On the theory of 4-th root Finsler metrics

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Abstract

In this paper, we consider exponential change of Finsler metrics. First, we find a condition under which the exponential change of a Finsler metric is projectively related to it. Then we restrict our attention to the 4-th root metric. Let $F=\sqrt[4]{A}$ be an 4-th root Finsler metric on an open subset $U\subset\mathbb{R}^n$ and $\bar{F}=e^{\beta/F}F$ be the exponential change of F. We show that $\bar{F}$ is locally projectively flat if and only if it is locally Minkowskian. Finally, we obtain necessary and sufficient condition under which $\bar{F}$ be locally dually flat.

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1 Introduction

Let (M,F) be a Finsler manifold. For a 1-form $\beta(x,y)=b_i(x)y^i$ on M, we have a change of the Finsler metric F which is defined by $F(x,y)\to\bar{F}(x,y)=f(F,\beta)$, where $f=f(F,\beta)$ is a positively homogeneous function of F. This is called a β-change of F. If $||\beta||_F:=\sup_{F(x,y)=1}|\beta|<1$, then it is easy to see that $\bar{F}$ is again a Finsler metric [10]. Indeed, $\bar{F}$ is positive and strong convex if $||\beta||_F<1$.

There is a special case of β-change, namely

$$\bar{F}(x,y)=e^{\frac{\beta}{F}}F(x,y) \tag{1}$$

which is called the exponential change. Here, we assume that $\beta\neq 0$. If $F=\alpha$ is a Riemannian metric, then $\bar{F}=e^{\frac{\beta}{\alpha}}\alpha$ is the exponential Finsler metric. Due to this reason, the transformation (1) has been called the exponential change of Finsler metrics. For other β-changes, see [21], [23] and [25].

For a Finsler metric $F=F(x,y)$, its geodesics curves are characterized by the system of ODE $\ddot{c}^i+2G^i(\dot{c})=0$, where the local functions $G^i=G^i(x,y)$ are called the spray coefficients. Two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if any geodesic of the first is also geodesic for the second and vice versa. In this case, there is a scalar function $P=P(x,y)$ defined on TM_0 such that $\bar{G}^i=G^i+Py^i$, where $\bar{G}^i$ and G^i are the geodesic spray coefficients of $\bar{F}$ and F, respectively.

In this paper, we find a condition under which the exponential change of a Finsler metric is projectively related to it. Let (M,F) be a Finsler manifold and $\beta=b_i(x)y^i$ a 1-form on M. Put

$$r_{ij}:=\frac{1}{2}(b_{i|j}+b_{j|i}),\quad s_{ij}:=\frac{1}{2}(b_{i|j}-b_{j|i}),\quad r_{00}=r_{ij}y^iy^j,$$

where " | " denotes the horizontal derivation with respect to the Berwald connection of F. Then we have the following.

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Theorem 1.1. Let (M,F) be Finsler manifold. Suppose that $\bar{F} = e^{\beta/F}F$ be the exponential change of F. Then $\bar{F}$ is projectively related to F if only if β satisfies

$$s_{ij} = \frac{1}{2}\big[\mathcal{A}_j r_{ik} - \mathcal{A}_i r_{jk}\big] y^k, \tag{2}$$

where $\mathcal{A}_i := (\beta/F)_{y^i}$. In this case, the projective factor is given by $P = \frac{1}{2F} r_{00}$.

The theory of m-th root Finsler metric has been developed by Shimada and Matsumoto and applied to Biology as an ecological metric [7, 11]. Let M be an n-dimensional C^∞ manifold, TM its tangent bundle. Let $F = \sqrt[m]{A}$ be a Finsler metric on M, where $A = A(x,y)$ is given by $A := a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ with $a_{i_1 \dots i_m}$ symmetric in all its indices. Then F is called an m-th root Finsler metric (see [11, 14, 16, 17, 18, 19, 20, 22, 25]). The special m-th root Finsler metric in the form $F = \sqrt[m]{y^1 y^2 \dots y^m}$ is called the Berwald-Moór metric [4, 5, 7]. Recently, physical studies due to Asanov, Pavlov and their co-workers emphasize the important role played by the Berwald-Moór metric in the theory of space-time structure and gravitation as well as in unified gauge field theories [2, 12, 13]. In [3], Balan prove that the Berwald-Moór structures are pseudo-Finsler of Lorentz type and for co-isotropic submanifolds of Berwald-Moór spaces present the Gauss-Weingarten, Gauss-Codazzi, Peterson-Mainardi and Ricci-Kühne equations.

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. It is known that a Finsler metric $F(x,y)$ on an open domain $U \subset \mathbb{R}^n$ is locally projectively flat if and only if $G^i = P y^i$, where $P = P(x,y)$ is called the projective factor and is a C^∞ scalar function on $TM_0 = TM \setminus \{0\}$ satisfying $P(x,\lambda y) = \lambda P(x,y)$ for all $\lambda > 0$.

Theorem 1.2. Let $F = \sqrt[4]{A}$ be an 4-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Suppose that $\bar{F} = e^{\beta/F}F$ be the exponential change of F. Then $\bar{F}$ is locally projectively flat if and only if F is locally Minkowskian.

In [1], Amari-Nagaoka introduced the notion of dually flat Riemannian metrics when they study the information geometry on Riemannian manifolds. In Finsler geometry, Shen extends the notion of locally dually flatness for Finsler metrics [9]. A Finsler metric F on a manifold M is said to be locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients are in the form $G^i = -\frac{1}{2} g^{ij} H_{y^j}$, where $H = H(x,y)$ is a positively homogeneous scalar function on TM_0.

Theorem 1.3. Let $F = \sqrt[4]{A}$ be an 4-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Suppose that $\bar{F} = e^{\beta/F}F$ be the exponential change of F. Then $\bar{F}$ is a locally dually flat Finsler metric if and only if the following hold

$$16(\beta_{0l} - 2\beta_{x^l})A^2 + 4(A_l\beta_0 + A_0 b_l)A - 4\beta(A_{0l} - 2A_{x^l})A + A_0 A_l \beta = 0, \tag{3}$$

$$2(A_{0l} - 2A_{x^l})A - A_0 A_l = 0, \tag{4}$$

$$16 b_l \beta_0 A^2 - 4\beta(A_0 b_l + \beta_0 A_l)A + \beta^2 A_0 A_l = 0. \tag{5}$$

Moreover, suppose that A is irreducible. Then, there exists a 1-form $\gamma = \gamma_i y^i$ on U such that (3),

(4) and (5) reduce to following

$$A_{x^l} = \frac{1}{6}\Big[2A\gamma_l + \gamma A_l\Big], \tag{6}$$

$$\beta_{x^l} = \frac{1}{12}\Big[2b_l\gamma + \beta\gamma_l\Big]. \tag{7}$$

2 Preliminary

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where

$$G^i := \frac{1}{4}g^{il}\Big[\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big], \qquad y \in T_xM. \tag{8}$$

$\mathbf{G}$ is called the spray associated to (M, F). In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$. Then two Finsler metrics F and $\bar{F}$ on a manifold M are called projectively related if any geodesic of the first is also geodesic for the second and vice versa. In this case, $G^i = \bar{G}^i + Py^i$, where G^i and $\bar{G}^i$ are the geodesic spray coefficients of F and $\bar{F}$, respectively, and $P = P(x, y)$ is a positively homogeneous scalar function on TM_0. Indeed, two regular metric spaces are projectively related if there is a diffeomorphism between them such that the pull-back metric is pointwise projective to another one. The following lemma plays an important role.

Lemma 2.1. (Rapcsák [8]) *Let F and $\bar{F}$ be two Finsler metrics on a manifold M. Then $\bar{F}$ is projectively related to F if and only if $\bar{F}$ satisfies*

$$\bar{F}_{|k,l}y^k - \bar{F}_{|l} = 0. \tag{9}$$

where "$|$" denotes the horizontal derivation with respect to the Berwald connection of F. In this case, the spray coefficients are related by $\bar{G}^i = G^i + Py^i$, where

$$P = \frac{\bar{F}_{|k}y^k}{2\bar{F}}. \tag{10}$$

The $P = P(x, y)$ is called the projective factor of $F(x, y)$.

It is known that a Finsler metric $F(x, y)$ on $\mathcal{U} \subset R^n$ is projective if and only if its geodesic coefficients G^i are in the form $G^i(x, y) = P(x, y)y^i$, where $P : T\mathcal{U} = \mathcal{U} \times \mathrm{R}^n \to \mathrm{R}$ is positively homogeneous of degree one with respect to y. Much earlier, in [6], G. Hamel proved that a Finsler metric F on $\mathcal{U} \subset \mathbb{R}^n$ is projectively flat if and only if it satisfies

$$F_{x^k y^l}y^k = F_{x^l}. \tag{11}$$

See [15], [22] and [24].

A Finsler metric $F = F(x, y)$ on a manifold M is said to be locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients are in the form $G^i = -\frac{1}{2}g^{ij}H_{y^j}$, where $H = H(x, y)$ satisfying $H(x, \lambda y) = \lambda^3 H(x, y)$ for all $\lambda > 0$. Such a coordinate system is called an adapted coordinate system. In [9], Shen proved that the Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is dually flat if and only if it satisfies

$$(F^2)_{x^k y^l} y^k = 2(F^2)_{x^l}. \tag{12}$$

In this case, $H = -\frac{1}{6}[F^2]_{x^m} y^m$.

3 Proof of the Theorem 1.1

Throughout this paper, we use the Berwald connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by " $|$ " and " $,$ " respectively. Now, let (M, F) be a Finsler manifold and $\beta = b_i(x)y^i$ a 1-form on M. Put

$$s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \qquad r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}),$$
$$r_{i0} := r_{ij}y^j, \qquad r_{00} := r_{ij}y^i y^j, \qquad s_{i0} := s_{ij}y^j,$$

where "$|$" denotes the horizontal derivation with respect to the Berwald connection of F.

Proof of Theorem 1.1: The following hold

$$b_{i|j} = \frac{\partial b_i}{\partial x^j} - b_s \Gamma^s{}_{ij}, \qquad \Gamma^s{}_{ij} = \Gamma^s{}_{ji},$$

where $\Gamma^i{}_{jk} = \Gamma^i{}_{jk}(x, y)$ are the Christoffel symbols of the Berwald connection of F. Then we have

$$s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}) = \frac{1}{2}\Big(\frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i}\Big), \tag{13}$$
$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}) = \frac{1}{2}\Big(\frac{\partial b_i}{\partial x^j} + \frac{\partial b_j}{\partial x^i} - 2b_s\Gamma^s{}_{ij}\Big). \tag{14}$$

By (13) and (14), we get

$$\beta_{|l} = b_{i|l}y^i = \Big(\frac{\partial b_i}{\partial x^l} - \Gamma^s{}_{il}b_s\Big)y^i, \tag{15}$$
$$\beta_{|k}y^k = b_{i|k}y^i y^k = (r_{ik} + s_{ik})y^i y^k = r_{00}, \tag{16}$$
$$\beta_{|k,l}y^k = \Big(\frac{\partial b_l}{\partial x^k} - b_s\Gamma^s{}_{lk}\Big)y^k. \tag{17}$$

(15) and (17) imply that

$$\beta_{|k,l}y^k - \beta_{|l} = s_{lk}y^k = 2s_{l0}. \tag{18}$$

For $\bar{F} = e^{\beta/F}F$, we have

$$\bar{F}_{|l} = \beta_{|l}e^{\frac{\beta}{F}}, \tag{19}$$
$$\bar{F}_{|k,l}y^k = \Big[\beta_{|k,l}y^k + \Big(\frac{\beta}{F}\Big)_l \beta_{|k}y^k\Big]e^{\frac{\beta}{F}}. \tag{20}$$

Then by (16), (18), (19) and (20), we get the following

$$\begin{aligned}\bar{F}_{|k,l}y^k - \bar{F}_{|l} &= \Big[\beta_{|k,l}y^k + \Big(\frac{\beta}{F}\Big)_l \beta_{|k}y^k - \beta_{|l}\Big]e^{\frac{\beta}{F}} \\ &= \Big[2s_{l0} + \Big(\frac{\beta}{F}\Big)_l r_{00}\Big]e^{\frac{\beta}{F}}. \end{aligned} \tag{21}$$

By the Lemma 2.1, $\bar{F}$ is projectively related to F if and only if

$$2s_{l0} + \Big(\frac{\beta}{F}\Big)_l r_{00} = 0. \tag{22}$$

Taking a vertical derivation of (22) yields

$$s_{li} = -\frac{1}{2}\Big[\Big(\frac{\beta}{F}\Big)_{li} r_{00} + 2\Big(\frac{\beta}{F}\Big)_l r_{0i}\Big]. \tag{23}$$

Since $s_{li} = -s_{il}$, then by (23) we get

$$\Big(\frac{\beta}{F}\Big)_{li} r_{00} + 2\Big(\frac{\beta}{F}\Big)_l r_{0i} = -\Big(\frac{\beta}{F}\Big)_{il} r_{00} - 2\Big(\frac{\beta}{F}\Big)_i r_{0l}$$

or

$$\Big(\frac{\beta}{F}\Big)_{li} r_{00} = -\Big(\frac{\beta}{F}\Big)_i r_{0l} - \Big(\frac{\beta}{F}\Big)_l r_{0i}. \tag{24}$$

By (23) and (24), we have

$$s_{ij} = \frac{1}{2}\Big[\Big(\frac{\beta}{F}\Big)_j r_{0i} - \Big(\frac{\beta}{F}\Big)_i r_{0j}\Big]. \tag{25}$$

By (25), we get (2).

Now, by (10) and (19) it follows that

$$P = \frac{\bar{F}_{|k}y^k}{2\bar{F}} = \frac{\beta_{|k}y^k e^{\frac{\beta}{F}}}{2\bar{F}} = \frac{1}{2F} r_{00}.$$

This completes the proof. Q.E.D.

4 Proof of the Theorem 1.2

In this section, we are going to prove the Theorem 1.2. First, we remark the following.

Lemma 4.1. Let $F = \sqrt[4]{A}$ be an 4-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Suppose that the equation holds

$$\Psi A^{\frac{7}{4}} + \Xi A^{\frac{5}{4}} + \Phi A^{\frac{3}{4}} + \Theta A^{\frac{1}{4}} + \Upsilon A^{\frac{-1}{4}} + \Omega = 0, \tag{26}$$

where $\Phi, \Psi, \Theta, \Upsilon, \Omega, \Xi$ are homogeneous polynomials in y. Then

$$\Psi A^2 + \Phi A + \Upsilon = 0, \tag{27}$$

$$\Xi A + \Theta = 0, \tag{28}$$

$$\Omega = 0. \tag{29}$$

Proof. By contracting (26) with $\sqrt[4]{A}$ we get

$$\Psi A^2 + \Phi A + \Upsilon + (\Xi A^{\frac{5}{4}} + \Theta A^{\frac{1}{4}} + \Omega)A^{\frac{1}{4}} = 0, \tag{30}$$

Thus

$$\Psi A^2 + \Phi A + \Upsilon = 0, \tag{31}$$
$$\Xi A^{\frac{5}{4}} + \Theta A^{\frac{1}{4}} + \Omega = 0 \tag{32}$$

Multiplying (32) with $A^{\frac{3}{4}}$ implies that

$$(\Xi A + \Theta)A + \Omega A^{\frac{3}{4}} = 0 \tag{33}$$

Then

$$\Xi A + \Theta = 0, \tag{34}$$
$$\Omega = 0 \tag{35}$$

This completes the proof. Q.E.D.

For an 4-th root metric $F = \sqrt[4]{A}$, let us put

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^j \partial y^j}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i} y^i, \quad A_{0l} = A_{x^k y^l} y^k = \frac{\partial^2 A}{\partial x^i \partial y^l} y^k.$$

Proof of the Theorem 1.2: The following holds For $\bar{F} = e^{\beta A^{-\frac{1}{4}}} \sqrt[4]{A}$, the following hold

$$[\bar{F}]_{x^l} = \frac{1}{4}\big[A_{x^l} A^{-\frac{3}{4}} - \beta A_{x^l} A^{-1} + 4\beta_{x^l}\big] e^{\beta A^{\frac{-1}{4}}}, \tag{36}$$

$$[\bar{F}]_{x^k y^l} y^k = \frac{1}{4}\Bigg[\frac{1}{4}\beta^2 A_0 A_l A^{-\frac{9}{4}} - \frac{3}{4} A_0 A_l A^{-\frac{7}{4}} - \beta(A_l \beta_0 + A_0 b_l) A^{-\frac{5}{4}} + A_{0l} A^{-\frac{3}{4}}$$
$$+4 b_l \beta_0 A^{-\frac{1}{4}} + \frac{3}{4}\beta A_0 A_l A^{-2} - \beta A_{0l} A^{-1} + 4\beta_{0l}\Bigg] e^{\beta A^{-\frac{1}{4}}}, \tag{37}$$

where

$$\beta_{x^i} := \frac{\partial \beta}{\partial x^i}, \qquad \beta_i := \frac{\partial \beta}{\partial y^i} = b_i, \qquad \beta_0 := \beta_{x^i} y^i, \qquad \beta_{0l} := \beta_{x^i y^l} y^i.$$

Since $\bar{F}$ is locally projectively flat metric, then (11) holds

$$[\bar{F}]_{x^k y^l} y^k - [\bar{F}]_{x^l} = 0. \tag{38}$$

By putting (36) and (37) in (38) and multiplying the result with A^2, we get

$$
\begin{aligned}
&-\frac{3}{4}A_0A_lA^{\frac{1}{4}} + A_{0l}A^{\frac{5}{4}} + \beta A_0A_l - A_{0l}A\beta + 4\beta_{0l}A^2\\
&-\frac{1}{4}A_0A_l\beta + 4\beta_0 b_l A^{\frac{7}{4}} - \beta\beta_0A_lA^2A^{\frac{3}{4}} - \beta b_lA_0A^{\frac{3}{4}}\\
&+\frac{1}{4}\beta^2A_0A_lA^{-\frac{1}{4}} - A_{x^l}A^{\frac{5}{4}} - 4\beta_{x^l}A^2 + \beta AA_{x^l} = 0, \qquad (39)
\end{aligned}
$$

Simplifying (39) results that

$$
\begin{aligned}
16b_l\beta_0A^{\frac{7}{4}} \;&+4(A_{0l} - A_{x^l})A^{\frac{5}{4}} - 4\beta(\beta_0A_l + b_lA_0)A^{\frac{3}{4}} - 3A_0A_lA^{\frac{1}{4}}\\
&+\beta^2A_0A_lA^{-\frac{1}{4}} + 16(\beta_{0l} - \beta_{x^l})A^2 - 4\beta(A_{0l} - A_{x^l})A + 3\beta A_0A_l = 0. \qquad (40)
\end{aligned}
$$

By Lemma 4.1 and (40), we have

$$16b_l\beta_0A^2 - 4\beta(\beta_0A_l + b_lA_0)A + \beta^2A_0A_l = 0, \qquad (41)$$

$$4(A_{0l} - A_{x^l})A = 3A_0A_l, \qquad (42)$$

$$16(\beta_{0l} - \beta_{x^l})A^2 - 4\beta(A_{0l} - A_{x^l})A + 3\beta A_0A_l = 0. \qquad (43)$$

By (42) and (43), we obtain

$$\beta_{0l} - \beta_{x^l} = 0. \qquad (44)$$

Since $\beta_{0l} - \beta_{x^l} = 2s_{l0}$, then by considering (22) it follows that $r_{00} = 0$ and then $P = 0$. Thus $G^i = 0$ and F reduces to a locally Minkowskian metric. Q.E.D.

5 Proof of the Theorem 1.3

In this section, we are going to characterize locally dually flat Finsler metrics which is obtained by an exponential change of m-th root metrics. First, we remark the following.

Lemma 5.1. Let $F = \sqrt[4]{A}$ be an 4-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Suppose that the following equation holds

$$\Theta A^{\frac{9}{4}} + \Phi A^{\frac{3}{2}} + \Psi A^{\frac{5}{4}} + \Upsilon A^{\frac{1}{2}} + \Xi A^{\frac{1}{4}} + \Omega = 0, \qquad (45)$$

where $\Phi, \Psi, \Theta, \Upsilon, \Omega$ and Ξ are homogeneous polynomials in y. Then

$$\Theta A^2 + \Psi A + \Xi = 0, \qquad (46)$$

$$\Phi A + \Upsilon = 0, \qquad (47)$$

$$\Omega = 0. \qquad (48)$$

Proof. By contracting (45) with $A^{\frac{3}{4}}$ we get

$$\Theta A^3 + \Phi A^{\frac{9}{4}} + \Psi A^2 + \Upsilon A^{\frac{5}{4}} + \Xi A + \Omega A^{\frac{3}{4}} = 0, \qquad (49)$$

Thus

$$\Theta A^2 + \Psi A + \Xi = 0, \qquad (50)$$

$$\Phi A^{\frac{6}{4}} + \Upsilon A^{\frac{2}{4}} + \Omega = 0 \qquad (51)$$

By (51), we have (47) and (48). Q.E.D.

Proof of the Theorem 1.3: The following holds

$$[\bar{F}^2]_{x^l} = \frac{1}{2}\Big[A_{x^l}A^{-\frac{1}{2}} + 4\beta_{x^l}A^{\frac{1}{4}} - \beta A_{x^l}A^{-\frac{3}{4}}\Big]e^{2\beta A^{-\frac{1}{4}}}, \tag{52}$$

$$\begin{aligned}[\bar{F}^2]_{x^k y^l}y^k = \frac{1}{2}\Bigg[& -\frac{1}{2}A_lA_0A^{-\frac{3}{2}} + A_{0l}A^{-\frac{1}{2}} + A_l\beta_0A^{-\frac{3}{4}} + 4\beta_{0l}A^{\frac{1}{4}} - b_lA_0A^{-\frac{3}{4}} \\ & +\frac{3}{4}A_lA_0\beta A^{-\frac{7}{4}} - A_{0l}\beta A^{-\frac{3}{4}} + 2A_0b_lA^{-\frac{3}{4}} - \frac{1}{2}\beta A_0A_lA^{-\frac{7}{4}} \\ & -2\beta\beta_0A_lA^{-1} - 2b_l\beta A_0A^{-1} + \frac{1}{2}\beta^2A_0A_lA^{-2} + 8b_l\beta_0\Bigg]e^{2\beta A^{-\frac{1}{4}}}.\end{aligned} \tag{53}$$

Since $\bar{F}$ is locally dually flat metric, then

$$[\bar{F}^2]_{x^k y^l}y^k - 2[\bar{F}^2]_{x^l} = 0. \tag{54}$$

Putting (52) and (53) in (54) and multiplying the result with A^2 imply that

$$\begin{aligned}&16(\beta_{0l} - 2\beta_{x^l})A^{\frac{9}{4}} + 4(A_{0l} - 2A_{x^l})A^{\frac{3}{2}} + 4\Big[A_l\beta_0 + A_0b_l + \beta(2A_{x^l} - A_{0l})\Big]A^{\frac{5}{4}} \\ &-2A_0A_lA^{\frac{1}{2}} + A_0A_l\beta A^{\frac{1}{4}} + 32b_l\beta_0A^2 - 8\beta(A_0b_l + \beta_0A_l)A + 2\beta^2A_0A_l = 0.\end{aligned} \tag{55}$$

Then by Lemma 5.1 and (55), we get (3), (4) and (5).

Now, suppose that A is irreducible. By (4), irreducibility of A and $deg(A_l) = 3$, it follows that there exists a 1-form $\gamma = \gamma_l y^l$ on U such that

$$A_0 = \gamma A. \tag{56}$$

Taking a vertical derivative of (56) implies that

$$A_{0l} = A\gamma_l + \gamma A_l - A_{x^l}. \tag{57}$$

By putting (56) and (57) in (4), we get (6), and by putting (56) in (5), we obtain

$$(4\beta_0 - \beta\gamma)(4b_lA - \beta A_l) = 0. \tag{58}$$

If $4b_lA - \beta A_l = 0$, then A is reducible which contradicts with our assumption. Then, we get

$$\beta_0 = \frac{1}{4}\beta\gamma. \tag{59}$$

Putting (4), (56) and (57) in (3) yield

$$16(\beta_{0l} - 2\beta_{x^l})A + (4\beta_0 - \beta\gamma)A_l + 4\gamma Ab_l = 0. \tag{60}$$

By (59) and (60), we have

$$4(\beta_{0l} - 2\beta_{x^l}) + b_l\gamma = 0. \tag{61}$$

On the other hand, taking a vertical derivation of (59) implies that

$$\beta_{0l} + \beta_{x^l} = \frac{1}{4}\Big[b_l\gamma + \beta\gamma_l\Big]. \tag{62}$$

By (61) and (62), one can obtain

$$\beta_{x^l} = \frac{1}{12}\Big[2b_l\gamma + \beta\gamma_l\Big]. \tag{63}$$

This completes the proof. Q.E.D.

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