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On generation and enumeration of orthogonal Chebyshev-Frolov lattices

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ABSTRACT. We discuss orthogonal Chebyshev-Frolov lattices, their generating matrices and their use in Frolov cubature formula. We give a detailed account on coordinate-permuted systems that lead to fast computation and enumeration of such lattices. In particular, we explain the recurrences identified in (K. Suzuki and T. Yoshiki, Hiroshima Math. J., 49(1):139–159, 2019) via a plain constructive approach exhibiting a new hierarchical basis of polynomials. Dual Chebyshev-Frolov lattices and their generating matrices are also studied. Lattices enumeration in axis-parallel boxes is discussed.

1. Introduction

Numerical integration in multi-dimension is a highly active research topic. In a variety of scientific and engineering contexts, the objective is to approximate integrals on general domains $\Omega \subset \mathbb{R}^d$ that cannot be handled analytically. For many applications, the geometries of Ω and the occurrence of singularities are the major sources of difficulty. That said, the approximation of integrals involving smooth integrands over regular domains is still a challenging task, especially in high dimensions. For instance, integrals of the form

$$\mathcal{Q}_d[f] := \int_{[0,1]^d} f(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \tag{1}$$

with the proper assumptions on f, arising in innumerable areas such as physics, data mining, finance, parametrized PDE, uncertainty quantification, etc. Sparse Grids, Monte-Carlo (MC) and Quasi Monte-Carlo (QMC) methods, see e.g. [1, 2, 3, 4] and references their-in, account for a major part of numerical integration procedures tailored to high dimensions.

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Ideally, one aims in computing an efficient and stable numerical approximation $\mathcal{Q}_{d,N}[f]$ to the above integral, which have reliable error guarantees for classical smoothness manifolds. For instance, through the worst-case error

$$e(N, \mathcal{K}) := \sup_{\|f\|_{\mathcal{K}} \le 1} |\mathcal{Q}_d[f] - \mathcal{Q}_{d,N}[f]|, \tag{2}$$

where $\mathcal{K}$ is a given smoothness manifold of d-variate functions. The integer N reflects a numerical budget, in general dominated by the number of queries of function f.

We are interested in a specific family of lattice rules called Frolov cubatures. In a nutshell, given M a fixed non singular $d \times d$ matrix and $N \in \mathbb{N}$ a scaling facor, we consider $M_N = (N \det(M))^{-1/d} M$ (it satisfies $\det(M_N) = 1/N$) and the associated d-dimensional lattice $M_N\mathbb{Z}^d = \{M_N\boldsymbol{k} : \boldsymbol{k} \in \mathbb{Z}^d\}$. Matrix M_N is called a *generating* matrix for the lattice. Any other generating matrix is necessarily equal to $M_N S$ with S uni-modular, i.e. in $\mathbf{SL}_d(\mathbb{Z}) := \{S \in \mathbb{Z}^{d\times d} : \det(S) = \pm 1\}$. We consider the cubature

$$\mathcal{Q}_{d,N}[f] := \frac{1}{N} \sum_{\boldsymbol{x} \in M_N\mathbb{Z}^d} f(\boldsymbol{x}). \tag{3}$$

The function f is assumed to be supported on a bounded domain Ω, thus only finitely many summands contribute to the sum, i.e. quadrature nodes picked on the grid $M_N\mathbb{Z}^d \cap \Omega$. The quadrature weights are all equal to $1/N$, yet the quadrature is not a Quasi-Monte Carlo method since in general $\mathcal{Q}_{d,N}[\mathbf{1}_\Omega] \neq 1$. We note however that $\mathcal{Q}_{d,N}[\mathbf{1}_\Omega] \to \mathrm{vol}(\Omega)$ as $N \to \infty$.

The description of Frolov quadrature is fairly straightforward. Furthermore, the convergence analysis in the sense of (2) is rather standard, especially through techniques of harmonic analysis, see e.g. [5, 6, 7, 8, 9, 10, 11, 12]. Frolov [5] established that under the mere *admissibility* condition

$$\mathrm{Nm}(M) := \inf_{\substack{\boldsymbol{x} \in M\mathbb{Z}^d \\ \boldsymbol{x} \neq \boldsymbol{0}}} |\Pi_{i=1}^d x_i| > 0, \tag{4}$$

there holds an optimal asymptotic worst-case behavior of (2) with respect to functions with L_p-bounded mixed derivative of order $r \in \mathbb{N}$ supported in $[0,1]^d$. We refer to the tutorial paper [10] for a detailed proof. We also refer to [8] for a survey on this optimality result in many classical function spaces on the cube. The inspection of all the aforementioned references confirms that the performance of the quadrature is strongly tied and can be quantified through quantities that are intrinsic to the lattice $M_N\mathbb{Z}^d$ (or equally $M\mathbb{Z}^d$). In particular, through invariants (invariance with respect to generating matrices) pertaining to its geometry. For instance $\mathrm{Nm}(M)$, $|\det(M)|$, $\inf\{\|MS\|_\infty : S \in \mathbf{SL}_d(\mathbb{Z})\}$, and other invariants.

The notion of lattice admissibility is central to Frolov cubatures. Another notion of utmost importance is that of *duality*. The dual of a lattice $\Gamma \subset \mathbb{R}^d$ denoted $\Gamma^\perp$ is defined by $\Gamma^\perp = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x}^\top \boldsymbol{y} \in \mathbb{Z} \ \forall \boldsymbol{y} \in \Gamma\}$. A matrix V is a generating matrix for Γ if and only if $V^\perp := (V^{-1})^\top$ is a generating matrix for $\Gamma^\perp$. The importance of duality resides mainly on two facts. On the one hand, it supplies a new optimal cubature, as admissibility of Γ implies that of $\Gamma^\perp$ (and vice versa), see e.g. [12]. On the other hand, the standard approach to studying the stability and accuracy of the quadrature associated with Γ through harmonic analysis requires a well understanding of geometric properties of the dual.

There exists a generic procedure to building admissible lattices in any dimension d. If P^* is a polynomial of degree d such that (i) P^* has leading coefficient 1, (ii) P^* has integer coefficients, (iii) P^* is irreducible over $\mathbb{Q}[X]$, and (iv) P^* has d distinct real roots $\xi_1, \dots, \xi_d$, then the Vandermonde matrix $V = (\xi_i^{j-1})_{1 \le i,j \le d}$ generates an admissible lattice, with

$$\mathrm{Nm}(V) = 1, \qquad \mathrm{Nm}(V^\perp) = |\det(V)|^{-2}. \tag{5}$$

We recall that $|\det(V)|^2 = \prod_{k \ne l} |\xi_k - \xi_l|$. For this special lattice, the nodes are the vectors $(P(\xi_1), \dots, P(\xi_d))^\top$ for P varying in $\mathbb{Z}_{d-1}[X]$ (polynomials in $\mathbb{Z}[X]$ of degree at most $d-1$). It is worthwhile to point out that given S a $d \times d$ unimodular matrix, $VS = (L_j(\xi_i))_{1 \le i,j \le d}$ where $L_1, \dots, L_d$ is the family of polynomials in $\mathbb{Z}_{d-1}[X]$ whose transition matrix from the canonical basis $\{1, X, \dots, X^{d-1}\}$ is S. As S varies in $\mathbf{SL}_d(\mathbb{Z})$, $\{L_1, \dots, L_d\}$ can be any basis of $\mathbb{Z}_{d-1}[X]$ (linear combinations with respect to $\mathbb{Z}$). In particular, for $L_1, \dots,$ L_d any *polynomial sequence* of $\mathbb{Z}_{d-1}[X]$ (L_j has degree $j-1$ and leading coefficient 1) the matrix $(L_j(\xi_i))_{1 \le i,j \le d}$ is a generating matrix for $V\mathbb{Z}^d$. Depending on a desired and within reach structure (orthogonality, recurrences, fast computation, etc) on the generating matrix, one can plug in the appropriate basis.

The algebraic construction is extremely useful since it systematically provides the invariants $\mathrm{Nm}(\cdot)$ and $|\det(\cdot)|$ for the lattice $\Gamma = V\mathbb{Z}^d$ and its dual $\Gamma^\perp = V^\perp\mathbb{Z}^d$. The estimation of $\mathrm{Nm}(\cdot)$ for an arbitrary matrix do not seem straightforward. Frolov [5] have used polynomials $P^*(x) := -1 + \prod_{j=1}^d (x - 2j + 1)$ in his construction. Constructions based on Chebyshev polynomials were considered in [6, 7, 11, 12, 13, 14] giving rise to the so called Chebyshev-Frolov lattices, i.e. $\{(P(\xi_1), \dots, P(\xi_d))^\top : P \in \mathbb{Z}_{d-1}[X]\}$ where $\xi_1, \dots, \xi_d$ are roots of specified irreducible factors of Chebyshev polynomials, e.g. [13] for more details.

In this paper, we study orthogonal Chebyshev-Frolov lattices. In §2, we recall succinctly definitions and properties. In §3, we investigate their generating matrices. We show in particular that using the appropriate reordering of rows and columns of the so-called Chebyshev-Vandermonde matrices, one is able to identify recurrences that are favorable to fast generation and enumeration of these lattices. We identify the polynomial sequence of $\mathbb{Z}[X]$ which allows to derive the recurrences derived in [14] thus providing an explicit constructive approach. In §4, we discuss the fast enumeration procedure of [14].

2. Chebyshev-Frolov lattices

We let $(T_j)_{j\geq 0}$ be Chebyshev polynomials of the first kind, defined by $T_j(\cos(\theta)) = \cos(j\theta)$. We then let $(\tilde{T}_j)_{j\geq 0}$ be the scaled Chebyshev polynomials whose leading coefficients are 1. They are given by $\tilde{T}_j(x) = 2T_j(x/2)$ for $j \geq 1$, which are Chebyshev polynomials rescaled to $[-2, 2]$. These polynomials have integer coefficients and simple real roots. Moreover, the irreducibility is well understood. A Chebyshev polynomial T_k (hence $\tilde{T}_k$) is irreducible if and only if k is a power of 2. The algebraic construction can thus be invoked. We denote the roots of Chebyshev polynomials T_{2^n} and $\tilde{T}_{2^n}$ for $n \geq 0$ by

$$\begin{array}{ll} \Xi_n = \{\xi_{n,0}, \dots, \xi_{n,2^n-1}\}, & \xi_{n,i} = \cos(\theta_{n,i}) \\ \tilde{\Xi}_n = \{\tilde{\xi}_{n,0}, \dots, \tilde{\xi}_{n,2^n-1}\}, & \tilde{\xi}_{n,i} = 2\cos(\theta_{n,i}) \end{array}, \qquad \theta_{n,i} := \frac{2i+1}{2 \times 2^n}\pi. \tag{6}$$

The roots of the scaled polynomials $\tilde{T}_{2^n}$ lie in $[-2, 2]$. The Vandermonde matrices $((\tilde{\xi}_{n,i})^j)_{0\leq i,j\leq 2^n-1}$ generate admissible Frolov lattices in dimensions $d = 2^n$. We note the use of 0-indexing of matrices rows and columns. For the sake of notational clarity, 0-indexing is used throughout the paper.

For a fixed integer n, $d = 2^n$ and abscissas Ξ_n, $\tilde{\Xi}_n$ as before, we denote by V_n and $\tilde{V}_n$ the associated Chebyshev-Vandermonde matrices (with respect to the families T and $\tilde{T}$), i.e. $V_n := (T_j(\xi_{n,i}))_{0\leq i,j\leq d-1}$, $\tilde{V}_n := (\tilde{T}_j(\tilde{\xi}_{n,i}))_{0\leq i,j\leq d-1}$. Since the polynomials $\tilde{T}_j$ have degrees j, integer coefficients and leading coefficients 1, the transition matrix from the canonical basis $\{1, X, \dots, X^{d-1}\}$ into $\{1, \tilde{T}_1, \dots, \tilde{T}_{d-1}\}$ is lower triangular with unit diagonal hence in $\mathbf{SL}_d$. As a consequence, the matrix $\tilde{V}_n$ is a generating matrix for the identified admissible lattice in dimension 2^n.

For the sake of clarity, we drop the subscript n in the notation of $\xi_{n,i}$, $\tilde{\xi}_{n,i}$, $\theta_{n,i}$ and $\tilde{\theta}_{n,i}$. The matrices V_n and $\tilde{V}_n$ are given by

$$\begin{pmatrix} 1 & T_1(\xi_0) & \cdots & T_{d-1}(\xi_0) \\ 1 & T_1(\xi_1) & \cdots & T_{d-1}(\xi_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & T_1(\xi_{d-1}) & \cdots & T_{d-1}(\xi_{d-1}) \end{pmatrix} = \begin{pmatrix} 1 & \cos(\theta_0) & \cdots & \cos((d-1)\theta_0) \\ 1 & \cos(\theta_1) & \cdots & \cos((d-1)\theta_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cos(\theta_{d-1}) & \cdots & \cos((d-1)\theta_{d-1}) \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & \tilde{T}_1(\tilde{\xi}_0) & \cdots & \tilde{T}_{d-1}(\tilde{\xi}_0) \\ 1 & \tilde{T}_1(\tilde{\xi}_1) & \cdots & \tilde{T}_{d-1}(\tilde{\xi}_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \tilde{T}_1(\tilde{\xi}_{d-1}) & \cdots & \tilde{T}_{d-1}(\tilde{\xi}_{d-1}) \end{pmatrix} = \begin{pmatrix} 1 & 2\cos(\theta_0) & \cdots & 2\cos((d-1)\theta_0) \\ 1 & 2\cos(\theta_1) & \cdots & 2\cos((d-1)\theta_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2\cos(\theta_{d-1}) & \cdots & 2\cos((d-1)\theta_{d-1}) \end{pmatrix},$$

respectively. The matrix V_n has entries in $[-1,1]$ while the matrix $\tilde{V}_n$ has entries in $[-2,2]$. We note that $\tilde{V}_n = V_n \operatorname{diag}[1,2,\ldots,2]$ and it can be checked that $\tilde{V}_n^\top V_n = dI_{2^n}$. Lattice $\tilde{\Gamma}_n := \tilde{V}_n\mathbb{Z}$ is orthogonal and $\tilde{V}_n^\top \tilde{V}_n = \operatorname{diag}[d, 2d, \ldots, 2d]$. The associated dual lattice $\tilde{\Gamma}_n^\perp$ is generated by $\tilde{V}_n^\perp = V_n/d$ and is therefore given by $\tilde{\Gamma}_n^\perp = \Gamma_n/d$ with $\Gamma_n := V_n\mathbb{Z}$. We will use the naming convention introduced in [12] and refer to $\tilde{\Gamma}_n$ as Chebyshev-Frolov lattices (CF-lattices for short). When necessary, we will refer to Γ_n as dual CF-lattices. In light of what precedes and invoking the conclusions of the algebraic construction, the invariants of lattices $\tilde{\Gamma}_n$ and Γ_n are given by

$$\begin{aligned} |\det(\tilde{V}_n)| &= (\sqrt{2d})^d/\sqrt{2}, & \operatorname{Nm}(\tilde{V}_n) &= 1, \\ |\det(V_n)| &= \sqrt{2}(\sqrt{d/2})^d, & \operatorname{Nm}(V_n) &= 2/2^d, \end{aligned} \tag{7}$$

while clearly the infimum of $\|\tilde{V}_n S\|_\infty$ and $\|V_n S\|_\infty$ over $S \in \mathbf{SL}_d(\mathbb{Z})$ are smaller than 1 and 2 respectively.

Enumeration of CF-lattices in hypercubes has already been addressed, see e.g. [11, 12, 13, 14]. For example, given a function f supported in the hypercube $\Omega = [-1/2, 1/2]^d$, one is interested in enumerating the quadrature nodes contributing to (3) with $M_N = (N \det(\tilde{V}_n))^{-1/d}\tilde{V}_n$. The previous amounts to enumerating $\tilde{V}_n\boldsymbol{k}$ with $\boldsymbol{k} \in \mathbb{Z}^d$ such that

$$-\lambda\mathbf{1} \le \tilde{V}_n\boldsymbol{k} \le \lambda\mathbf{1}, \qquad \lambda := \sqrt{\frac{d}{2}}\left(\frac{N}{\sqrt{2}}\right)^{1/d}, \tag{8}$$

where $\mathbf{1} = (1,\ldots,1)^\top \in \mathbb{R}^d$ and $\le$ is coordinate-wise comparaison. By simply using $\boldsymbol{k} = \tilde{V}_n^{-1}\tilde{V}_n\boldsymbol{k} = V_n^\top\tilde{V}_n\boldsymbol{k}/d$, one derives $\|\boldsymbol{k}\|_\infty \le \|\tilde{V}_n\boldsymbol{k}\|_\infty$. This shows that $\boldsymbol{k}$ belongs to $\{-\lfloor\lambda\rfloor, \ldots, +\lfloor\lambda\rfloor\}^d$. The previous isotropic grid has cardinality $(2\lfloor\lambda\rfloor + 1)^d$ which is of order $(\sqrt{2d})^d(N/\sqrt{2})$. For small dimensions d (e.g. $d = 2^1, 2^2, 2^3$) and small scaling factors N, one can simply enumerate all the integer vectors $\boldsymbol{k}$ in the grid and verifies if $-\lambda\mathbf{1} \le \tilde{V}_n\boldsymbol{k} \le \lambda\mathbf{1}$. This plain procedure is also disposed to parallelization by enumerating independently the components of a partition of the grid. In spite of this, it is unpractical for higher dimensions, for instance $(\sqrt{2d})^d$ becomes very large ($(\sqrt{2d})^d = 8^{32} > 10^{28}$ for $d = 32$).

In [11, 12], an enumeration procedure based on orthogonality of CF-lattices was introduced. Since $\|\tilde{V}_n\boldsymbol{k}\|_2^2 = d(k_1^2 + 2k_2^2 + \cdots + 2k_d^2)$ for $\boldsymbol{k} = (k_1, k_2, \dots, k_d)$ and $[-\lambda, \lambda]^d \subset \{\boldsymbol{x} \in \mathbb{R}^d : \|\boldsymbol{x}\|_2 \le \sqrt{d}\lambda\}$, then $\tilde{V}_n\boldsymbol{k} \in [-\lambda, \lambda]^d$ implies that $(k_1^2 + 2k_2^2 + \cdots + 2k_d^2) \le \lambda^2$. The previous encodes integers in $\mathbb{Z}^d$ located within an ellipse and which can be enumerated using a nested loop. We refer to [11, 12] for more details on this strategy and its performance.

In the recent paper [14], an optimal enumeration procedure is described, with optimality in the sense only the desired integers $\boldsymbol{k}$ are touched during the procedure. It is based on generating matrices A_n that are more suitable for enumeration. Moreover, the procedure is more general since it covers enumeration of $\tilde{V}_n\mathbb{Z}^d$ in any axis-parallel box $[\boldsymbol{b}, \boldsymbol{c}] := \{\boldsymbol{z} \in \mathbb{R}^{2^n} : \boldsymbol{b} \le \boldsymbol{z} \le \boldsymbol{c}\}$ for $\boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^{2^n}$. This procedure will be discussed in §4.

Remark 1. *Enumeration of dual CF-lattices $V_n\mathbb{Z}$ is computationally equivalent to that of CF-lattices $\tilde{V}_n\mathbb{Z}$. Strategies discussed above can be examined for matrices V_n. Otherwise, one can use $2V_n\mathbb{Z}^d = \tilde{V}_n\mathbb{Y}$ with $\mathbb{Y} = \mathrm{diag}[2, 1, \dots, 1]\mathbb{Z}^d$ and adapt the strategies accordingly. More precisely,*

$$\boldsymbol{b} \le V_n\boldsymbol{k} \le \boldsymbol{c} \quad \Longleftrightarrow \quad 2\boldsymbol{b} \le \tilde{V}_n\tilde{\boldsymbol{k}} \le 2\boldsymbol{c}, \tag{9}$$

where $\boldsymbol{k}, \tilde{\boldsymbol{k}} \in \mathbb{Z}^d$ are related by $\tilde{k}_1 = 2k_1$, $\tilde{k}_2 = k_2, \dots, \tilde{k}_d = k_d$. Enumerating nodes $V_n\boldsymbol{k}$ ($\boldsymbol{k} \in \mathbb{Z}^d$) in $[\boldsymbol{b}, \boldsymbol{c}]$ amounts to enumerating nodes $\tilde{V}_n\tilde{\boldsymbol{k}}$ ($\tilde{\boldsymbol{k}} \in \mathbb{Y}$) in $[2\boldsymbol{b}, 2\boldsymbol{c}]$ then normalizing by 2. An illustration of this stratagem is given in §4.

3. New coordinates-permuted systems

3.1. Two specific families of permutations. We let $(\mathscr{I}_n)_{n\ge0}$ and $(\mathscr{J}_n)_{n\ge0}$ be the "ordered" sets of indices, defined recursively by: $\mathscr{I}_0 = \mathscr{J}_0 = \{0\}$, and

$$\begin{aligned} \mathscr{I}_{n+1} &= \mathscr{I}_n \wedge \{2^{n+1} - 1 - \mathscr{I}_n\} \\ \mathscr{J}_{n+1} &= 2\mathscr{J}_n \wedge \{2\mathscr{J}_n + 1\} \end{aligned}, \qquad n \ge 0, \tag{10}$$

where $2^{n+1} - 1 - \mathscr{I}_n := \{2^{n+1} - 1 - i : i \in \mathscr{I}_n\}$, $2\mathscr{J}_n := \{2j : j \in \mathscr{J}_n\}$, $2\mathscr{J}_n + 1 := \{2j + 1 : j \in \mathscr{J}_n\}$ and $\wedge$ is the concatenation operation. For instance,

$$\begin{aligned} &\mathscr{I}_1 = \{0, 1\}, \quad \mathscr{I}_2 = \{0, 1, 3, 2\}, \quad \mathscr{I}_3 = \{0, 1, 3, 2, 7, 6, 4, 5\} \\ &\mathscr{J}_1 = \{0, 1\}, \quad \mathscr{J}_2 = \{0, 2, 1, 3\}, \quad \mathscr{J}_3 = \{0, 4, 2, 6, 1, 5, 3, 7\} \end{aligned}, \dots. \tag{11}$$

The nested sets of indices $\mathscr{I}_0 \subset \mathscr{I}_1 \subset \cdots$ and $\mathscr{J}_0 \subset \mathscr{J}_1 \subset \cdots$, reflect two specific ways of re-ordering the nested sets of indices $\{0, \dots, 2^n - 1\}$, $n \ge 0$. In particular, in every $\mathscr{J}_n$ the first 2^{n-1} numbers are the even numbers ordered according to decreasing "largest" dividing power of 2, the ordering of the odd numbers is accordingly implied.

The sets $\mathscr{I}_n$ and $\mathscr{J}_n$ can be described by permutations over $\{0, 1, \dots, 2^n - 1\}$. We denote $\pi_n^{(|)}$ and $\pi_n^{(-)}$ such permutations, i.e. $\mathscr{I}_n = \{\pi_n^{(|)}(i) : i =$

$0, \ldots, 2^n-1\}$ and $\mathscr{J}_n = \{\pi_n^{(-)}(j) : j = 0, \ldots, 2^n-1\}$. The recursions (10) can be readily reflected in these permutations. Indeed,

- $\pi_0^{(|)}$ is the identity permutation over $\{0\}$, then having $\pi_n^{(|)}$ computed,

$$\begin{aligned} \pi_{n+1}^{(|)}(i) &= \pi_n^{(|)}(i) \\ \pi_{n+1}^{(|)}(2^n+i) &= 2^{n+1}-1-\pi_n^{(|)}(i) \end{aligned}, \qquad i = 0, \ldots, 2^n-1. \tag{12}$$

- $\pi_0^{(-)}$ is the identity permutation over $\{0\}$, then having $\pi_n^{(-)}$ computed,

$$\begin{aligned} \pi_{n+1}^{(-)}(j) &= 2\pi_n^{(-)}(j) \\ \pi_{n+1}^{(-)}(2^n+j) &= 2\pi_n^{(-)}(j)+1 \end{aligned}, \qquad j = 0, \ldots, 2^n-1. \tag{13}$$

Remark 2. *The permutations $\pi_n^{(|)}$ are the restriction of $\sigma : \mathbb{N} \to \mathbb{N}$ a fixed permutation of $\mathbb{N}$. This permutation can be described using a "bit-flip" procedure; $\sigma(0) = 0$, $\sigma(1) = 1$ then for $2^n \leq k < 2^{n+1}$ and $\sigma(k-2^n) = \sum_{j=0}^{n-1} \varepsilon_j 2^j$, $\sigma(k) = 2^{n+1}-1-\sigma(k-2^n) = 2^n + \sum_{j=0}^{n-1}(1-\varepsilon_j)2^j$. As for the permutations $\pi_n^{(-)}$, they can be produced using the "bit-reversal" Vander-Corput sequence $(c_k)_{k\geq 0}$. This sequence takes value in $[0,1[$ and is defined by $c_0 = 0$ and*

$$c_k = \frac{1}{2}\sum_{j=0}^{n-1} a_j 2^{-j}, \qquad k = \sum_{j=0}^{n-1} a_j 2^j. \tag{14}$$

One can verify that $k \mapsto 2^n c_k$ define a permutation over $\{0, \ldots, 2^n-1\}$ which satisfies the same recursion as $\pi_n^{(-)}$, hence $\pi_n(k) = 2^n c_k$ for $k \in \{0, \ldots, 2^n-1\}$. From this identification, we observe that $\pi_n^{(-)}$ have order 2, i.e.

$$\pi_n^{(-)} \circ \pi_n^{(-)}(j) = j, \qquad n \geq 0,\ j = 0, \ldots, 2^n-1. \tag{15}$$

We let $P_n^{(|)}, P_n^{(-)} \in \{0,1\}^{2^n \times 2^n}$ be the permutation matrices associated with $\pi_n^{(|)}$, $\pi_n^{(-)}$ (i.e. $P_n^{(-)} = (\delta_{i,\pi_n^{(-)}(j)})_{0\leq i,j\leq d-1}$ and $P_n^{(|)} = (\delta_{i,\pi_n^{(|)}(j)})_{0\leq i,j\leq d-1}$). Based on (12) and (13), one can derive recursions for these matrices. Such recursions will not be relevant in our analysis. We note however in light of (15) that $P_n^{(-)}$ are symmetric and

$$P_n^{(-)} \times P_n^{(-)} = I_{2^n}, \qquad n \geq 0, \tag{16}$$

where I_{2^n} is the $2^n \times 2^n$ identity matrix. Also, note that $P_0^{(-)} = I_1$, $P_1^{(-)} = I_2$.

For later use, we introduce the $2^n \times 2^n$ matrices Q_n defined by: $Q_0 = (1)$, $Q_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, otherwise for $n \geq 1$

$$Q_n = P_n^{(-)} J_{2^n} P_n^{(-)}, \qquad J_{2^n} := \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{pmatrix} \in \{0,1\}^{2^n \times 2^n}. \tag{17}$$

Matrices Q_n have at most two units along every row or column and clearly $Q_n \in \mathbf{SL}_{2^n}(\mathbb{Z})$. The inverse matrices $Q_n^{-1} = P_n^{(-)} J_{2^n}^{-1} P_n^{(-)}$ have their entries in $\{-1, 0, 1\}$ but are relatively full (having $(2^n+1)2^n/2$ non-zero entries). We note in view of (16) that for any $n \geq 0$

$$Q_n = I_{2^n} + (\delta_{\pi_n^{(-)}(i), \pi_n^{(-)}(j)+1})_{0 \leq i,j \leq 2^n-1}. \tag{18}$$

Building on this, we are able to derive a recurrence for Q_n.

Lemma 1. *There holds* $Q_0 = (1)$ *and for* $n \geq 0$,

$$Q_{n+1} - I_{2^{n+1}} = \begin{pmatrix} 0 & Q_n - I_{2^n} \\ I_{2^n} & 0 \end{pmatrix}. \tag{19}$$

Proof. We use the shorthands $\pi_m = \pi_m^{(-)}$ for simplicity. In view of (13), $\pi_{n+1}(2^n + j) = \pi_{n+1}(j) + 1$ for any $0 \leq j \leq 2^n - 1$, hence the lower block I_{2^n} in (19). For $2^n \leq j \leq 2^{n+1} - 1$, one has $\pi_{n+1}(j) = 2\pi_n(j - 2^n) + 1$ hence $\pi_{n+1}(j) + 1 = 2(\pi_n(j - 2^n) + 1)$ which is equal to $\pi_{n+1}(i)$ where $0 \leq i \leq 2^n - 1$ is such that $\pi_n(j - 2^n) + 1 = \pi_n(i)$. This implies the upper block $Q_n - I_{2^n}$. □

We introduce the $2^n \times 2^n$ matrices $\tilde{Q}_n$ defined by: $\tilde{Q}_0 = (2)$, $\tilde{Q}_1 = \left(\begin{smallmatrix} 2 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, otherwise for $n \geq 1$

$$\tilde{Q}_n = P_n^{(-)} \tilde{J}_2^n P_n^{(-)} \qquad \tilde{J}_{2^n} := \begin{pmatrix} 2 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{pmatrix} \in \{0, 1, 2\}^{2^n \times 2^n}. \tag{20}$$

Since the first row and column of $P_n^{(-)}$ consist in $(1, 0, \ldots)$ for any $n \geq 1$, then $P_n^{(-)}$ and $\tilde{I}_{2^n} := \mathrm{diag}[2, 1, \ldots, 1]$ commute and $\tilde{Q}_n = \tilde{I}_{2^n} Q_n$ for any $n \geq 1$. Compared to Q_n only the unit at position $(i, j) = (0, 0)$ is different and it is equal to 2 $(\tilde{Q}_n = Q_n + E_{0,0})$.

Remark 3. *In view of* (19), *matrices* $\tilde{Q}_n - \tilde{I}_{2^n} = Q_n - I_{2^n} =: R_n$ *satisfy the recursion* $R_0 = (0)$ *and* $R_{n+1} = \left(\begin{smallmatrix} 0 & R_n \\ I_{2^n} & 0 \end{smallmatrix}\right)$ *for* $n \geq 0$. *The action of* R_n *is rather straightforward, in view of* (15) *and* (18), *for* $\boldsymbol{x} = (x_0, \ldots, x_{2^n-1})$, $\boldsymbol{y} = R_n \boldsymbol{x}$ *is given by* $y_0 = 0$ *and* $y_i = x_j$ *with* $j = \pi_n^{(-)}(\pi_n^{(-)}(i) - 1)$ *otherwise.*

The families of permutations $\pi_n^{(|)}$ and $\pi_n^{(-)}$ will be used for permuting rows and columns of generating matrices. Matrices Q_n and $\tilde{Q}_n$ are transition matrices associated with Chebyshev bases.

3.2. Coordinate-permuted Chebyshev abscissas. The roots of Chebyshev polynomials T_{2^n} and $\tilde{T}_{2^n}$, introduced in (6), $\Xi_n = \{\xi_{n,0}, \ldots, \xi_{n,2^n-1}\}$ and $\tilde{\Xi}_n =$

$\{\tilde{\xi}_{n,0}, \dots, \tilde{\xi}_{n,2^n-1}\}$ are formulated using a standard order. Subsequently, we consider the order implied by $\mathscr{I}_n$, $\Xi_n = \{\xi_{n,i}\}_{i \in \mathscr{I}_n}$ and $\tilde{\Xi}_n = \{\tilde{\xi}_{n,i}\}_{i \in \mathscr{I}_n}$. We recall that $\mathscr{I}_{n+1} = \mathscr{I}_n \wedge \mathscr{I}_n'$ with $\mathscr{I}_n' = 2^{n+1} - 1 - \mathscr{I}_n$, see (10). This shows that $\Xi_{n+1} = \Xi_{n+1}^+ \wedge -\Xi_{n+1}^+$ and $\tilde{\Xi}_{n+1} = \tilde{\Xi}_{n+1}^+ \wedge -\tilde{\Xi}_{n+1}^+$ with $\Xi_{n+1}^+ = \{\xi_{n+1,i}\}_{i \in \mathscr{I}_n}$ and $\tilde{\Xi}_{n+1}^+ = \{\tilde{\xi}_{n+1,i}\}_{i \in \mathscr{I}_n}$ comprising only roots that are > 0. We condense the main interest in introducing permuted roots on the following: for $k = 0, \dots, 2^n - 1$, $i = \pi_{n+1}^{(|)}(k) \in \mathscr{I}_n$ and $i' = \pi_{n+1}^{(|)}(2^n + k) = 2^{n+1} - 1 - i \in \mathscr{I}_n'$,

$$\begin{aligned} \xi_{n+1,i'} &= -\xi_{n+1,i} \\ T_2(\xi_{n+1,i'}) &= T_2(\xi_{n+1,i}) = \xi_{n,i} \end{aligned} . \tag{21}$$

We recall that $T_2(x) = \cos(2\theta) = 2x^2 - 1$ for $x = \cos(\theta)$. The same holds with abscissas $\tilde{\xi}_{n,i}$ up to changing T_2 by $\tilde{T}_2$ ($\tilde{T}_2(x) = x^2 - 2$).

We denote by D_n the $2^n \times 2^n$ diagonal matrices

$$D_n := \operatorname{diag}[(\tilde{\xi}_{n+1,i})_{i \in \mathscr{I}_n}] = 2 \times \operatorname{diag}[(\xi_{n+1,i})_{i \in \mathscr{I}_n}]. \tag{22}$$

For example $D_0 = (\sqrt{2})$ and $D_1 = \operatorname{diag}[2\cos(\pi/8), 2\cos(3\pi/8)]$. The numbers on the diagonal belong to $]0, 2[$. Since

$$\tilde{T}_{2^{n+1}}(x) = (x - \tilde{\xi}_{n+1,0}) \dots (x - \tilde{\xi}_{n+1,2^{n+1}-1}),$$

substituting by $x = 0 = 2\cos(\pi/2)$ implies $\det(D_n) = \sqrt{2}$ for any $n \geq 0$.

We let $L_0, \dots, L_{2^{n+1}-1}$ be an arbitrary family of polynomials such that every L_k has the same parity as k (as functions, L_{2j} are even while L_{2j+1} are odd). We then let $\alpha_0, \dots, \alpha_{2^n-1}$, $\beta_0, \dots, \beta_{2^n-1}$ and $\gamma_0, \dots, \gamma_{2^n-1}$ be the families defined by $L_{2j}(x) = \alpha_j(y)$, $L_{2j+1}(x) = 2x\beta_j(y)$ and $2xL_{2j+1}(x) = \gamma_j(y)$ with $y = T_2(x) = 2x^2 - 1$. We introduce the matrices

$$V_L := (L_j(\xi_{n+1,i}))_{\substack{i \in \mathscr{I}_{n+1} \\ j \in \mathscr{J}_{n+1}}}, \tag{23}$$

$$V_\alpha := (\alpha_j(\xi_{n,i}))_{\substack{i \in \mathscr{I}_n \\ j \in \mathscr{J}_n}}, \qquad V_\beta := (\beta_j(\xi_{n,i}))_{\substack{i \in \mathscr{I}_n \\ j \in \mathscr{J}_n}}, \qquad V_\gamma := (\gamma_j(\xi_{n,i}))_{\substack{i \in \mathscr{I}_n \\ j \in \mathscr{J}_n}}. \tag{24}$$

Lemma 2. *There holds*

$$V_L = \begin{pmatrix} V_\alpha & D_n V_\beta \\ V_\alpha & -D_n V_\beta \end{pmatrix} = \begin{pmatrix} V_\alpha & D_n^{-1} V_\gamma \\ V_\alpha & -D_n^{-1} V_\gamma \end{pmatrix}. \tag{25}$$

Proof. The recurrences on the sets of indices $\mathscr{I}_n$ and $\mathscr{J}_n$ imply a block representation (with $\mathscr{I}_n' := 2^{n+1} - 1 - \mathscr{I}_n$) of the form

$$\mathscr{I}_{n+1}\left\{ \overbrace{\quad V_L \quad}^{\mathscr{J}_{n+1}} \right. = \begin{array}{r} \\ \mathscr{I}_n \{ \\ \mathscr{I}_n' \{ \end{array} \begin{array}{cc} \overbrace{\quad\quad}^{2\mathscr{J}_n} & \overbrace{\quad\quad}^{2\mathscr{J}_n+1} \\ X_1 & Y_1 \\ X_2 & Y_2 \end{array}.$$

For $i \in \mathscr{I}_n$ arbitrary, we have $i = \pi_{n+1}^{(|)}(k)$ for some $0 \le k < 2^n$, then we introduce $i' := \pi_{n+1}^{(|)}(2^n + k) \in \mathscr{I}_n'$. By (21), there holds $\xi_{n+1,i} = -\xi_{n+1,i'}$ and $T_2(\xi_{n+1,i}) = T_2(\xi_{n+1,i'}) = \xi_{n,i}$. On the one hand, polynomials L_k have the same parity as k, hence $X_2 = X_1$ and $Y_2 = -Y_1$. On the other hand, from the definitions of α_j, β_j, γ_j, $L_{2j}(\xi_{n+1,i}) = \alpha_j(\xi_{n,i})$, $L_{2j+1}(\xi_{n+1,i}) = 2\xi_{n+1,i}\beta_j(\xi_{n,i})$ and $2\xi_{n+1,i}L_{2j+1}(\xi_{n+1,i}) = \gamma_j(\xi_{n,i})$. Therefore, $X_1 = V_\alpha$ and $Y_1 = D_n V_\beta = D_n^{-1} V_\gamma$. The proof is complete. $\square$

Matrices Q_n and $\tilde{Q}_n$ are transition matrices associated with Chebyshev bases. Indeed, in the new coordinate-permuted system, there holds

$$((T_j + T_{j+1})(\xi_{n,i}))_{\substack{i \in \mathscr{I}_n \\ j \in \mathscr{I}_n}} = (T_j(\xi_{n,i}))_{\substack{i \in \mathscr{I}_n \\ j \in \mathscr{I}_n}} \times Q_n. \tag{26}$$

This can be seen by changing back and forth into $0 \le j \le 2^n - 1$ and using that $T_{2^n}(\xi_{n,i}) = 0$ for all i. Similarly, given the family of polynomials defined by $\beta_0 = 1/2$ and $\beta_j + \beta_{j-1} = T_j$ for $j \ge 1$, one has

$$(\beta_j(\xi_{n,i}))_{\substack{i \in \mathscr{I}_n \\ j \in \mathscr{I}_n}} \times \tilde{Q}_n^\top = (T_j(\xi_{n,i}))_{\substack{i \in \mathscr{I}_n \\ j \in \mathscr{I}_n}}. \tag{27}$$

Actually $\beta_j(x) = V_j(x)/2$ where V_j are Chebyshev polynomials of the third kind, i.e. $V_j(x) = \cos((2j+1)\theta/2)/\cos(\theta/2)$ for $x = \cos(\theta)$.

3.3. Coordinate-permuted Chebyshev-Frolov lattices. In the present section, we denote by V_n and $\tilde{V}_n$ the $2^n \times 2^n$ Chebyshev-Vandermonde matrices, formulated in the new coordinate systems, i.e.

$$V_n := (T_j(\xi_{n,i}))_{\substack{i \in \mathscr{I}_n \\ j \in \mathscr{I}_n}}, \qquad \tilde{V}_n := (\tilde{T}_j(\tilde{\xi}_{n,i}))_{\substack{i \in \mathscr{I}_n \\ j \in \mathscr{I}_n}}. \tag{28}$$

Compared to plain matrices in §2, rows and columns are permuted according to $\pi_n^{(|)}$ and $\pi_n^{(-)}$ respectively, i.e. $V_n = (P_n^{(|)})^{-1} V_n^* P_n^{(-)}$ and $\tilde{V}_n = (P_n^{(|)})^{-1} \tilde{V}_n^* P_n^{(-)}$ (* to distinguish plain matrices of §2). This rearrangement is highly relevant for deriving simple recurrences for such matrices. We note that $\tilde{V}_n = V_n^{-\top}/2^n$ still holds for any $n \ge 0$ (since permutation matrices are orthogonal). Also, since $\mathscr{I}_n = \{0, \dots\}$, the first columns of $\tilde{V}_n$ and of V_n are still both equal to $(1, \dots, 1)^\top$ implying that $\tilde{V}_n = V_n \operatorname{diag}[1, 2, \dots, 2]$ holds. We derive recurrences for V_n and recurrences for $\tilde{V}_n$, with the latter being simply implied. We recall that $\tilde{Q}_n = \operatorname{diag}[2, 1, \dots, 1] Q_n$ for any $n \ge 1$.

The verifications $V_0 = (1)$ and $V_1 = \begin{pmatrix} 1 & \sqrt{2}/2 \\ 1 & -\sqrt{2}/2 \end{pmatrix}$ are immediate.

Lemma 3. *The following recurrences hold: for $n \ge 0$,*

$$V_{n+1} = \begin{pmatrix} V_n & D_n^{-1} V_n Q_n \\ V_n & -D_n^{-1} V_n Q_n \end{pmatrix}, \qquad V_{n+1} = \begin{pmatrix} V_n & D_n V_n \tilde{Q}_n^{-\top} \\ V_n & -D_n V_n \tilde{Q}_n^{-\top} \end{pmatrix}. \tag{29}$$

Proof. One has $T_{2j}(x) = T_j(2x^2 - 1)$ and $2xT_{2j+1}(x) = T_{2j}(x) + T_{2j+2}(x) = (T_j + T_{j+1})(2x^2 - 1)$ for any j. We may apply Lemma 2 with $\alpha_j = T_j$ and $\gamma_j = (T_j + T_{j+1})$ which in view of (26) implies the first recursion. On the other hand, if $\beta_0, \dots, \beta_{2^n-1}$ are such that $T_{2j+1}(x) = 2x\beta_j(2x^2 - 1)$, then $\beta_0 \equiv 1/2$ and $\beta_{j-1} + \beta_j = T_j$ for $j \geq 1$ (derived using $T_{2j-1}(x) + T_{2j+1}(x) = 2xT_{2j}(x) = 2xT_j(2x^2 - 1)$). We again apply Lemmas 2 (with these β_j) which in view of (27) implies the second recurrence. The proof is complete. $\square$

The verifications $\tilde{V}_0 = (1)$ and $\tilde{V}_1 = \begin{pmatrix} 1 & \sqrt{2} \\ 1 & -\sqrt{2} \end{pmatrix}$ are immediate.

Lemma 4. *The following recurrences hold: for $n \geq 0$,*

$$\tilde{V}_{n+1} = \begin{pmatrix} \tilde{V}_n & D_n^{-1}\tilde{V}_n\tilde{Q}_n \\ \tilde{V}_n & -D_n^{-1}\tilde{V}_n\tilde{Q}_n \end{pmatrix}, \qquad \tilde{V}_{n+1} = \begin{pmatrix} \tilde{V}_n & D_n\tilde{V}_nQ_n^{-\top} \\ \tilde{V}_n & -D_n\tilde{V}_nQ_n^{-\top} \end{pmatrix}. \tag{30}$$

In summary, the computation of the direct/transpose/inverse actions $V_n\boldsymbol{x}$, $V_n^\top\boldsymbol{x}$, $V_n^{-1}\boldsymbol{x}$, $V_n^{-\top}\boldsymbol{x}$ and $\tilde{V}_n\boldsymbol{x}$, $\tilde{V}_n^\top\boldsymbol{x}$, $\tilde{V}_n^{-1}\boldsymbol{x}$, $\tilde{V}_n^{-\top}\boldsymbol{x}$, for $\boldsymbol{x} \in \mathbb{R}^{2^n}$ can all be computed very efficiently. Actually, they are all related and can be implied from $\boldsymbol{x} \mapsto V_n\boldsymbol{x}$. In particular, such actions will mainly involve recursive applications of matrices Q_j which can be done in constant time, see Remark 3. As for Fast Fourier Transform (FFT), all the above listed actions can be optimized to have complexity $\mathcal{O}(d\log(d))$.

It is possible to derive recurrences along the same lines of (29) and (30) with matrices resulting from plugging in Chebyshev polynomials of second kind in the Vandermonde systems. These polynomials are defined by $U_k(\cos(\theta)) = \sin((k+1)\theta)/\sin(\theta)$ (if scaled according to $\tilde{U}_k(x) = U_k(x/2)$ they belong to $\mathbb{Z}[X]$ and have leading coefficient 1). However, such recurrences are not advantageous over the already identified recurrences.

Recurrences (29) and (30) can already be "laboriously" used for fast enumerations of lattices $\tilde{V}_n\mathbb{Z}$ and $V_n\mathbb{Z}$ within axis-parallel boxes. For instance using the techniques introduced in [14]. In the mentioned paper, the analysis is carried out on simpler recurrences (where basically Q_n is eliminated in (30)), which we derive shortly. In the next section, we exhibit the polynomial sequence which when plugged in the Vandermonde systems, yields directly such recurrences.

We let S_n, $\tilde{S}_n$ be the $2^n \times 2^n$ matrices defined recursively by $\tilde{S}_0 = S_0 = (1)$ and for $n \geq 0$

$$S_{n+1} = \begin{pmatrix} S_n & 0 \\ 0 & Q_n^{-1}S_n \end{pmatrix}, \qquad \tilde{S}_{n+1} = \begin{pmatrix} \tilde{S}_n & 0 \\ 0 & Q_n^\top\tilde{S}_n \end{pmatrix}. \tag{31}$$

These matrices are unimodular, $S_n, \tilde{S}_n \in \mathbf{SL}_{2^n}$ (in fact $\tilde{S}_n = S_n^{-\top}$) for any $n \geq 0$. Moreover, $A_n := \tilde{V}_n\tilde{S}_n$, $B_n := V_nS_n$ satisfy $A_0 = B_0 = (1)$ and for $n \geq 0$

$$A_{n+1} = \begin{pmatrix} A_n & D_n A_n \\ A_n & -D_n A_n \end{pmatrix}, \qquad B_{n+1} = \begin{pmatrix} B_n & D_n^{-1} B_n \\ B_n & -D_n^{-1} B_n \end{pmatrix}. \tag{32}$$

The matrix A_n generate the CF-lattice $\tilde{V}_n \mathbb{Z}^{2^n}$ while B_n generate the dual CF-lattice $V_n \mathbb{Z}^{2^n}$. We note in view of $\tilde{V}_n^{-\top} = V_n/2^n$, that $\tilde{A}_n^{-\top} = B_n/2^n$.

3.4. A new polynomial sequence. We let $(\tilde{H}_k)_{j \geq 0}$ be the sequence of polynomials defined by $\tilde{H}_0 \equiv 1$, then

$$\tilde{H}_k = \prod_{\substack{j=0 \\ a_j \neq 0}}^{n} \tilde{T}_{2^j}(x), \qquad k = \sum_{j=0}^{n} a_j 2^j, \quad a_j \in \{0, 1\}. \tag{33}$$

These polynomials have integer coefficients, have leading coefficients 1 and every $\tilde{H}_k$ has degree exactly k. In particular, the family $\tilde{H}_0, \tilde{H}_1, \ldots$ is a hierarchical basis for $\mathbb{Z}[X]$. This family can be used in the generation of CF-lattices. In addition, we observe that every polynomial $\tilde{H}_k$ have the same parity as k. More precisely, in view of $\tilde{T}_1(x) = x$ and $\tilde{T}_{2m}(x) = \tilde{T}_m(x^2 - 2)$ for any $m \geq 0$, a recurrence holds: $\tilde{H}_0 \equiv 1$,

$$\begin{matrix} \tilde{H}_{2k}(x) = \tilde{H}_k(x^2 - 2) \\ \tilde{H}_{2k+1}(x) = x\tilde{H}_k(x^2 - 2) \end{matrix}, \qquad k \geq 0. \tag{34}$$

The polynomial $\tilde{H}_k$ considered on the domain $[-2, 2]$ is uniformly bounded by $2^{\sigma_1(k)}$ where $\sigma_1(k)$ is the number of ones in the binary expansion of k. In particular $\sup_{-2 \leq x \leq 2} |\tilde{H}_k(x)| \leq (k+1)$ for any $k \geq 0$.

We let A_n be the Vandermonde matrices associated with the introduced polynomials $\tilde{H}_k$ and the Chebyshev abscissas $\tilde{\xi}_{n,i}$, on the coordinate-permuted systems described in the previous section, i.e.

$$A_n = (\tilde{H}_j(\tilde{\xi}_{n,i}))_{\substack{i \in \mathscr{I}_n \\ j \in \mathscr{I}_n}}, \qquad n \geq 0. \tag{35}$$

The verifications $A_0 = (1)$, $A_1 = \begin{pmatrix} 1 & \sqrt{2} \\ 1 & -\sqrt{2} \end{pmatrix}$ are immediate. Then, by a direct application of (34) and the arguments leading to Lemma 2.

Lemma 5. *The following recurrences hold: for $n \geq 0$,*

$$A_{n+1} = \begin{pmatrix} A_n & D_n A_n \\ A_n & -D_n A_n \end{pmatrix}. \tag{36}$$

It is immediate to derive similar recurrences for the matrices A_n^{-1}, $A_n^{-\top}$.

Lemma 6. *The following recurrence holds, $A_0^{-\top} = (1)$ and for $n \geq 0$*

$$A_{n+1}^{-\top} = \frac{1}{2} \begin{pmatrix} A_n^{-\top} & D_n^{-1} A_n^{-\top} \\ A_n^{-\top} & -D_n^{-1} A_n^{-\top} \end{pmatrix}. \tag{37}$$

Recurrences (36) and (37) are similar up the factor $1/2$ and the change of D_n into D_n^{-1} which is also diagonal with $D_n^{-1} = \mathrm{diag}[(1/\tilde{\xi}_{n+1,i})_{i\in\mathscr{I}_n}]$. In light of this observation, we have the following lemma

Lemma 7. *Matrices $A_n^{-\top}$ satisfy*

$$A_n^{-\top} = \frac{1}{2^n}\left(\frac{1}{\tilde{H}_j(\tilde{\xi}_{n,i})}\right)_{\substack{i\in\mathscr{I}_n\\ j\in\mathscr{I}_n}}, \qquad n \geq 0. \tag{38}$$

Proof. Let Δ_n be the matrix in the right side of (38). We easily verify that $\Delta_0 = A_0^{-\top}$ and $\Delta_1 = A_1^{-\top}$ using $\tilde{H}_0 \equiv 1$, $\tilde{H}_1 \equiv x$ and $\{\tilde{\xi}_{1,0}, \tilde{\xi}_{n,1}\} = \{\sqrt{2}, -\sqrt{2}\}$. In general, using (34) and the arguments leading to Lemma 2, one has for any $n \geq 0$, $\Delta_{n+1} = \frac{1}{2}\begin{pmatrix}\Delta_n & D_n^{-1}\Delta_n\\ \Delta_n & -D_n^{-1}\Delta_n\end{pmatrix}$. Matrices Δ_n satisfy the same recurrence as $A_n^{-\top}$, thus the equality. $\square$

The identification $(A_n^{-\top})^\top A_n = I_{2^n}$ provides the following identities

$$\sum_{i=0}^{2^n-1} \frac{\tilde{H}_k(\tilde{\xi}_{n,i})}{\tilde{H}_l(\tilde{\xi}_{n,i})} = 2^n \delta_{k,l}, \qquad 0 \leq k, l \leq 2^n - 1. \tag{39}$$

Visibly, these identities can be easily verified for $k = l$ and for k and l of different parities. The verification for k and l having the same parity uses the recursions (21) formulated for $\tilde{\xi}_{n,i}$ and recursions (34) for $\tilde{H}_k$.

Matrices $B_n := 2^n A_n^{-\top}$ satisfy $B_0 = (1)$, $B_1 = \begin{pmatrix}1 & 1/\sqrt{2}\\ 1 & -1/\sqrt{2}\end{pmatrix}$ and

$$B_{n+1} = \begin{pmatrix}B_n & D_n^{-1}B_n\\ B_n & -D_n^{-1}B_n\end{pmatrix}, \qquad n \geq 0. \tag{40}$$

It goes without saying, matrices A_n and B_n are as in the previous section, i.e. $A_n = \tilde{V}_n\tilde{S}_n$, $B_n = V_nS_n$. In particular, the matrix $\tilde{S}_n$ is the transition matrix from the basis $\{\tilde{T}_j\}_{j\in\mathscr{I}_n}$ into the basis $\{\tilde{H}_j\}_{j\in\mathscr{I}_n}$. The analysis made so far can be summarized on the following diagram

$$\begin{tikzcd} \tilde{V}_n \arrow[r, "\tilde{S}_n"] \arrow[d, "\perp"'] & A_n \arrow[d, "\perp"] \\ V_n/2^n \arrow[r, "\tilde{S}_n^{-\top}"'] & B_n/2^n. \end{tikzcd}$$

We conclude this section with few practical remarks.

Remark 4. *We note in view of* (31) *that $S_1 = \tilde{S}_1 = I_2$ and for any $n \geq 1$, the leading two columns/rows of matrices S_n and $\tilde{S}_n$ are the Kronecker vectors e_1 and e_2 (by induction). Matrices A_n and B_n are "almost" orthogonal in the* *sense $A_n^\top A_n$ and $B_n^\top B_n$ are block diagonal. The blocks are (2^n) (of size 1×1) and blocks of size $2^j \times 2^j$ for $j = 0, \dots, n-1$ in this order (by induction).*

4. Sequential enumeration of Chebyshev-Frolov lattices

In this section, the notation $\boldsymbol{z} = (\boldsymbol{z}_1\ \ \boldsymbol{z}_2) \in \mathbb{R}^{2d}$ stands for $\boldsymbol{z}_1, \boldsymbol{z}_2 \in \mathbb{R}^d$ and $\boldsymbol{z}$ being the vertical concatenation of $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$. For $n \geq 0$, $d = 2^n$, we introduce the functions $\boldsymbol{\rho}_n : \mathbb{R}^{2d} \to \mathbb{R}^d$ and $\boldsymbol{\phi}_n, \boldsymbol{\psi}_n : \mathbb{R}^d \times \mathbb{R}^{2d} \times \mathbb{R}^{2d} \to \mathbb{R}^d$ by

$$\boldsymbol{\rho}_n(\boldsymbol{z}) = \frac{\boldsymbol{z}_1 + \boldsymbol{z}_2}{2}, \qquad \begin{matrix} \boldsymbol{\phi}_n(\boldsymbol{z}, \boldsymbol{b}, \boldsymbol{c}) = D_n^{-1} \max(\boldsymbol{b}_1 - \boldsymbol{z}, -(\boldsymbol{c}_2 - \boldsymbol{z})) \\ \boldsymbol{\psi}_n(\boldsymbol{z}, \boldsymbol{b}, \boldsymbol{c}) = D_n^{-1} \min(\boldsymbol{c}_1 - \boldsymbol{z}, -(\boldsymbol{b}_2 - \boldsymbol{z})) \end{matrix}, \tag{41}$$

where $\boldsymbol{z} = (\boldsymbol{z}_1\ \ \boldsymbol{z}_2)$ in the definition of $\boldsymbol{\rho}_n$ while $\boldsymbol{b} = (\boldsymbol{b}_1\ \ \boldsymbol{b}_2)$, $\boldsymbol{c} = (\boldsymbol{c}_1\ \ \boldsymbol{c}_2)$ in the definitions of $\boldsymbol{\phi}_n$ and $\boldsymbol{\psi}_n$ and max and min are meant coordinate-wise. One has

Lemma 8. *For $\boldsymbol{x} = (\boldsymbol{x}_1\ \ \boldsymbol{x}_2)$, $\boldsymbol{b} = (\boldsymbol{b}_1\ \ \boldsymbol{b}_2)$, $\boldsymbol{c} = (\boldsymbol{c}_1\ \ \boldsymbol{c}_2) \in \mathbb{R}^{2d}$,*

$$\boldsymbol{b} \leq A_{n+1}\boldsymbol{x} \leq \boldsymbol{c} \quad \Longleftrightarrow \quad \begin{matrix} \boldsymbol{b}' \leq A_n\boldsymbol{x}_1 \leq \boldsymbol{c}' \\ \boldsymbol{b}'' \leq A_n\boldsymbol{x}_2 \leq \boldsymbol{c}'' \end{matrix}, \tag{42}$$

where $\boldsymbol{b}' = \boldsymbol{\rho}_n(\boldsymbol{b})$, $\boldsymbol{c}' = \boldsymbol{\rho}_n(\boldsymbol{c})$, $\boldsymbol{b}'' = \boldsymbol{\phi}_n(A_n\boldsymbol{x}_1, \boldsymbol{b}, \boldsymbol{c})$ and $\boldsymbol{c}'' = \boldsymbol{\psi}_n(A_n\boldsymbol{x}_1, \boldsymbol{b}, \boldsymbol{c})$.

Proof. Since $\boldsymbol{y} = A_{n+1}\boldsymbol{x} = (\boldsymbol{y}_+\ \ \boldsymbol{y}_-)$ with $\boldsymbol{y}_\pm = A_n\boldsymbol{x}_1 \pm D_nA_n\boldsymbol{x}_2 \in \mathbb{R}^d$,

$$\boldsymbol{b} \leq \boldsymbol{y} \leq \boldsymbol{c} \quad \Longleftrightarrow \quad \begin{matrix} \boldsymbol{b}_1 + \boldsymbol{b}_2 \leq \boldsymbol{y}_+ + \boldsymbol{y}_- \leq \boldsymbol{c}_1 + \boldsymbol{c}_2 \\ \boldsymbol{b}_1 \leq \boldsymbol{y}_+ \leq \boldsymbol{c}_1,\ \boldsymbol{b}_2 \leq \boldsymbol{y}_- \leq \boldsymbol{c}_2 \end{matrix}.$$

We then use the definitions of $\boldsymbol{\rho}_n$, $\boldsymbol{\phi}_n$, $\boldsymbol{\psi}_n$ and $D_n > 0$ entry-wise. □

Enumeration of lattices $A_m\mathbb{Z}^{2^m}$ in axis-parallel boxes is well disposed to recursion. For instance, by introducing $\mathscr{P}_n(\boldsymbol{b}, \boldsymbol{c}) := \{\boldsymbol{k} \in \mathbb{Z}^{2^n} : \boldsymbol{b} \leq A_n\boldsymbol{k} \leq \boldsymbol{c}\}$,

$$\boldsymbol{x} \in \mathscr{P}_{n+1}(\boldsymbol{b}, \boldsymbol{c}) \quad \Longleftrightarrow \quad \boldsymbol{x}_1 \in \mathscr{P}_n(\boldsymbol{b}', \boldsymbol{c}'),\ \boldsymbol{x}_2 \in \mathscr{P}_n(\boldsymbol{b}'', \boldsymbol{c}''). \tag{43}$$

The dependence of $\boldsymbol{b}''$ and $\boldsymbol{c}''$ in $\boldsymbol{x}_1$ can be alleviated by considering the recursion

$$\boldsymbol{x} \in \mathscr{P}_{n+1}(\boldsymbol{b}, \boldsymbol{c}) \quad \Longleftrightarrow \quad \begin{matrix} \boldsymbol{x}_1 \in \mathscr{P}_n(\boldsymbol{b}', \boldsymbol{c}'),\ \boldsymbol{x}_2 \in \mathscr{P}_n(\boldsymbol{b}'', \boldsymbol{c}'') \\ \boldsymbol{b}_1 \leq \boldsymbol{y}_1 + D_n\boldsymbol{y}_2 \leq \boldsymbol{c}_1 \\ \boldsymbol{b}_2 \leq \boldsymbol{y}_1 - D_n\boldsymbol{y}_2 \leq \boldsymbol{c}_2 \end{matrix}, \tag{44}$$

where now $\boldsymbol{b}'' = D_n^{-1}\frac{\boldsymbol{b}_1 - \boldsymbol{c}_2}{2}$, $\boldsymbol{c}'' = D_n^{-1}\frac{\boldsymbol{c}_1 - \boldsymbol{b}_2}{2}$ and $\boldsymbol{y}_i = A_n\boldsymbol{x}_i$, $i = 1, 2$. This is easily checked as the above proof. The enumeration in $d = 1$ is immediate since $\mathscr{P}_0(b, c) = \{\lceil b \rceil, \dots, \lfloor c \rfloor\}$ for $b, c \in \mathbb{R}$. In view of the above, there holds $\mathscr{P}_m(\boldsymbol{b},\boldsymbol{c}) \subset \mathscr{P}^*_m(\boldsymbol{b},\boldsymbol{c})$, the latter being the tensor-product grid (in $\mathbb{Z}^{2^m}$) defined recursively by $\mathscr{P}^*_0(b,c) = \{\lceil b \rceil, \ldots, \lfloor c \rfloor\}$ and $\mathscr{P}^*_{n+1}(\boldsymbol{b},\boldsymbol{c}) = \mathscr{P}^*_n(\boldsymbol{b}',\boldsymbol{c}') \otimes \mathscr{P}^*_n(\boldsymbol{b}'',\boldsymbol{c}'')$. However, such grids can still be relatively sizable.

For more insights on the previous, we expound the analysis for "dilated" symmetric hypercubes, i.e. enumerating $\mathscr{P}_n(\boldsymbol{v}) := \{\boldsymbol{k} \in \mathbb{Z}^{2^n} : -\boldsymbol{v} \le A_n\boldsymbol{k} \le +\boldsymbol{v}\}$. To this end, we introduce the new functions $\boldsymbol{\rho}_n^- : \mathbb{R}^{2d} \to \mathbb{R}^d$ by

$$\boldsymbol{\rho}_n^-(\boldsymbol{z}) = \frac{\boldsymbol{z}_1 - \boldsymbol{z}_2}{2}. \tag{45}$$

Using that $2\max(x,y) = (x+y) + |x-y|$ and $2\min(x,y) = (x+y) - |x-y|$ for any $x, y \in \mathbb{R}$, we derive the following convenient formulas,

$$\left.\begin{array}{l} D_n\boldsymbol{\phi}_n(\boldsymbol{z},-\boldsymbol{v},\boldsymbol{v}) = -\boldsymbol{\rho}_n(\boldsymbol{v}) + |\boldsymbol{z} + \boldsymbol{\rho}_n^-(\boldsymbol{v})| \\ D_n\boldsymbol{\psi}_n(\boldsymbol{z},-\boldsymbol{v},\boldsymbol{v}) = +\boldsymbol{\rho}_n(\boldsymbol{v}) - |\boldsymbol{z} - \boldsymbol{\rho}_n^-(\boldsymbol{v})| \end{array}\right. \in \mathbb{R}^d, \tag{46}$$

where $|\cdot|$ is meant coordinate-wise. The boxes $[\boldsymbol{\phi}_n(\boldsymbol{z},-\boldsymbol{v},\boldsymbol{v}), \boldsymbol{\psi}_n(\boldsymbol{z},-\boldsymbol{v},\boldsymbol{v})]$ are uniformly contained within $[-\boldsymbol{v}'', +\boldsymbol{v}'']$ with $\boldsymbol{v}'' := D_n^{-1}\boldsymbol{v}'$ and $\boldsymbol{v}' := \boldsymbol{\rho}_n(\boldsymbol{v})$. For a vector $\boldsymbol{v} = (\boldsymbol{v}_1\ \ \boldsymbol{v}_2) \in \mathbb{R}^{2d}$, the recursion (44) is reformulated as

$$\boldsymbol{x} \in \mathscr{P}_{n+1}(\boldsymbol{v}) \iff \begin{array}{c} \boldsymbol{x}_1 \in \mathscr{P}_n(\boldsymbol{v}'),\ \boldsymbol{x}_2 \in \mathscr{P}_n(\boldsymbol{v}'') \\ -\boldsymbol{v}_1 \le \boldsymbol{y}_1 + D_n\boldsymbol{y}_2 \le \boldsymbol{v}_1 \\ -\boldsymbol{v}_2 \le \boldsymbol{y}_1 - D_n\boldsymbol{y}_2 \le \boldsymbol{v}_2 \end{array}. \tag{47}$$

In the particular settings $\boldsymbol{v} = \alpha\mathbf{1} \in \mathbb{R}^{2d}$ which are of interest, there holds $\boldsymbol{\rho}_n(\alpha\mathbf{1}) = \alpha\mathbf{1} \in \mathbb{R}^d$ and $\boldsymbol{\rho}_n^-(\alpha\mathbf{1}) = \mathbf{0} \in \mathbb{R}^d$. Formulas (46) become

$$\left.\begin{array}{l} D_n\boldsymbol{\phi}_n(\boldsymbol{z},-\alpha\mathbf{1},+\alpha\mathbf{1}) = -(\alpha\mathbf{1} - |\boldsymbol{z}|) \\ D_n\boldsymbol{\psi}_n(\boldsymbol{z},-\alpha\mathbf{1},+\alpha\mathbf{1}) = +(\alpha\mathbf{1} - |\boldsymbol{z}|) \end{array}\right. \in \mathbb{R}^{2^n}. \tag{48}$$

The boxes $[\boldsymbol{\phi}_n(\boldsymbol{z},-\alpha\mathbf{1},\alpha\mathbf{1}), \boldsymbol{\psi}_n(\boldsymbol{z},-\alpha\mathbf{1},\alpha\mathbf{1})]$ are symmetric with respect to $\mathbf{0}$ and are uniformly contained within $[\boldsymbol{\phi}_n(\mathbf{0},-\alpha\mathbf{1},\alpha\mathbf{1}), \boldsymbol{\psi}_n(\mathbf{0},-\alpha\mathbf{1},\alpha\mathbf{1})]$. The idea here is that the section $\{\boldsymbol{x}_2 : (\mathbf{0}\ \ \boldsymbol{x}_2) \in \mathscr{P}_{n+1}(\alpha\mathbf{1})\} = \mathscr{P}_n(\alpha D_n^{-1}\mathbf{1})$ contains all the other sections $\{\boldsymbol{x}_2 : (\boldsymbol{x}_1\ \ \boldsymbol{x}_2) \in \mathscr{P}_{n+1}(\alpha\mathbf{1})\}$. If one enumerates $\boldsymbol{k}$ in $\mathscr{P}_n(\alpha D_n^{-1}\mathbf{1})$ and have associated $A_n\boldsymbol{k}$, then by simple lookup one can enumerate any other such section.

Despite the potential simplifications on recursive enumerations, in practice numerical overheads (memory usage, table lookup, etc) may slow down these procedures. A better alternative, developed in [14], consists in breaking down the enumeration process into a nested loop. As a matter of fact, inspection of Lemma 8 shows that the condition on x_i the i^{th} coordinate of $\boldsymbol{x}$ will only depend on the preceding coordinates $x_0, \ldots, x_{i-1}$. We describe in a nutshell this enumeration procedure.

For $\boldsymbol{z} = (z_0, \ldots, z_{2^n-1})^\top \in \mathbb{R}^{2^n}$, we introduce the "slicing" notation

$$\boldsymbol{z}_{m,p} = (z_{2^m p}, \ldots, z_{2^m(p+1)-1})^\top \in \mathbb{R}^{2^m}, \qquad \begin{array}{l} 0 \le m \le n \\ 0 \le p < 2^{n-m} \end{array}. \tag{49}$$

We note that $\boldsymbol{z}_{0,p} = z_p$ for any p and $\boldsymbol{z}_{n,0} = \boldsymbol{z}$. Also, $\boldsymbol{z}_{m,p} = (\boldsymbol{z}_{m-1,2p}\ \ \boldsymbol{z}_{m-1,2p+1})$. Given $\boldsymbol{x}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^{2^n}$ fixed, we introduce the sets of vectors in $\mathbb{R}^{2^n}$:

$$\boldsymbol{X} = \{\boldsymbol{x}^{(0)}, \ldots, \boldsymbol{x}^{(n)}\}, \qquad \boldsymbol{B} = \{\boldsymbol{b}^{(0)}, \ldots, \boldsymbol{b}^{(n)}\}, \qquad \boldsymbol{C} = \{\boldsymbol{c}^{(0)}, \ldots, \boldsymbol{c}^{(n)}\}. \tag{50}$$

- $\boldsymbol{x}^{(0)} = \boldsymbol{x}$, $\boldsymbol{x}^{(1)}$ is the concatenation of $A_1 \boldsymbol{x}_{1,p}$ for $0 \le p < 2^{n-1}$, $\boldsymbol{x}^{(2)}$ is the concatenation of $A_2 \boldsymbol{x}_{2,p}$ for $0 \le p < 2^{n-2}$, etc. In other words, $\boldsymbol{x}^{(m)}_{m,p} = A_m \boldsymbol{x}_{m,p}$ for $m = 0, \ldots, n$ and $p = 0, \ldots, 2^{n-m} - 1$. We note that $\boldsymbol{x}^{(n)} = A_n \boldsymbol{x}$.
- $\boldsymbol{b}^{(n)} = \boldsymbol{b}$, $\boldsymbol{c}^{(n)} = \boldsymbol{c}$, then for $m = n-1, \ldots, 0$, the vectors $\boldsymbol{b}^{(m)}$ and $\boldsymbol{c}^{(m)}$ are computed by backward recursion according to

$$\begin{array}{ll} \boldsymbol{b}^{(m)}_{m,2q} = \boldsymbol{\rho}_m(\boldsymbol{b}^{(m+1)}_{m+1,q}) & \boldsymbol{b}^{(m)}_{m,2q+1} = \boldsymbol{\phi}_m(\boldsymbol{x}^{(m)}_{m,2q}, \boldsymbol{b}^{(m+1)}_{m+1,q}, \boldsymbol{c}^{(m+1)}_{m+1,q}) \\ \boldsymbol{c}^{(m)}_{m,2q} = \boldsymbol{\rho}_m(\boldsymbol{c}^{(m+1)}_{m+1,q}) \end{array}, \quad \begin{array}{l} \\ \boldsymbol{c}^{(m)}_{m,2q+1} = \boldsymbol{\psi}_m(\boldsymbol{x}^{(m)}_{m,2q}, \boldsymbol{b}^{(m+1)}_{m+1,q}, \boldsymbol{c}^{(m+1)}_{m+1,q}) \end{array}, \tag{51}$$

for $0 \le q < 2^{n-(m+1)}$. Applying backward induction with Lemma 8, one shows

Lemma 9. *There holds* $\boldsymbol{b}^{(n)} \le \boldsymbol{x}^{(n)} \le \boldsymbol{c}^{(n)} \iff \cdots \iff \boldsymbol{b}^{(0)} \le \boldsymbol{x}^{(0)} \le \boldsymbol{c}^{(0)}$.

We are mainly interested in $\boldsymbol{b}^{(0)} = (b^{(0)}_0, \ldots, b^{(0)}_{2^n-1})$, $\boldsymbol{c}^{(0)} = (c^{(0)}_0, \ldots, c^{(0)}_{2^n-1})$. The identities in (51) yield the following: first, $b^{(0)}_0 = \rho_0 \circ \cdots \circ \rho_{n-1}(\boldsymbol{b})$ and $c^{(0)}_0 = \rho_0 \circ \cdots \circ \rho_{n-1}(\boldsymbol{c})$ are simply the arithmetic means of $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, i.e. $b^{(0)}_0 = (b_0 + \cdots + b_{2^n-1})/2^n$ and $c^{(0)}_0 = (c_0 + \cdots + c_{2^n-1})/2^n$. Then, for $j = 2^r p$ with $p = 2q+1$ odd, the numbers $b^{(0)}_j$ and $c^{(0)}_j$ are computed via $b^{(0)}_j = \rho_0 \circ \cdots \circ \rho_{r-1}(\boldsymbol{b}^{(r)}_{r,p})$ and $c^{(0)}_j = \rho_0 \circ \cdots \circ \rho_{r-1}(\boldsymbol{c}^{(r)}_{r,p})$, with

$$\boldsymbol{b}^{(r)}_{r,p} = \boldsymbol{\phi}_r(\boldsymbol{x}^{(r)}_{r,2q}, \boldsymbol{b}^{(r+1)}_{r+1,q}, \boldsymbol{c}^{(r+1)}_{r+1,q}), \qquad \boldsymbol{c}^{(r)}_{r,p} = \boldsymbol{\phi}_r(\boldsymbol{x}^{(r)}_{r,2q}, \boldsymbol{b}^{(r+1)}_{r+1,q}, \boldsymbol{c}^{(r+1)}_{r+1,q}). \tag{52}$$

Computing $\boldsymbol{x}^{(r)}_{r,2q} = (\boldsymbol{x}^{(r)}_{j-2^r}, \ldots, \boldsymbol{x}^{(r)}_{j-1})$ only requires $\boldsymbol{x}_{r,2q} = (x_{j-2^r}, \ldots, x_{j-1})$ since $\boldsymbol{x}^{(r)}_{r,2q} = A_r \boldsymbol{x}_{r,2q}$. One can show by induction on (51) that computing $\boldsymbol{b}^{(m)}_{m,p}$ and $\boldsymbol{c}^{(m)}_{m,p}$ only requires $\boldsymbol{b}$, $\boldsymbol{c}$ and $x_0, \ldots, x_{2^m p-1}$. Thus, computing $\boldsymbol{b}^{(r+1)}_{r+1,q}$ and $\boldsymbol{c}^{(r+1)}_{r+1,q}$ only require $x_0, \ldots, x_{j-2^r-1}$. The set of inequalities $\boldsymbol{b} \le A_n \boldsymbol{x} \le \boldsymbol{c}$ reduces to 2^n simultaneous inequalities $b^{(0)}_j \le x_j \le c^{(0)}_j$ for $0 \le j \le 2^n - 1$ with $b^{(0)}_0$ and $c^{(0)}_0$ independent on $\boldsymbol{x}$, otherwise $b^{(0)}_j$ and $c^{(0)}_j$ depend on $x_0, \ldots, x_{j-1}$.

For integers $m \ge 1$, we denote by $\boldsymbol{\tau}_m$ the linear applications defined over $\mathbb{R}^{2^m}$ by $\boldsymbol{\tau}_m(\boldsymbol{z}) = (\boldsymbol{z}_1 + D_{m-1}\boldsymbol{z}_2\ \ \boldsymbol{z}_1 - D_{m-1}\boldsymbol{z}_2)$ for $\boldsymbol{z} = (\boldsymbol{z}_1\ \ \boldsymbol{z}_2)$. We have in particular $A_m \boldsymbol{z} = \boldsymbol{\tau}_m((A_{m-1}\boldsymbol{z}_1\ \ A_{m-1}\boldsymbol{z}_2))$. Also, using introduced notation, $\boldsymbol{x}^{(m)}_{m,p} = \boldsymbol{\tau}_m(\boldsymbol{x}^{(m-1)}_{m-1,2p})$ for any $m = 1, \ldots, n$ and any $p = 0, \ldots, 2^{n-m} - 1$.

An implementation of the main enumeration algorithm in [14] can now be outlined. It consist on an outer loop "loop 0" and 2^n-nested inner loops (innermost loop is indexed 2^n) as follows; we first compute $b_0^{(0)}$, $c_0^{(0)}$, then

- **Loop** 0: we iterate x_0 in $\{\lceil b_0^{(0)}\rceil, \dots, \lfloor c_0^{(0)}\rfloor\}$.
- ...
- **Loop** j: we have known $x_0, \dots, x_{j-1}$ and write $j = 2^r p$ with p odd number. Then we perform the following
  - "Forward" pass: we update $\boldsymbol{x}^{(m)}_{m,p_m} = A_m \boldsymbol{x}_{m,p_m}$ for $m = 0, \dots, r$ and $p_m = i/2^m - 1$ (using applications $\boldsymbol{\tau}_m$).
  - "Backward" pass: we update $\boldsymbol{b}^{(r)}_{r,p}$ and $\boldsymbol{c}^{(r)}_{r,p}$ (using (52)), then update $\boldsymbol{b}^{(m)}_{m,2^{r-m}p}$ and $\boldsymbol{c}^{(m)}_{m,2^{r-m}p}$ for $m = r-1, \dots, 0$ (using applications $\boldsymbol{\rho}_m$).
  - we iterate x_j in $\{\lceil b_j^{(0)}\rceil, \dots, \lfloor c_j^{(0)}\rfloor\}$ and go to iteration $j+1$.
- ...
- **Loop** 2^n: we have known $x_0, \dots, x_{2^n-1}$. We update $\boldsymbol{x}^{(m)}_{m,p_m} = A_m \boldsymbol{x}_{m,p_m}$ for $m = 0, \dots, n$ and $p_m = 2^{n-m} - 1$. We store the node $\boldsymbol{x}^{(n)}$ ($\boldsymbol{x}^{(n)} = A_n \boldsymbol{x}^{(0)} \in [\mathbf{b}, \mathbf{c}]$ with $\boldsymbol{x}^{(0)} = (x_0, \dots, x_{2^n-1})$).

Although not structurally relevant, we may think of $\boldsymbol{X}$ as a $2^n \times (n+1)$ matrix, $\boldsymbol{X} = [\boldsymbol{x}^{(0)}| \dots |\boldsymbol{x}^{(n)}]$ (and the same for $\boldsymbol{B}$ and $\boldsymbol{C}$). In the inner loops numbered $j = 1, \dots, 2^n - 1$, the forward pass propagates changes in a $2^r \times (r+1)$ sub-matrix of $\boldsymbol{X}$ (rows numbered $2^r(p-1), \dots, 2^r p - 1$ and columns numbered $0, \dots, r$) while the backward pass propagates change in sub-matrices of $\boldsymbol{B}$ and $\boldsymbol{C}$ associated with the same rows and columns.

The detailed implementation of the algorithm is given in [14]. We note in particular that nested loops can be implemented as recursive functions. The algorithm is optimal, in the sense exactly the nodes belonging to $\mathscr{P}_n(\boldsymbol{b}, \boldsymbol{c})$ are touched in the course of the enumeration. Since action $\boldsymbol{x} \mapsto A_n \boldsymbol{x}$ has complexity $\mathcal{O}(d \log(d))$, it is safe to say the algorithm has complexity $\mathcal{O}(d \log(d) \#(\mathscr{P}_n(\boldsymbol{b}, \boldsymbol{c}))$. We must also draw attention on the optimality in memory usage. One only alters the fixed "data-structures" encoding $\boldsymbol{X}$, $\boldsymbol{B}$, $\boldsymbol{C}$ during the execution of the algorithm.

Enumerations of dual CF-lattices $V_n \mathbb{Z}^{2^n}$ in axis-parallel boxes is equally disposed to sequential enumeration. Indeed, the discussed algorithm can be applied with matrices B_n. The sole difference is changing D_m by D_m^{-1} in the functions $\boldsymbol{\phi}_m$, $\boldsymbol{\psi}_m$ and $\boldsymbol{\tau}_m$. Otherwise, we can simply use the same algorithm. Indeed, in view of Remark 1, enumerating $V_n \boldsymbol{k}$ in $[\boldsymbol{b}, \boldsymbol{c}]$ amounts to enumerating $\tilde{V}_n \tilde{\boldsymbol{k}}$ ($\tilde{\boldsymbol{k}} \in \mathbb{Y}$) in $[2\boldsymbol{b}, 2\boldsymbol{c}]$ then normalizing by 2, where $\mathbb{Y} = \mathrm{diag}[2, 1, \dots, 1]\mathbb{Z}^d$. Since $\tilde{V}_n = A_n S_n^\top$ and the leading column/row of S_n is $(1, 0, \dots, 0)^\top \in \mathbb{R}^{2^n}$, then $S_n^\top \mathbb{Y} = \mathbb{Y}$ and $\tilde{V}_n \mathbb{Y} = A_n \mathbb{Y}$ for any $n \geq 1$. One can simply implement the discussed algorithm for $2\boldsymbol{b}$ and $2\boldsymbol{c}$ with two minor modifications (i) in **loop** 0, one iterates over even integers (ii) in **loop** 2^n, one stores nodes $\boldsymbol{x}^{(n)}/2$.

5. Conclusion

We have discussed CF-lattices and dual CF-lattices. Given the families of permutations $\pi_n^{(|)}$, $\pi_n^{(-)}$ as in (12) and (13), the Chebyshev abscissas $\tilde{\xi}_{n,i}$ as in (6) and the polynomials $\tilde{H}_j$ as in (34), we have shown that matrices

$$A_n = (a_{i,j}^{(n)})_{0 \le i,j \le 2^n - 1}, \qquad B_n = (1/a_{i,j}^{(n)})_{0 \le i,j \le 2^n - 1}, \tag{53}$$

with $a_{i,j}^{(n)} := \tilde{H}_{\pi_n^{(-)}(j)}(\tilde{\xi}_{n,\pi_n^{(|)}(i)})$ generate CF-lattices $\tilde{V}_n\mathbb{Z}^d (= A_n\mathbb{Z}^d)$ and dual CF-lattices $V_n\mathbb{Z}^d (= B_n\mathbb{Z}^d)$. Sequential enumeration of these lattices in axis-parallel boxes is computationally optimal. The analysis is founded on a generic algebraic construction, however it also motivates its bypassing for the benefit of direct construction of generating matrices via recurrence.

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