Hiroshima Math. J., **51** (2021), 155–161
doi:10.32917/h2020073

CFA modules and the finiteness of coassociated primes of local homology modules

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(Received August 7, 2020)
(Revised January 15, 2021)

Abstract. We introduce the concept CFA modules and their applications in investigation the coassociated primes of local homology modules. The main result of this paper says that if M is a CFA linearly compact R-module and t is a non-negative integer such that $H_i^I(M)$ is CFA for all $i < t$, then $R/I \otimes_R H_t^I(M)$ is CFA. Hence, the set $\mathrm{Coass}_R\, H_t^I(M)$ is finite.

1. Introduction

In this paper, R is a Noetherian commutative ring with identity, I is an ideal of R and M is an R-module. The theory of local homology was initiated by Matlis [8] in 1974. This theory was studied and improved by Simon [13], Greenlees and May [5], Tarrío, López and Lipman [1]. In [2], Cuong and Nam defined the local homology modules $H_i^I(M)$ of M with respect to I by

$$H_i^I(M) = \varprojlim_t \mathrm{Tor}_i^R(R/I^t, M).$$

This concept in some sense is dual to Grothendieck's definition of local cohomology modules. Also, this definition of local homology modules coincides with the definition of Greenlees and May [5] when M is a linearly compact R-module. We recall the concept of linearly compact modules by using the terminology of Macdonald [6]. A topological module M over a topological ring R is said to be linearly topologized if M has a nuclear base $\mathscr{M}$ consisting of open submodules which satisfies the condition: given $x \in M$ and $N \in \mathscr{M}$, there exists a nucleus U of R such that $Ux \subseteq N$. A Hausdorff linearly topologized R-module M is said to be linearly compact if M has the following property: if $\mathscr{F}$ is a family of closed cosets (i.e., the cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathscr{F}$ have a non-empty intersection. A Hausdorff linearly topologized R-module M is

2010 *Mathematics Subject Classification*. Primary 13D07, 13E05; Secondary 13D99.
Key words and phrases. Coassociated primes, cosupport, local homology.

called semi-discrete if every submodule of M is closed. If M is an Artinian R-module, then M is semi-discrete linearly compact.

In [14], an R module L is called *cocyclic* if it is a submodule of the injective hull $E(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$ of R. Yassemi defined the set of *coassociated prime ideals* (respectively the *cosupport*) of an R-module M, and denoted by $\operatorname{Coass}_R M$ (respectively $\operatorname{Cosupp}_R M$) to be the set of prime ideals $\mathfrak{p}$ such that there exists a cocyclic homomorphic image L of M with $\operatorname{Ann}_R L = \mathfrak{p}$ (respectively $\operatorname{Ann}_R L \subseteq \mathfrak{p}$). Thus $\operatorname{Coass}_R M \subseteq \operatorname{Cosupp}_R M$ and if M is an Artinian R-module, then $\operatorname{Coass}_R M$ is a finite set. If $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules, then

$$\operatorname{Coass}_R C \subseteq \operatorname{Coass}_R B \subseteq \operatorname{Coass}_R A \cup \operatorname{Coass}_R C$$

and

$$\operatorname{Cosupp}_R B = \operatorname{Cosupp}_R A \cup \operatorname{Cosupp}_R C.$$

Let N be a finitely generated R-module and M an R-module. Then

$$\operatorname{Coass}_R(N \otimes_R M) = \operatorname{Supp}_R N \cap \operatorname{Coass}_R M.$$

It follows from [3, Corollary 3.7] and [9, Theorem 3.8] that if M is a linearly compact R-module, then $\operatorname{Coass}_R H_i^I(M) \subseteq V(I)$ for all $i \geq 0$.

An interesting problem in commutative algebra is determining when the set of associated primes of the ith local cohomology module $H_I^i(M)$ of M is finite. There is a similar question: when is the set of coassociated primes of a local homology module finite? In [3, Theorem 4.5], Cuong and Nam showed that if M is a semi-discrete linearly compact R-module and the local homology R-module $H_i^I(M)$ is Artinian for all $0 \neq i < t$, then the set $\operatorname{Coass}_R H_t^I(M)$ is finite. In [10], Nam introduced the concept of modules satisfying the finiteness condition for coassociated primes if the set of coassociated primes of any its submodule is finite. He proved that if M is a linearly compact R-module and local homology module $H_i^I(M)$ satisfies the finiteness condition for coassociated primes for all $i < t$, then the set $\operatorname{Coass}_R H_t^I(M)$ is finite. He also proved in [11] that if M is a semi-discrete linearly compact R-module such that M/IM is Artinian and $H_i^I(M)$ is minimax for all $i < t$, then the set $\operatorname{Coass}_R H_t^I(M)$ is finite. Mafi and Saremi [7] showed that if M is an Artinian R-module and $\operatorname{Cosupp}_R H_i^I(M)$ is finite for all $i < t$, then $\operatorname{Coass}_R H_t^I(M)$ is finite.

The aim of this paper is to introduce the concept CFA modules and their applications in investigation the coassociated primes of local homology modules. The main result of this paper is Theorem 1 which says that if M is a CFA linearly compact R-module and t is a non-negative integer such that

$H_i^I(M)$ is CFA for all $i < t$, then $R/I \otimes_R H_t^I(M)$ is CFA. Thus, the set $\mathrm{Coass}_R\, H_t^I(M)$ is finite.

2. Main results

Definition 1. An R-module M is called CFA if there is a submodule N such that $\mathrm{Cosupp}_R\, N$ is a finite set and M/N is an Artinian R-module.

Example 1. (i) The class of CFA modules contains the class of Artinian modules.

(ii) All R-modules with finite cosupport are CFA.

(iii) Let M be a finitely generated R-module. If R is a semi-local ring or $\mathrm{Supp}_R\, M$ is a finite set, then M is a CFA R-module by [14, Theorem 2.10].

(iv) In [17], an R-module M is called minimax if it has a finitely generated R-module N such that M/N is Artinian. Hence, if M is a minimax module over a semi-local ring, then M is CFA.

(v) In [16, Theorem], we see that semi-discrete linearly compact modules are minimax modules. Therefore, if M is a semi-discrete linearly compact module over a semi-local ring, then M is CFA.

(vi) Let $(R, \mathfrak{m})$ be a local ring and $E_R(R/\mathfrak{m})$ be the injective hull of $R/\mathfrak{m}$. Let $\hat{R}$ denote the $\mathfrak{m}$-adic completion of R and $M = \hat{R} \oplus E_R(R/\mathfrak{m})$. We will prove that M is a CFA module. It is sufficient to show that $\mathrm{Cosupp}_R\, \hat{R}$ is a finite set. It follows from [14, Theorem 3.2(a)] that

$$\mathrm{Cosupp}_R\, \hat{R} = \mathrm{Supp}_R\, \mathrm{Hom}_R(\hat{R}, E_R(R/\mathfrak{m})).$$

On the other hand, one sees that $\hat{R} \cong \mathrm{Hom}_R(E_R(R/\mathfrak{m}), E_R(R/\mathfrak{m}))$ and $E_R(R/\mathfrak{m})$ is an Artinian R-module. By [12, Theorem 1.6 (5)], we have $\mathrm{Hom}_R(\hat{R}, E_R(R/\mathfrak{m})) \cong E_R(R/\mathfrak{m})$. This implies that $\mathrm{Cosupp}_R\, \hat{R} = \{\mathfrak{m}\}$ is a finite set.

The following results provide us some properties of CFA modules. It should be noted that a Serre subcategory of the category of R-modules is a class of R-modules which is closed under taking submodules, quotient modules and extensions.

Proposition 1. *The following statements hold:*

(i) *If M is a CFA R-module, then* $\mathrm{Coass}_R\, M$ *is a finite set.*

(ii) *The class of CFA R-modules is a Serre subcategory of the category of R-modules.*

(iii) *Let N be a finitely generated R-module and M a CFA R-module. Then* $\operatorname{Ext}_R^i(N, M)$ *and* $\operatorname{Tor}_i^R(N, M)$ *are CFA for all* $i \geq 0$.

Proof. (i) By [14, Lemma 1.22], the set of coassociated prime ideals of an Artinian module is finite. Hence, the assertion follows from the definition.

(ii) First, we see that the class of R-modules with finite cosupport is a Serre subcategory of the category of R-modules. On the other hand, the class of Artinian R-modules is a Serre subcategory which is closed under injective hulls. It follows from [15, Corollary 3.5] that the class of CFA R-modules is a Serre subcategory of the category of R-modules.

(iii) We only prove the claim for the Ext modules, and the proof for the Tor modules is similar. Since R is a Noetherian ring and N is a finitely generated R-module, the module N has a free resolution

$$\mathbf{F}: \quad \cdots F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0,$$

where F_i is finitely generated free for all $i \geq 0$. Consequently, for each non-negative integer i, there is a positive integer t such that $\operatorname{Hom}_R(F_i, M) = \bigoplus^t M$. We know that $\operatorname{Ext}_R^i(N, M) = H^i(\operatorname{Hom}_R(\mathbf{F}, M))$ which is a subquotient of the CFA module $\operatorname{Hom}_R(F_i, M)$. Hence, the assertion follows from (ii). □

We see in [14, Definition 1.12] that if M is an R-module, then the subset $\mathrm{w}(M)$ of R is defined by

$$\mathrm{w}(M) = \{a \in R \mid M \xrightarrow{.a} M \text{ is not surjective}\}.$$

Lemma 1. *Let I be an ideal of R and N a CFA R-module. The following statements are equivalent:*

(i) *There is an element $x \in I$ such that $xN = N$;*

(ii) *$N = IN$.*

Proof. (i) ⇒ (ii). We have $N = xN \subseteq IN$ as $x \in I$. Hence, one asserts that $IN = N$.

(ii) ⇒ (i). By [14, Theorem 1.13], there is an equality

$$\mathrm{w}(N) = \bigcup_{\mathfrak{p} \in \operatorname{Coass}_R N} \mathfrak{p}.$$

Suppose that $N \neq xN$ for any $x \in I$. Then $I \subseteq \mathrm{w}(N)$. Proposition 1 (i) shows that $\operatorname{Coass}_R N$ is a finite set. Therefore, one gets that $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Coass}_R N$. Let K be a submodule of N such that N/K is Artinian and $\mathfrak{p} = \operatorname{Ann}(N/K)$ by [14, Theorem 1.6]. Consequently, one obtains that $IN \subset \mathfrak{p}N \subseteq K \subsetneq N$, which is a contradition. □

Now, we are going to state and prove the main result of this paper.

Theorem 1. *Let M be a linearly compact R-module and t a non-negative integer. If M and $H_i^I(M)$ are CFA for all $i < t$, then*

$$R/I \otimes_R H_t^I(M)$$

is CFA. Thus, $\operatorname{Coass}_R H_t^I(M)$ *is finite.*

Proof. The proof is by induction on t. Let $t = 0$. The short exact sequence

$$0 \to \bigcap_{i>0} I^i M \to M \to H_0^I(M) \to 0$$

induces the following exact sequence

$$R/I \otimes_R \bigcap_{i>0} I^i M \to R/I \otimes_R M \to R/I \otimes_R H_0^I(M) \to 0.$$

Proposition 1 shows that $R/I \otimes_R H_0^I(M)$ is CFA.

Let $t > 0$. Since $H_t^I(M) \cong H_t^I(\bigcap_{i>0} I^i M)$ by [4, Lemma 3.10], we can replace M by $N := \bigcap_{i>0} I^i M$. It is clear that $IN = N$. Hence, there is an element $x \in I$ such that $xN = N$ by Lemma 1. The short exact sequence

$$0 \to 0 :_N x \to N \xrightarrow{x} N \to 0$$

induces a long exact sequence

$$\cdots \to H_t^I(N) \xrightarrow{x} H_t^I(N) \xrightarrow{\alpha} H_{t-1}^I(0 :_N x) \xrightarrow{\beta} H_{t-1}^I(N) \to \cdots \to H_0^I(N) \to 0.$$

By the assumption, $H_i^I(0 :_N x)$ is CFA for all $i < t - 1$. The exact sequences

$$H_t^I(N) \xrightarrow{x} H_t^I(N) \to \operatorname{Im} \alpha \to 0$$

and

$$0 \to \operatorname{Im} \alpha \to H_{t-1}^I(0 :_N x) \to \operatorname{Im} \beta \to 0$$

lead the isomorphism

$$R/I \otimes_R H_t^I(N) \cong R/I \otimes_R \operatorname{Im} \alpha$$

and the long exact sequence

$$\operatorname{Tor}_1^R(R/I, \operatorname{Im} \beta) \to R/I \otimes_R \operatorname{Im} \alpha \to R/I \otimes_R H_{t-1}^I(0 :_N x).$$

It follows from Proposition 1 that $\operatorname{Im} \beta$ is CFA. Hence, so is $\operatorname{Tor}_1^R(R/I, \operatorname{Im} \beta)$. The inductive hypothesis shows that $R/I \otimes_R H_{t-1}^I(0 :_N x)$ is a CFA R-module. Therefore, one gets that $R/I \otimes_R \operatorname{Im} \alpha$ is a CFA R-module and this completes the proof. □

Corollary 1. *Let M be a linearly compact R-module and t a non-negative integer. If M is a CFA R-module and $H_i^I(M)$ is Artinian or* $\operatorname{Cosupp}_R H_i^I(M)$ *is finite for all $i < t$, then* $\operatorname{Coass}_R H_t^I(M)$ *is a finite set.*

Corollary 2. *Let M be an Artinian R-module and t a non-negative integer. If $H_i^I(M)$ is Artinian or* $\operatorname{Cosupp}_R H_i^I(M)$ *is finite for all $i < t$, then* $\operatorname{Coass}_R H_t^I(M)$ *is a finite set.*

In [10], an R-module M satisfies the finite condition for coassociated primes if the set of coassociated primes of any submodule of M is finite.

Remark 1. *A submodule of a CFA module is CFA, since the class of CFA modules is a Serre subcategory (Proposition* 1(ii)*) and a Serre subcategory is closed under taking submodules. This yields that a CFA module satisfies the finite condition for coassociated primes.*

Hence, the main result of this paper (Theorem 1) is a corollary of [10, Theorem 3.1].

Acknowledgement

The author is deeply grateful to the referee for a very careful reading of the manuscript and many valuable suggestions and for drawing the author's attention to Remark 1.

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