of Corollary 1.2.

Proof (of Corollary 1.2). From our main theorem, we can just assume that $\{e_1, \ldots, e_{2n}\}$ is a symplectic basis and $\omega = \omega_0$ is the canonical form. Define $\mathfrak{l} = \operatorname{Span}\{e_1, \ldots, e_n\}$. Then it is easy to show that $\mathfrak{l}$ is a Lagrangian ideal. □

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