Generic distance-squared mappings on plane curves

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Abstract. A distance-squared function is one of the most significant functions in the application of singularity theory to differential geometry. Moreover, distance-squared mappings are naturally extended mappings of distance-squared functions, wherein each component is a distance-squared function. In this paper, compositions of a given plane curve and generic distance-squared mappings on the plane into the plane are investigated from the viewpoint of stability.

1. Introduction

Throughout this paper, let \( l \) and \( n \) stand for positive integers. In this paper, unless otherwise stated, all manifolds and mappings belong to class \( C^\infty \) and all manifolds are without boundary. Let \( q = (q_1, \ldots, q_n) \in \mathbb{R}^n \) be a given point. The mapping \( d_q : \mathbb{R}^n \to \mathbb{R} \) defined by
\[
d_q(x) = \sum_{i=1}^{n} (x_i - q_i)^2
\]
is called a distance-squared function, where \( x = (x_1, \ldots, x_n) \). In [5], the following notion is investigated.

Definition 1. Let \( p_1, \ldots, p_{\ell} \) be \( \ell \) given points in \( \mathbb{R}^n \). Set \( p = (p_1, \ldots, p_{\ell}) \in (\mathbb{R}^n)^{\ell} \). The mapping \( D_p : \mathbb{R}^n \to \mathbb{R}^\ell \) defined by
\[
D_p = (d_{p_1}, \ldots, d_{p_{\ell}})
\]
is called a distance-squared mapping.

We have the following motivation for investigating distance-squared mappings. Height functions and distance-squared functions have been investigated in detail so far, and they are useful tools in the applications of singularity theory to differential geometry (see [1]). A mapping in which each
component is a height function is nothing but a projection. Projections as well as height functions or distance-squared functions have been investigated so far. For example, in [6] (resp., [2]), compositions of generic projections and embeddings (resp., stable mappings) are investigated from the viewpoint of stability (for the definition of stability, refer to [3]). On the other hand, a mapping in which each component is a distance-squared function is a distance-squared mapping. Therefore, it is natural to investigate distance-squared mappings as well as projections.

In this paper, compositions of a given plane curve and generic distance-squared mappings on the plane into the plane are investigated from the viewpoint of stability.

A mapping \( f : \mathbb{R}^n \to \mathbb{R}^l \) is said to be \( \mathcal{A} \)-equivalent to a mapping \( g : \mathbb{R}^n \to \mathbb{R}^l \) if there exist diffeomorphisms \( \varphi : \mathbb{R}^n \to \mathbb{R}^n \) and \( \psi : \mathbb{R}^l \to \mathbb{R}^l \) such that \( \psi \circ f \circ \varphi^{-1} = g \). For given points \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n \), set

\[
\overline{xy} = (y_1 - x_1, \ldots, y_n - x_n).
\]

Given \( \ell \) points \( p_1, \ldots, p_\ell \in \mathbb{R}^n \) (1 \( \leq \ell \leq n + 1 \)) are said to be in general position if \( p_1, p_2, \ldots, p_{\ell+1} \) (2 \( \leq \ell \leq n + 1 \)) are linearly independent.

In [5], a characterization of distance-squared mappings is given as follows:

**Proposition 1 ([5]).** (1) Let \( \ell, n \) be integers such that 2 \( \leq \ell \leq n \), and let \( p_1, \ldots, p_\ell \in \mathbb{R}^n \) be in general position. Then, \( D_p : \mathbb{R}^n \to \mathbb{R}^l \) is \( \mathcal{A} \)-equivalent to the mapping defined by \( (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{\ell-1}, x_\ell^2 + \cdots + x_n^2) \).

(2) Let \( \ell, n \) be integers such that 1 \( \leq n < \ell \), and let \( p_1, \ldots, p_\ell \in \mathbb{R}^n \) be \( \ell \) points such that \( p_1, \ldots, p_{n+1} \) are in general position. Then, \( D_p : \mathbb{R}^n \to \mathbb{R}^l \) is \( \mathcal{A} \)-equivalent to the inclusion \( (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0, \ldots, 0) \).

In the following, by \( N \), we denote a manifold of dimension 1. A mapping \( f : N \to \mathbb{R}^2 \) is called a mapping with normal crossings if the mapping \( f \) satisfies the following conditions.

(1) For any \( y \in \mathbb{R}^2 \), \( |f^{-1}(y)| \leq 2 \), where \( |A| \) is the number of elements of the set \( A \).

(2) For any two distinct points \( q_1, q_2 \in N \) satisfying \( f(q_1) = f(q_2) \), we have \( \dim(df_{q_1}(T_{q_1}N) + df_{q_2}(T_{q_2}N)) = 2 \).

From Corollary 8 in [4], we have the following.

**Proposition 2 ([4]).** Let \( \gamma : N \to \mathbb{R}^2 \) be an injective immersion, where \( N \) is a manifold of dimension 1. Then, the set

\[
\{ p \in \mathbb{R}^2 \times \mathbb{R}^2 \mid D_p \circ \gamma : N \to \mathbb{R}^2 \text{ is an immersion with normal crossings} \}
\]

is dense in \( \mathbb{R}^2 \times \mathbb{R}^2 \).
On the other hand, the purpose of this paper is to investigate whether the set
\[ \{ p \in \gamma(N) \times \gamma(N) \mid D_p \circ \gamma : N \to \mathbb{R}^2 \text{ is an immersion with normal crossings} \} \]
is dense in \( \gamma(N) \times \gamma(N) \) or not. Here, note that \( O \) is an open set of \( \gamma(N) \times \gamma(N) \) if there exists an open set \( O' \) of \( \mathbb{R}^2 \times \mathbb{R}^2 \) satisfying \( O = O' \cap (\gamma(N) \times \gamma(N)) \).

Let \( \gamma : N \to \mathbb{R}^2 \) be an immersion. We say that \( \kappa : U \to \mathbb{R} \) is called the \textit{curvature} of \( \gamma \) on a coordinate neighborhood \((U, t)\) of \( N \) if
\[
\kappa(t) = \det \begin{pmatrix}
\frac{d\gamma_1}{dt}(t) & \frac{d^2\gamma_1}{dt^2}(t) \\
\frac{d\gamma_2}{dt}(t) & \frac{d^2\gamma_2}{dt^2}(t)
\end{pmatrix}
\left(\left(\frac{d\gamma_1}{dt}(t)\right)^2 + \left(\frac{d\gamma_2}{dt}(t)\right)^2\right)^{3/2},
\]
where \( \gamma = (\gamma_1, \gamma_2) \). Note that for a given point \( q \in N \), whether \( \kappa(q) = 0 \) or not does not depend on the choice of a coordinate neighborhood.

**Definition 2.** Let \( N \) be a manifold of dimension 1. We say that an immersion \( \gamma : N \to \mathbb{R}^2 \) satisfies \((\ast)\) if for any non-empty open set \( U \) of \( N \), there exists a point \( q \in U \) satisfying \( \kappa(q) \neq 0 \), where \( \kappa \) is the curvature of \( \gamma \) on a coordinate neighborhood around \( q \).

The main result in this paper is the following.

**Theorem 1.** Let \( \gamma : N \to \mathbb{R}^2 \) be an injective immersion satisfying \((\ast)\), where \( N \) is a manifold of dimension 1. Then, the set
\[ \{ p \in \gamma(N) \times \gamma(N) \mid D_p \circ \gamma : N \to \mathbb{R}^2 \text{ is an immersion with normal crossings} \} \]
is dense in \( \gamma(N) \times \gamma(N) \).

If we drop the hypothesis \((\ast)\) in Theorem 1, then the conclusion of Theorem 1 does not necessarily hold (see Examples 1 and 2 in Section 2).

In Theorem 1, if the mapping \( D_p \circ \gamma : N \to \mathbb{R}^2 \) is proper, then the immersion with normal crossings \( D_p \circ \gamma : N \to \mathbb{R}^2 \) is necessarily stable (see [3], p. 86). Thus, from Theorem 1, we get the following.

**Corollary 1.** Let \( N \) be a compact manifold of dimension 1. Let \( \gamma : N \to \mathbb{R}^2 \) be an embedding satisfying \((\ast)\). Then, the set
\[ \{ p \in \gamma(N) \times \gamma(N) \mid D_p \circ \gamma : N \to \mathbb{R}^2 \text{ is stable} \} \]
is dense in \( \gamma(N) \times \gamma(N) \).
In Section 2, Examples 1 and 2 are given. In Section 3, preliminaries for the proof of Theorem 1 are given. Section 4 is devoted to the proof of Theorem 1.

2. Dropping the hypothesis (\(\ast\)) in Theorem 1

In this section, we will give two examples such that Theorem 1 without the hypothesis (\(\ast\)) does not hold (see Examples 1 and 2).

Firstly, we prepare the following proposition, which is used in Example 1.

**Proposition 3.** Let \(g : N \to \mathbb{R}^2\) be a mapping, where \(N\) is a manifold of dimension 1. Let \(p_1, p_2\) be two points of \(\mathbb{R}^2\). Then, a point \(q \in A(N)\) is a singular point of the mapping \(D_p \circ \gamma : N \to \mathbb{R}^2\) \((p = (p_1, p_2))\) if and only if

\[
\frac{p_1\gamma(q)}{dt}(q) \cdot \frac{d\gamma}{dt}(q) = 0 \quad \text{and} \quad \frac{p_2\gamma(q)}{dt}(q) \cdot \frac{d\gamma}{dt}(q) = 0,
\]

where \(t\) is a local coordinate around the point \(q\) and ‘·’ stands for the inner product in \(\mathbb{R}^2\), that is, \(p_1\) and \(p_2\) are on the line normal to the curve \(\gamma(N)\) at \(\gamma(q)\).

**Proof.** Let \(q\) be a point of \(N\). The composition of \(\gamma : N \to \mathbb{R}^2\) and \(D_p : \mathbb{R}^2 \to \mathbb{R}^2\) is given as follows:

\[
D_p \circ \gamma(q) = ((\gamma_1(q) - p_{11})^2 + (\gamma_2(q) - p_{12})^2, (\gamma_1(q) - p_{21})^2 + (\gamma_2(q) - p_{22})^2),
\]

where \(p_1 = (p_{11}, p_{12}), p_2 = (p_{21}, p_{22})\) and \(\gamma = (\gamma_1, \gamma_2)\).

Then, we have

\[
\frac{dD_p \circ \gamma}{dt}(q) = 2\left((\gamma_1(q) - p_{11}) \frac{d\gamma_1}{dt}(q) + (\gamma_2(q) - p_{12}) \frac{d\gamma_2}{dt}(q),
\right.
\]

\[
(\gamma_1(q) - p_{21}) \frac{d\gamma_1}{dt}(q) + (\gamma_2(q) - p_{22}) \frac{d\gamma_2}{dt}(q)\bigg)
\]

\[
= 2\left(\frac{p_1\gamma(q)}{dt}(q) \cdot \frac{d\gamma}{dt}(q), \frac{p_2\gamma(q)}{dt}(q) \cdot \frac{d\gamma}{dt}(q)\right),
\]

where \(t\) is a local coordinate around the point \(q\). Hence, a point \(q\) is a singular point of the mapping \(D_p \circ \gamma\) if and only if

\[
\left(\frac{p_1\gamma(q)}{dt}(q), \frac{p_2\gamma(q)}{dt}(q) \cdot \frac{d\gamma}{dt}(q)\right) = (0, 0).
\]

**Example 1.** In this example, we use Proposition 3. Let \(\gamma : S^1 \to \mathbb{R}^2\) be an embedding such that \(\gamma(S^1)\) is given by Figure 1. Here, note that there
exists an open set $U$ of $N$ such that for any $q \in U$, $\kappa(q) = 0$ (see $\gamma(U)$ in Figure 1). Namely, $\gamma$ does not satisfy $(\ast)$.

Let $p = (p_1, p_2) \in \gamma(U) \times \gamma(U)$ be any point. Then, we will show that the mapping $D_p \circ \gamma$ is not an immersion. From Figure 1, it is clearly seen that

$$\frac{p_1 \gamma(q')}{dt} \cdot \frac{d\gamma}{dt}(q') = 0 \quad \text{and} \quad \frac{p_2 \gamma(q')}{dt} \cdot \frac{d\gamma}{dt}(q') = 0,$$

where $\gamma(q')$ is the point in Figure 1 and $t$ is a local coordinate around the point $q'$. By Proposition 3, the point $q'$ is a singular point of $D_p \circ \gamma$. Namely, for any $p = (p_1, p_2) \in \gamma(U) \times \gamma(U)$, the mapping $D_p \circ \gamma$ is not an immersion. Since $\gamma(U) \times \gamma(U)$ is a non-empty open set of $\gamma(S^1) \times \gamma(S^1)$, the conclusion of Theorem 1 does not hold.

**Example 2.** Let $I_1$, $I_2$ and $I_3$ be open intervals $(0, 1)$, $(1, 2)$ and $(2, 3)$ of $\mathbb{R}$, respectively. Let $\gamma : I_1 \cup I_2 \cup I_3 \rightarrow \mathbb{R}^2$ be the mapping given by

$$\gamma(t) = \begin{cases} (t, -1), & t \in I_1, \\ (t - 1, 0), & t \in I_2, \\ (t - 2, 1), & t \in I_3. \end{cases}$$

For the image of $\gamma$, see Figure 2. Here, note that $\gamma$ does not satisfy $(\ast)$. Let $p = (p_1, p_2) \in \gamma(I_2) \times \gamma(I_2)$ be any point. Then, we will show that $D_p \circ \gamma$ is not a mapping with normal crossings. Since $p_1 = (p_{11}, p_{12})$, $p_2 = (p_{21}, p_{22}) \in \gamma(I_2)$, we have $p_{12} = p_{22} = 0$. Thus, we obtain

$$D_p(x_1, x_2) = ((x_1 - p_{11})^2 + x_2^2, (x_1 - p_{21})^2 + x_2^2).$$
Let \( t_0 \in I_1 \) be any element. Then, it follows that \( t_0 + 2 \in I_3 \) and
\[
(D_p \circ \gamma)(t_0) = (D_p \circ \gamma)(t_0 + 2).
\]
Since
\[
(D_p \circ \gamma)|_{I_1}(t) = ((t - p_{11})^2 + 1, (t - p_{21})^2 + 1),
\]
\[
(D_p \circ \gamma)|_{I_2}(t) = ((t - 2 - p_{11})^2 + 1, (t - 2 - p_{21})^2 + 1),
\]
we get
\[
d(D_p \circ \gamma)|_{t_0} = 2\left(\frac{t - p_{11}}{t - p_{21}}\right)_{t = t_0},
\]
\[
d(D_p \circ \gamma)|_{t_0 + 2} = 2\left(\frac{t - 2 - p_{11}}{t - 2 - p_{21}}\right)_{t = t_0 + 2}.
\]
Since the rank of the \( 2 \times 2 \) matrix \((d(D_p \circ \gamma)|_{t_0}, d(D_p \circ \gamma)|_{t_0 + 2})\) is less than two, \( D_p \circ \gamma \) is not a mapping with normal crossings. Hence, for any \( p = (p_1, p_2) \in \gamma(I_2) \times \gamma(I_2) \), \( D_p \circ \gamma \) is not a mapping with normal crossings.

**Remark 1.** There is an example such that Theorem 1 without the hypothesis (*) holds. Let \( \gamma : \mathbb{R} \to \mathbb{R}^2 \) be the mapping defined by \( \gamma(t) = (t, 0) \). Set
\[ A = \{ p \in \gamma(\mathbb{R}) \times \gamma(\mathbb{R}) \mid D_p \circ \gamma : \mathbb{R} \to \mathbb{R}^2 \text{ is an immersion with normal crossings} \}. \]

We will show that \( A \) is dense in \( \gamma(\mathbb{R}) \times \gamma(\mathbb{R}) \). Let \( p_1 = (p_{11}, p_{12}), p_2 = (p_{21}, p_{22}) \in \gamma(\mathbb{R}) \) (\( = \mathbb{R} \times \{0\} \)) be arbitrary points. Then, we have

\[
D_p \circ \gamma(t) = ((t - p_{11})^2, (t - p_{21})^2),
\]

where \( p = (p_1, p_2) \). It is not hard to see that if \( p_{11} \neq p_{21} \), then there exists a diffeomorphism \( H : \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( H \circ D_p \circ \gamma(t) = (t, 0) \). Namely, if \( p_{11} \neq p_{21} \), then \( D_p \circ \gamma \) is an immersion with normal crossings. On the other hand, if \( p_{11} = p_{21} \), then \( D_p \circ \gamma \) is not an immersion with normal crossings. Hence,

\[
A = \{ p \in \gamma(\mathbb{R}) \times \gamma(\mathbb{R}) \mid p_{11} \neq p_{21} \}.
\]

Thus, \( A \) is dense in \( \gamma(\mathbb{R}) \times \gamma(\mathbb{R}) \).

### 3. Preliminaries for the proof of Theorem 1

For the proof of Theorem 1, we prepare Proposition 4 and Lemma 1.

**Proposition 4.** Let \( L \) be a straight line of \( \mathbb{R}^2 \). For any \( p_1, p_2 \in L \) \( (p_1 \neq p_2) \) and for any \( \bar{p}_1, \bar{p}_2 \in L \) \( (\bar{p}_1 \neq \bar{p}_2) \), there exists an affine transformation \( H : \mathbb{R}^2 \to \mathbb{R}^2 \) such that

\[
H \circ D_p = D_{\bar{p}},
\]

where \( p = (p_1, p_2) \) and \( \bar{p} = (\bar{p}_1, \bar{p}_2) \).

**Proof.** Set \( p_1 = (p_{11}, p_{12}), p_2 = (p_{21}, p_{22}), \bar{p}_1 = (\bar{p}_{11}, \bar{p}_{12}) \) and \( \bar{p}_2 = (\bar{p}_{21}, \bar{p}_{22}) \).

Let \( H_1 : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation defined by

\[
H_1(X_1, X_2) = (X_1, X_1 - X_2).
\]

Then, we have

\[
H_1 \circ D_p(x_1, x_2) = ((x_1 - p_{11})^2 + (x_2 - p_{12})^2, 2((p_{21} - p_{11})x_1 + (p_{22} - p_{12})x_2) + c_1),
\]

where \( c_1 \) is a constant term.

Let \( H_2 : \mathbb{R}^2 \to \mathbb{R}^2 \) be the affine transformation defined by

\[
H_2(X_1, X_2) = (X_1, X_2 - c_1).
\]
Then, we get
\[ H_2 \circ H_1 \circ D_p(x_1, x_2) = ((x_1 - p_{11})^2 + (x_2 - p_{12})^2, 2((p_{21} - p_{11})x_1 + (p_{22} - p_{12})x_2)). \]

Since \( p_1, p_2, \bar{p}_1, \bar{p}_2 \in L \) and \( p_1 \neq p_2 \), there exist \( \lambda_1, \lambda_2 \in \mathbb{R} \) satisfying
\[ \bar{p}_1 = p_1 + \lambda_1 \bar{p}_1 \bar{p}_2, \quad (1) \]
\[ \bar{p}_2 = p_1 + \lambda_2 \bar{p}_1 \bar{p}_2. \quad (2) \]

Since \( \bar{p}_1 \neq \bar{p}_2 \), we get \( \lambda_1 \neq \lambda_2 \).

Let \( H_3 : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear transformation defined by
\[ H_3(X_1, X_2) = (X_1 - \lambda_1 X_2, X_1 - \lambda_2 X_2). \]

Then, we get
\[
H_3 \circ H_2 \circ H_1 \circ D_p(x_1, x_2) \\
= (x_1^2 - 2(p_{11} + \lambda_1 (p_{21} - p_{11}))x_1 + x_2^2 - 2(p_{12} + \lambda_1 (p_{22} - p_{12}))x_2 + d_1, \\
x_1^2 - 2(p_{11} + \lambda_2 (p_{21} - p_{11}))x_1 + x_2^2 - 2(p_{12} + \lambda_2 (p_{22} - p_{12}))x_2 + d_2),
\]
where \( d_1, d_2 \) are constant terms. By (1) and (2), we also get
\[
H_3 \circ H_2 \circ H_1 \circ D_p(x_1, x_2) \\
= (x_1^2 - 2\bar{p}_{11}x_1 + x_2^2 - 2\bar{p}_{12}x_2 + d_1, x_1^2 - 2\bar{p}_{21}x_1 + x_2^2 - 2\bar{p}_{22}x_2 + d_2) \\
= ((x_1 - \bar{p}_{11})^2 + (x_2 - \bar{p}_{12})^2 + d_1', (x_1 - \bar{p}_{21})^2 + (x_2 - \bar{p}_{22})^2 + d_2'),
\]
where \( d_1', d_2' \) are constant terms.

Let \( H_4 : \mathbb{R}^2 \to \mathbb{R}^2 \) be the affine transformation defined by
\[ H_4(X_1, X_2) = (X_1 - d_1', X_2 - d_2'). \]

Then, we have
\[
H_4 \circ H_3 \circ H_2 \circ H_1 \circ D_p(x_1, x_2) \\
= ((x_1 - \bar{p}_{11})^2 + (x_2 - \bar{p}_{12})^2, (x_1 - \bar{p}_{21})^2 + (x_2 - \bar{p}_{22})^2) \\
= D_p(x_1, x_2).
\]

This completes the proof of Proposition 4.

**Lemma 1.** Let \( \gamma : N \to \mathbb{R}^2 \) be an immersion satisfying \((*)\), where \( N \) is a manifold of dimension 1. Then, for any non-empty open set \( U_1 \times U_2 \) of \( N \times N \),
there exists an element \((q_1, q_2) \in U_1 \times U_2\) such that

\[
\det \begin{pmatrix}
\frac{d\gamma_1}{dt_1}(q_1) & \gamma_1(q_2) - \gamma_1(q_1) \\
\frac{d\gamma_2}{dt_1}(q_1) & \gamma_2(q_2) - \gamma_2(q_1)
\end{pmatrix} \neq 0,
\]

where \(\gamma = (\gamma_1, \gamma_2)\) and \(t_1\) is a local coordinate around \(q_1\).

**Proof.** Let \(U_1 \times U_2\) be any non-empty open set of \(N \times N\). Then, there exists a coordinate neighborhood \((U'_1 \times U'_2, (t_1, t_2))\) satisfying \(U'_1 \times U'_2 \subset U_1 \times U_2\). Fix \(q'_1 \in U'_1\).

Now, suppose that for any point \(t_2 \in U'_2\),

\[
\det \begin{pmatrix}
\frac{d\gamma_1}{dt_1}(q'_1) & \gamma_1(t_2) - \gamma_1(q'_1) \\
\frac{d\gamma_2}{dt_1}(q'_1) & \gamma_2(t_2) - \gamma_2(q'_1)
\end{pmatrix} = 0,
\]

where \(\gamma = (\gamma_1, \gamma_2)\). By (3), we have

\[
\frac{d\gamma_1}{dt_1}(q'_1)(\gamma_2(t_2) - \gamma_2(q'_1)) - \frac{d\gamma_2}{dt_1}(q'_1)(\gamma_1(t_2) - \gamma_1(q'_1)) = 0,
\]

for any point \(t_2 \in U'_2\). Hence, we get

\[
\frac{d\gamma_1}{dt_1}(q'_1)\frac{d\gamma_2}{dt_2}(t_2) - \frac{d\gamma_2}{dt_1}(q'_1)\frac{d\gamma_1}{dt_2}(t_2) = 0,
\]

for any point \(t_2 \in U'_2\). By (4) and (5), we have

\[
\begin{pmatrix}
\frac{d\gamma_2}{dt_2}(t_2) & -\frac{d\gamma_1}{dt_2}(t_2) \\
\frac{d^2\gamma_2}{dt_2^2}(t_2) & -\frac{d^2\gamma_1}{dt_2^2}(t_2)
\end{pmatrix}
\begin{pmatrix}
\frac{d\gamma_1}{dt_1}(q'_1) \\
\frac{d\gamma_2}{dt_1}(q'_1)
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

for any point \(t_2 \in U'_2\). Since \(\gamma\) is an immersion, it follows that

\[
\begin{pmatrix}
\frac{d\gamma_1}{dt_1}(q'_1) \\
\frac{d\gamma_2}{dt_1}(q'_1)
\end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
By (6) and (7), we have

$$\det \begin{pmatrix} \frac{d\gamma_2}{dt_2}(t_2) & -\frac{d\gamma_1}{dt_2}(t_2) \\ \frac{d^2\gamma_2}{dt_2^2}(t_2) & -\frac{d^2\gamma_1}{dt_2^2}(t_2) \end{pmatrix} = 0$$

for any point $t_2 \in U_2'$. This contradicts the hypothesis that $\gamma$ satisfies (*).

**Remark 2.** It is clearly seen that Lemma 1 does not depend on the choice of a coordinate neighborhood containing a point $q_1$ of $N$.

### 4. Proof of Theorem 1

Let $O$ be any non-empty open set of $\gamma(N) \times \gamma(N)$. Then, there exist non-empty open sets $O_1$ and $O_2$ of $\gamma(N)$ satisfying $O_1 \times O_2 \subset O$. For the proof, it is sufficient to show that there exist points $p_1 \in O_1$ and $p_2 \in O_2$ such that $D_p \circ \gamma : N \to \mathbb{R}^2$ is an immersion with normal crossings, where $p = (p_1, p_2)$. Since $\gamma$ is continuous, there exist coordinate neighborhoods $(U_1, t_1)$ and $(U_2, t_2)$ of $N$ such that $\gamma(U_1) \subset O_1$ and $\gamma(U_2) \subset O_2$.

Now, let $I_1$ (resp., $I_2$) be an open interval containing 0 (resp., 1) of $\mathbb{R}$, and let $\Phi : U_1 \times U_2 \times I_1 \times I_2 \to \mathbb{R}^4$ be the mapping defined by

$$\Phi(t_1, t_2, s_1, s_2) = (\gamma(t_1) + s_1\gamma(t_1)\gamma(t_2), \gamma(t_1) + s_2\gamma(t_1)\gamma(t_2))$$

$$= \left( (1 - s_1)\gamma_1(t_1) + s_1\gamma_1(t_2), (1 - s_1)\gamma_2(t_1) + s_1\gamma_2(t_2),
(1 - s_2)\gamma_1(t_1) + s_2\gamma_1(t_2), (1 - s_2)\gamma_2(t_1) + s_2\gamma_2(t_2) \right),$$

where $\gamma = (\gamma_1, \gamma_2)$. Then, we get

$$J\Phi(t_1, t_2, s_1, s_2) = \begin{pmatrix}
(1 - s_1)\frac{d\gamma_1}{dt_1}(t_1) & s_1\frac{d\gamma_1}{dt_2}(t_2) & \gamma_1(t_2) - \gamma_1(t_1) & 0 \\
(1 - s_1)\frac{d\gamma_2}{dt_1}(t_1) & s_1\frac{d\gamma_2}{dt_2}(t_2) & \gamma_2(t_2) - \gamma_2(t_1) & 0 \\
(1 - s_2)\frac{d\gamma_1}{dt_1}(t_1) & s_2\frac{d\gamma_1}{dt_2}(t_2) & 0 & \gamma_1(t_2) - \gamma_1(t_1) \\
(1 - s_2)\frac{d\gamma_2}{dt_1}(t_1) & s_2\frac{d\gamma_2}{dt_2}(t_2) & 0 & \gamma_2(t_2) - \gamma_2(t_1)
\end{pmatrix}.$$
Generic distance-squared mappings on plane curves

\begin{equation*}
J\Phi_{(t_1,t_2,0,1)} = \begin{pmatrix}
\frac{d\gamma_1}{dt_1}(t_1) & 0 & \gamma_1(t_2) - \gamma_1(t_1) & 0 \\
\frac{d\gamma_2}{dt_1}(t_1) & 0 & \gamma_2(t_2) - \gamma_2(t_1) & 0 \\
0 & \frac{d\gamma_1}{dt_2}(t_2) & 0 & \gamma_1(t_2) - \gamma_1(t_1) \\
0 & \frac{d\gamma_2}{dt_2}(t_2) & 0 & \gamma_2(t_2) - \gamma_2(t_1)
\end{pmatrix}.
\end{equation*}

Let us first show that there exists an element \((t_1, t_2)\in U_1 \times U_2\) such that \(\det d\Phi_{(t_1, t_2, 0, 1)} \neq 0\). Let \(\varphi_1 : U_1 \times U_2 \to \mathbb{R}\) and \(\varphi_2 : U_1 \times U_2 \to \mathbb{R}\) be the functions defined by

\begin{align*}
\varphi_1(t_1, t_2) &= \det \begin{pmatrix}
\frac{d\gamma_1}{dt_1}(t_1) & \gamma_1(t_2) - \gamma_1(t_1) \\
\frac{d\gamma_2}{dt_1}(t_1) & \gamma_2(t_2) - \gamma_2(t_1)
\end{pmatrix}, \\
\varphi_2(t_1, t_2) &= \det \begin{pmatrix}
\frac{d\gamma_1}{dt_2}(t_2) & \gamma_1(t_2) - \gamma_1(t_1) \\
\frac{d\gamma_2}{dt_2}(t_2) & \gamma_2(t_2) - \gamma_2(t_1)
\end{pmatrix},
\end{align*}

respectively. Note that the function \(\varphi_1\) (resp., \(\varphi_2\)) is defined by the entries of the 1st column vector and the 3rd column vector of \(J\Phi_{(t_1, t_2, 0, 1)}\) (resp., the 2nd column vector and the 4th column vector of \(J\Phi_{(t_1, t_2, 0, 1)}\)). In order to show that there exists an element \((t_1, t_2)\in U_1 \times U_2\) such that \(\det d\Phi_{(t_1, t_2, 0, 1)} \neq 0\), it is sufficient to show that there exists an element \((t_1, t_2)\in U_1 \times U_2\) satisfying \(\varphi_1(t_1, t_2) \neq 0\) and \(\varphi_2(t_1, t_2) \neq 0\). By Lemma 1, there exists \((t_1', t_2') \in U_1 \times U_2\) such that \(\varphi_1(t_1', t_2') \neq 0\). Since the function \(\varphi_1\) is continuous, there exists an open neighborhood \(U_1' \times U_2' \subset U_1 \times U_2\) such that \(\varphi_1(t_1', t_2') \neq 0\) for any \((t_1, t_2)\in U_1' \times U_2'\). Moreover, by Lemma 1, there exists \((t_1, t_2)\in U_1' \times U_2'\) such that \(\varphi_2(t_1, t_2) \neq 0\). Namely, there exists an element \((t_1, t_2)\in U_1 \times U_2\) such that \(\det d\Phi_{(t_1, t_2, 0, 1)} \neq 0\).

Now, by the inverse function theorem, there exists an open neighborhood \(V\) of \((t_1, t_2, 0, 1)\in U_1 \times U_2 \times I_1 \times I_2\) such that \(\Phi : V \to \Phi(V)\) is a diffeomorphism. Let \(\Sigma \subset \mathbb{R}^2 \times \mathbb{R}^2\) be the set consisting of points \(p = (p_1, p_2) \in \mathbb{R}^4\) such that \(Dp \circ \gamma : N \to \mathbb{R}^2\) is not an immersion with normal crossings. Note that by Proposition 2, the set \(\mathbb{R}^4 - \Sigma\) is dense in \(\mathbb{R}^4\). Set

\[\mathcal{A} = \{(y_1, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2 | y_1 = y_2\} \subset \mathbb{R}^4 - \Sigma\].

Since \(\Phi(V)\) is an open set of \(\mathbb{R}^4\) and the set \(\mathcal{A}\) is a proper algebraic set of \(\mathbb{R}^4\), there exists an element \(p' = (p'_1, p'_2) \in \Phi(V) - \Sigma \cup \mathcal{A}\). As \(p' \notin \Sigma\), the com-
position $D_{p'} \circ \gamma : N \to \mathbb{R}^2$ is an immersion with normal crossings. Set $(t'_1, t'_2, s'_1, s'_2) = (\Phi|_Y)^{-1}(p'_1, p'_2)$. Then, we have

$$p'_1 = \gamma(t'_1) + s'_1 \gamma(t'_1) \gamma(t'_2),$$

$$p'_2 = \gamma(t'_1) + s'_2 \gamma(t'_1) \gamma(t'_2).$$

Since $p'_1 \neq p'_2$, we get $\gamma(t'_1) \neq \gamma(t'_2)$. Let $L$ be the straight line defined by

$$L = \{ \gamma(t'_1) + s \gamma(t'_1) \gamma(t'_2) \mid s \in \mathbb{R} \}.$$

Set $\tilde{p}_1 = \gamma(t'_1)$ and $\tilde{p}_2 = \gamma(t'_2)$. Then, it is clearly seen that $\tilde{p}_1 \in O_1$ and $\tilde{p}_2 \in O_2$. Since $p'_1, p'_2 \in L$ ($p'_1 \neq p'_2$) and $\tilde{p}_1, \tilde{p}_2 \in L$ ($\tilde{p}_1 \neq \tilde{p}_2$), by Proposition 4, there exists an affine transformation $H : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$H \circ D_{p'} = D_p,$$

where $\tilde{p} = (\tilde{p}_1, \tilde{p}_2)$. Since $D_{p'} \circ \gamma : N \to \mathbb{R}^2$ is an immersion with normal crossings, $D_p \circ \gamma : N \to \mathbb{R}^2$ is also an immersion with normal crossings. \hfill \square

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References


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