The ring of modular forms for the even unimodular lattice of signature (2,18)

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Abstract. We show that the ring of modular forms with characters for the even unimodular lattice of signature (2,18) is obtained from the invariant ring of \(\text{Sym}(\text{Sym}^3(V) \oplus \text{Sym}^2(V))\) with respect to the action of \(\text{SL}(V)\) by adding a Borcherds product of weight 132 with one relation of weight 264, where \(V\) is a 2-dimensional \(\mathbb{C}\)-vector space. The proof is based on the study of the moduli space of elliptic K3 surfaces with a section.

1. Introduction

Let \(U\) be the even unimodular hyperbolic lattice of rank 2. A \(U\)-polarized K3 surface in the sense of [Nik79] is a pair \((Y, j)\) of a K3 surface \(Y\) and a primitive lattice embedding \(j : U \hookrightarrow \text{Pic} Y\). As explained, e.g., in [Huy], an elliptic K3 surface with a section corresponds naturally to a pseudo-ample \(U\)-polarized K3 surface. Fix a primitive embedding of \(U\) to the K3 lattice \(L = U \oplus U \oplus E_8 \oplus E_8\), which is unique up to the left action of \(O(L)\), and let \(T = U \oplus E_8 \oplus E_8\) be the orthogonal lattice. As explained in [Dol96, Section 3], the global Torelli theorem [PSŠ71, BR75] and the surjectivity of the period map [Tod80] show that the period map gives an isomorphism from the coarse moduli scheme of pseudo-ample \(U\)-polarized K3 surfaces to the quotient \(M := \Gamma \backslash \mathcal{D}\) of the bounded Hermitian domain

\[ \mathcal{D} := \{ [\Omega] \in \mathbb{P}(T \otimes \mathbb{C}) \mid \langle \Omega, \Omega \rangle = 0, \langle \Omega, \mathcal{D} \rangle > 0 \} \]  

(1.1)
of type IV by \(\Gamma := O(T)\).

The moduli space of elliptic K3 surfaces with a section attracts much attention recently, not only from the point of view of modular compactification (see e.g. [AB, ABE] and references therein), but also because of the relation with tropical geometry and mirror symmetry [HU19, OO].

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A modular form on $\mathcal{D}$ with respect to $\Gamma$ of weight $k \in \mathbb{Z}$ and character $\chi \in \text{Char}(\Gamma) := \text{Hom}(\Gamma, \mathbb{C}^\times)$ is a holomorphic function $f: \mathcal{D} \to \mathbb{C}$ on the total space

$$\mathcal{D} := \{ \Omega \in \mathcal{O} \otimes \mathbb{C} | (\Omega, \Omega) = 0, (\Omega, \partial \bar{\Omega}) > 0 \}$$  \hspace{1cm} (1.2)

of a principal $\mathbb{C}^\times$-bundle on $\mathcal{D}$ satisfying

(i) $f(az) = a^{-k}f(z)$ for any $a \in \mathbb{C}^\times$, and

(ii) $f(\gamma z) = \chi(\gamma)f(z)$ for any $\gamma \in \Gamma$.

The vector spaces $A_k(\Gamma, \chi)$ of modular forms constitute the ring

$$\tilde{A}(\Gamma) := \bigoplus_{k=0}^\infty \bigoplus_{\chi \in \text{Char}(\Gamma)} A_k(\Gamma, \chi)$$  \hspace{1cm} (1.3)

of modular forms. We also write the subring of modular forms without characters as

$$A(\Gamma) := \bigoplus_{k=0}^\infty A_k(\Gamma).$$  \hspace{1cm} (1.4)

Let $V := \text{Spec} \mathbb{C}[x, w]$ be a 2-dimensional affine space over $\mathbb{C}$. For $k \in \mathbb{N}$, we write the $k$-th symmetric product of $V$ as $\text{Sym}^k V$. The special linear group $\text{SL}_2$ acts naturally on $S := \text{Sym}^8 V \times \text{Sym}^{12} V$ considered as an affine variety, whose coordinate ring will be denoted by

$$\mathbb{C}[S] = \mathbb{C}[u_{8,0}, u_{7,1}, \ldots, u_{0,8}, u_{12,0}, u_{11,1}, \ldots, u_{0,12}].$$  \hspace{1cm} (1.5)

We let $\mathbb{G}_m$ act on $S$ in such a way that $u_{i,j}$ has weight $(i+j)/2$. This $\mathbb{G}_m$-action commutes with the $\text{SL}_2$-action, so that the invariant subring $\mathbb{C}[S]^{\text{SL}_2}$ has an induced $\mathbb{G}_m$-action.

Building on [Mir81], it is shown in [OO, Theorem 7.9] that the period map induces an isomorphism from $\text{Proj} \mathbb{C}[S]^{\text{SL}_2}$ to the Satake–Baily–Borel compactification of $\Gamma \backslash \mathcal{D}$. As we explain in Section 2, the period map also gives an isomorphism

$$A(\Gamma) \cong \mathbb{C}[S]^{\text{SL}_2}$$  \hspace{1cm} (1.6)

of graded rings.

Note that we have $\text{Char}(\Gamma) = \{ \text{id}, \det \}$ (cf. e.g. [GHS09, Corollary 1.8]). The main result of this paper is the following:

**Theorem 1.** One has

$$\tilde{A}(\Gamma) \cong (\mathbb{C}[S]^{\text{SL}_2})[s_{132}]/(s_{132}^2 - A_{264}),$$  \hspace{1cm} (1.7)
where $s_{132}$ is an element of weight 132 and $A_{264} \in \mathbb{C}[S]^{SL_2}$ is an element of weight 264.

The proof is based on the construction of an algebraic stack which is isomorphic to the orbifold quotient $[O(T) \backslash \mathscr{D}]$ in codimension 1. The same strategy has been used in [HU] and [NU] to determine the rings of modular forms with characters for the lattices $\mathbb{U} \perp \mathbb{U} \perp \mathbb{E}_8$ and $\mathbb{U} \perp \mathbb{U} \perp \mathbb{A}_1 \perp \mathbb{A}_1$, respectively.

The modular form $s_{132}$ is constructed in [FSM07, Lemma 5.1]. It can also be obtained either as the quasi pull-back [GHS13, Theorem 8.2] of the Borcherds form $F_{12}$ associated with the even unimodular lattice of signature $(2,26)$ [Bor95, Section 10, Example 2], or by applying [Bor95, Theorem 10.1] to the nearly holomorphic modular form

$$\frac{1728E_4}{E_4^3 - E_6^2} = \frac{1}{q} + 264 + 8244q + 139520q^2 + \cdots, \quad (1.8)$$

where

$$E_4 = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n} = 1 + 240q + 2160q^2 + \cdots, \quad (1.9)$$

$$E_6 = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n} = 1 - 504q - 6632q^2 + \cdots. \quad (1.10)$$

In particular, it is a cusp form with character $\det$ admitting an infinite product expansion. See also [DKW19, Section 5] and references therein for the case of the even unimodular lattice of signature $(2,10)$.

Since $SL_2$ is reductive, the invariant ring $\mathbb{C}[S]^{SL_2}$ is finitely generated, and there exists an algorithm for computing a finite generating set (see e.g. [Stu08] and references therein). The element $A_{264}$ can also be computed algorithmically, and it is an interesting problem to describe them explicitly.

2. The coarse moduli space of U-polarized K3 surfaces

As is well known (cf. e.g. [SS10, Section 4]), a U-polarized K3 surface admits a Weierstrass model of the form

$$z^2 = y^3 + g_2(x, w; u)y + g_3(x, w; u) \quad (2.1)$$

in $\mathbb{P}(1,4,6,1)$, where

$$g_2(x, w; u) = \sum_{i=0}^{8} u_{8-i,i} x^{8-i} w^i \quad (2.2)$$

$$= u_{8,0} x^8 + u_{7,1} x^7 w + \cdots + u_{0,8} w^8, \quad (2.3)$$
\[ g_3(x, w; u) = \sum_{i=0}^{12} u_{12-i} x^{12-i} w^i \]

for

\[ u = ((u_{8,0}, \ldots, u_{0,8}), (u_{12,0}, \ldots, u_{0,12})) \in S. \]

The hypersurface in \( \mathbb{P}(1, 4, 6, 1) \) defined by (2.1) has a singularity worse than rational double points on the fiber at \( a \in \mathbb{P}^1 \) if and only if \( \text{ord}_a(g_2) \geq 4 \) and \( \text{ord}_a(g_3) \geq 6 \) (see e.g. [Mir89, Proposition III.3.2]). Let \( U \subset S \) be the open subscheme parametrizing hypersurfaces with at worst rational double points.

The parameter \( u \) describing a given \( U \)-polarized K3 surface is unique up to the action of \( \text{SL}_2 \times \mathbb{G}_m \), where \( \mathbb{G}_m \) acts on \( \mathbb{P}(1, 4, 6, 1) / \text{Sym}_8 V \times \text{Sym}^{12} V \) by

\[ \mathbb{G}_m \ni \lambda : ((x, y, z, w), (u_{i,j})) \mapsto ((x, \lambda^2 y, \lambda^3 z, w), (\lambda^{i+j}/u_{i,j})) \]

rescaling the holomorphic volume form

\[ \Omega = \text{Res} \frac{w \, dx \wedge dy \wedge dz}{z^2 - y^3 - g_2(x, w; u) y - g_3(x, w; u)} \]

as

\[ \Omega_{\lambda u} = \text{Res} \frac{w \, dx \wedge d(\lambda^2 y) \wedge d(\lambda^3 z)}{(\lambda^3 z)^2 - (\lambda^2 y)^3 - g_2(x, w; \lambda \cdot u) (\lambda^2 y) - g_3(x, w; \lambda \cdot u)} = \lambda^{-1} \Omega_u. \]

The categorical quotient \( T := U / \text{SL}_2 \) is the coarse moduli scheme of pairs \((Y, \Omega)\) consisting of a \( U \)-polarized K3 surface \( Y \) and a holomorphic volume form \( \Omega \in H^0(\omega_Y) \) on \( Y \). The fact that the codimension of \( S \setminus U \) is greater than 2 implies an isomorphism

\[ \mathbb{C}[S]^{\text{SL}_2} \cong \mathbb{C}[T] \]

of graded rings. Since the character of \( \mathbb{C}[S] \) as a \( \text{SL}_2 \times \mathbb{G}_m \)-module is given by

\[ \prod_{i=0}^{8} (1 - q^{2i-8} t^4)^{-1} \prod_{i=0}^{12} (1 - q^{2i-12} t^6)^{-1}, \]

the Hilbert series of the invariant ring is given by
\[ \sum_{i=0}^{\infty} \dim(C[S]^{SL_2}, t^i) = \text{Res}_{q=0} \left( (q^{-1} - q) \prod_{i=0}^{8} (1 - q^{2i-8} t^4)^{-1} \prod_{i=0}^{12} (1 - q^{2i-12} t^6)^{-1} \right) \]  

(2.12)

as explained, e.g., in [Muk03, Section 4.4]. It follows from the global Torelli theorem and the surjectivity of the period map that the period map induces a ring isomorphism

\[ A(G) \cong C[T], \]  

(2.13)

which preserves the grading by (2.9). The isomorphism (1.6) follows from (2.10) and (2.13).

3. Modular forms with characters

The coarse moduli space \( M \) of \( U \)-polarized K3 surfaces is an open subvariety of its Satake–Baily–Borel compactification \( \text{Proj} A(G) \cong \mathbb{P}(4^9, 6^{13})/\text{SL}_2 \). Although \( M = \Gamma \backslash \mathcal{D} \) and the orbifold quotient \( \mathcal{M} := [\Gamma \backslash \mathcal{D}] \) are closely related, the canonical morphism \( \mathcal{M} \to M \) is not an isomorphism even in codimension 1. In order to obtain an orbifold which is isomorphic to \( \mathcal{M} \) in codimension 1 (so that the total coordinate rings are isomorphic), consider the stack

\[ \mathbb{P} := [\mathbb{P}(4^9, 6^{13})/\text{SL}_2], \]  

(3.1)

declared as the quotient of \( C^{22} \setminus 0 \) by the action of \( \text{SL}_2 \times \text{G}_m \). The morphism \( \mathcal{M} \to M \) lifts to a morphism \( \mathcal{M} \to \mathbb{P} \), which is an isomorphism in codimension 0, since the generic stabilizers are \( \{ \pm \text{id} \} \) on both sides.

Stabilizers of \( \mathcal{M} \) along divisors come from reflections. One divisor with a generic stabilizer comes from the reflection with respect to a \( (\pm C_0) \)-vector whose reflection hyperplane corresponds to the locus where the Picard lattice contains \( U \perp A_1 \). In order to describe this locus, first consider the discriminant

\[ h(x, w; u) := 4g_2(x, w; u)^3 + 27g_3(x, w; u)^2 \]  

(3.2)

of \( y^3 + g_2(x, w; u)y + g_3(x, w; u) \) as a polynomial in \( y \), which is homogeneous of degree 24 in \( (x, w) \) and degree 12 in \( u \). Note that the discriminant of a polynomial \( \sum_{i=0}^{n} a_i x^i w^{n-i} \) with respect to \( (x, w) \) is homogeneous of degree \( 2(n-1) \) in \( \mathbb{Z}[a_0, \ldots, a_n] \) if \( \deg a_0 = \cdots = \deg a_n = 1 \). It follows that the discriminant \( k_{552}(u) \) of \( h(x, w; u) \) with respect to \( (x, w) \) is a homogeneous polynomial of degree \( 2 \cdot 23 \cdot 12 = 552 \) in \( u \). A general point on the divisor \( D_{552} \) of \( \mathbb{P} \) defined by \( k_{552}(u) \) corresponds to the locus where two fibers of Kodaira type \( I_1 \) collapse into one fiber. This divisor has two components; a general point
on one component corresponds to the case when there exists a point \( p = [x : w] \) on \( \mathbb{P}^1 \) such that neither \( g_2 \) nor \( g_3 \) vanishes at \( p \), and a general point on the other component corresponds to the case when both \( g_2 \) and \( g_3 \) vanishes at \( p \). In the former case, the resulting singular fiber is of Kodaira type \( I_2 \), and the surface acquires an \( A_1 \)-singularity. In the latter case, the resulting singular fiber is of Kodaira type \( I_1 \), and the surface does not acquire any new singularity. The defining equation of the latter component is the resultant of \( g_2 \) and \( g_3 \). It is given as the determinant

\[
\begin{vmatrix}
  u_{8,0} & u_{7,1} & \cdots & u_{0,8} \\
  u_{8,0} & \cdots & \cdots & u_{1,7} & u_{0,8} \\
  \vdots & \vdots & \ddots & \vdots & \ddots \\
  u_{12,0} & u_{11,1} & \cdots & \cdots & u_{1,11} & u_{0,12} \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \ddots \\
  u_{12,0} & u_{11,1} & \cdots & \cdots & u_{1,11} & u_{0,12}
\end{vmatrix}
\]  

(3.3)
of the Sylvester matrix, which is homogeneous of degree

\[
12 \times 4 + 8 \times 6 = 96.
\]  

(3.4)
As shown in [HU, Lemma 6.1], the polynomial \( k_{552}(u) \) is divisible by \( r_{96}(u)^3 \), and the quotient

\[
A_{264}(u) := k_{552}(u)/r_{96}(u)^3
\]  

(3.5)
defines the reflection hyperplane along a \((-2)\)-vector.

Recall from [AGV08, Cad07] that the root construction is an operation which adds a stabilizer along a divisor. Let \( \mathbb{T} \) be the stack obtained from \( \mathbb{P} \) by the root construction of order 2 along the divisor on \( \mathbb{P} \) defined by \( A_{264}(t) \), which is the quotient of the double cover of \( \mathbb{P} \) branched along \( A_{264}(t) \) by the group \( G \) of deck transformations. The Picard group of \( \mathbb{T} \) (or the \( G \)-equivariant Picard group of \( \mathbb{P} \)) is generated by the pull-back \( \mathcal{C}_\mathbb{T}(1) := p^*\mathcal{C}_\mathbb{P}(1) \) of the generator \( \mathcal{C}_\mathbb{P}(1) \) of the Picard group of \( \mathbb{P} \) by the structure morphism \( p : \mathbb{T} \to \mathbb{P} \) and the line bundle \( \mathcal{C}_\mathbb{T}(D_{132}) \) such that the space \( H^0(\mathcal{C}_\mathbb{T}(D_{132})) \) is generated by an element \( s_{132} \) satisfying \( s_{132}^2 = A_{264} \in H^0(\mathcal{C}_\mathbb{T}(264)) \equiv H^0(\mathcal{C}_\mathbb{P}(264)) \). Note also that \( \omega_\mathbb{P} \cong \mathcal{C}_\mathbb{P}(a) \), where

\[
a = -\sum_{i=0}^{8} \deg u_{8-i,i} - \sum_{i=0}^{12} \deg u_{12-i,i} = -9 \times 4 - 13 \times 6 = -114.
\]  

(3.6)
The ramification formula for the canonical bundle gives

\[ \omega_T \cong p^* \omega_P \otimes \mathcal{O}_T(D_{132}) \]  
(3.7)

\[ \cong \mathcal{O}_T(-114) \otimes \mathcal{O}_T(132 + (-132 + D_{132})) \]  
(3.8)

\[ \cong \mathcal{O}_T(18) \otimes \mathcal{O}_T(-132 + D_{132}). \]  
(3.9)

Note that \( \mathcal{O}_T (-132 + D_{132}) \) is an element of order two in \( \text{Pic} \, T \). By comparing (3.9) with

\[ \omega_M \cong \mathcal{O}_M(\text{dim} \, M) \otimes \text{det} = \mathcal{O}_M(18) \otimes \text{det} \]  
(3.10)

which follows from (the proof of) [HU, Proposition 5.1], one concludes that \( M \) has no further stabilizer along a divisor, so that the lift \( M \to T \) of \( M \to P \) is an isomorphism in codimension 1. It follows that the injective map \( \mathbb{Z} \times \text{Char}(\Gamma) \to \text{Pic} \, M, \, (i, \chi) \mapsto \mathcal{O}_M(i) \otimes \chi \) is surjective, and the total coordinate ring (also known as the Cox ring) of \( M \) is given by

\[ \bigoplus_{\mathcal{L} \in \text{Pic} \, M} H^0(\mathcal{L}) \cong \bigoplus_{i=0}^{\infty} H^0(\mathcal{O}_M(i)) / (s^2_{132} - A_{264}(i)). \]  
(3.11)

References


[FSM07] Eberhard Freitag and Riccardo Salvati Manni, Modular forms for the even modular lattice of signature \( (2,10) \), J. Algebraic Geom. 16 (2007), no. 4, 753–791. MR 2357689
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