On certain new means generated by generalized trigonometric functions

József Sándor[1] and Barkat Ali Bhayo[2]

[1]Babeş-Bolyai University, Department of Mathematics, Str. Kogalniceanu nr. 1, 400084 Cluj-Napoca, Romania
[2]Department of Mathematics, Sukkur IBA University, Airport road Sukkur, Sindh, Pakistan
[2]Faculty of Natural Sciences, Sabanci University, 34956 Tuzla/Istanbul, Turkey
E-mail: jsandor@math.ubbcluj.ro[1], barkat.bhayo@iba-suk.edu.pk[2]

Abstract

In this paper, authors generalize logarithmic mean L, Neuman-Sándor M, two Seiffert means P and T as an application of generalized trigonometric and hyperbolic functions. Moreover, several two-sided inequalities involving these generalized means are established.

2010 Mathematics Subject Classification. **34L10**. 33E05, 33C75
Keywords. generalized trigonometric functions, hypergeometric functions, p-Laplacian, logarithmic mean, two Seiffert means.

1 Introduction

For the definition of new means involved in our formulation we introduce some special functions and notation. The *Gaussian hypergeometric function* is defined by

$$F\,(a,b;c;z) = {}_2F_1\,(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{z^n}{n!}, \quad |z|<1,$$

where (a,n) denotes the shifted factorial function

$$(a,n) = a(a+1)(a+2)\dots(a+n-1), \quad n=1,2,3,\dots,$$

and $(a,0)=1$ for $a\neq 1$. For the applications of this function in various fields of the mathematical and natural sciences, reader is referred to [4].

Special functions, such as classical *gamma function* Γ, the *digamma function* ψ and the *beta function* $B(.,.)$ have close relation with hypergeometric function. For $x,y>0$, these functions are defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt, \quad \psi(x) = \frac{\Gamma^{'}(x)}{\Gamma(x)}, \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

respectively. The hypergeometric function can be represented in the integral form as follows

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt. \tag{1.1}$$

The eigenfunction $\sin_p$ of the so-called one-dimensional p-Laplacian problem [18]

$$-\Delta_p u = -\left(|u'|^{p-2}u'\right)' = \lambda |u|^{p-2}u,\ u(0)=u(1)=0, \quad p>1,$$

Tbilisi Mathematical Journal 12(1) (2019), pp. 1–16.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 04 July 2018 .
Accepted for publication: 19 November 2018.

is the inverse function of $F_p : [0,1] \to \left[0, \frac{\pi_p}{2}\right]$, defined as

$$F_p(x) = \text{arcsin}_p(x) = \int_0^x (1-t^p)^{-\frac{1}{p}}\, dt,$$

where

$$\pi_p = 2\text{arcsin}_p(1) = \frac{2}{p}\int_0^1 (1-s)^{-\frac{1}{p}} s^{\frac{1}{p}-1} ds = \frac{2}{p}\, B\left(1-\frac{1}{p}, \frac{1}{p}\right) = \frac{2\pi}{p\, \sin\left(\frac{\pi}{p}\right)}\,.$$

The function arcsin_p is called the generalized inverse sine function, and its inverse function $\sin_p$: $[0, \pi_p/2] \to [0,1]$ is called generalized sine function. For $x \in [\pi_p/2, \pi_p]$, one can extends the function $\sin_p$ to $[0, \pi_p]$ by defining $\sin_p(x) = \sin(\pi_p - x)$, and further extension cab be achieved on $\mathbb{R}$ by oddness and 2π-periodicity. The range of p is restricted to $(1,\infty)$ because only in this case $\sin_p(x)$ can be made periodic like usual sine function.

Similarly, the other generalized inverse trigonometric and hyperbolic functions $\text{arccos}_p : (-1,1) \to (-a_p, a_p)$, $\text{arctan}_p : (-\infty,\infty) \to (-a_p, a_p)$, $\text{arcsinh}_p : (-\infty,\infty) \to (-\infty,\infty)$, $\text{arctanh}_p : (-1,1) \to (-\infty,\infty)$ are defined as follows

$$\begin{aligned} \text{arccos}_p(x) &= \int_0^{(1-x^p)^{\frac{1}{p}}} (1-|t|^p)^{-\frac{1}{p}} dt, \quad \text{arctan}_p(x) = \int_0^x (1+|t|^p)^{-1} dt, \\ \text{arcsinh}_p(x) &= \int_0^x (1+|t|^p)^{-\frac{1}{p}} dt, \quad \text{arctanh}_p(x) = \int_0^x (1-|t|^p)^{-1} dt, \end{aligned} \tag{1.2}$$

where $a_p = \pi_p/2$. Above inverse generalized trigonometric and hyperbolic functions coincide with usual trigonometric and hyperbolic functions for $p = 2$.

2 Generalization of means and main result

For two positive real numbers a and b, we define arithmetic mean A, geometric mean G, logarithmic mean L, two Seiffert means P and T, and Neuman-Sándor mean M introduced in [24] as follows,

$$A = A(a,b) = \frac{a+b}{2}, \quad G = G(a,b) = \sqrt{ab},$$

$$L = L(a,b) = \frac{a-b}{\log(a)-\log(b)}, \quad a \neq b,$$

$$P = P(a,b) = \frac{a-b}{2\text{arcsin}\left(\frac{a-b}{a+b}\right)},$$

$$T = T(a,b) = \frac{a-b}{2\text{arctan}\left(\frac{a-b}{a+b}\right)},$$

$$M = M(a,b) = \frac{a-b}{2\text{arcsinh}\left(\frac{a-b}{a+b}\right)}.$$

The arithmetic-geometric mean $AG(a,b)$ of two real numbers a and b is defined as follows: Let us consider the sequences $\{a_n\}$ and $\{b_n\}$ satisfying

$$a_{n+1}=\frac{a_n+b_n}{2},\quad b_{n+1}=\sqrt{a_nb_n},\quad n=0,1,2,\ldots$$

with $a_0=a$ and $b_0=b$.

In [8], Bhatia and Li generalized the logarithmic mean L and arithmetic-geometric mean $AG(a,b)$ by introducing an interpolating family of means $\mathbb{M}_p(a,b)$, defined by

$$\frac{1}{\mathbb{M}_p(a,b)}=n_p\int_0^\infty\frac{dt}{((t^p+a^p)(t^p+b^p))^{1/p}},\quad p\in(0,\infty),$$

where $n_p=\int_0^\infty\frac{dt}{(1+t^p)^{2/p}}$. Moreover,

$$\mathbb{M}_0(a,b)=\lim_{p\to 0}\mathbb{M}_p(a,b)=\sqrt{ab},$$

$$\mathbb{M}_1(a,b)=L(a,b)\quad\text{and}\quad\mathbb{M}_2(a,b)=AG(a,b).$$

In [26, 27], Neuman generalized the logarithmic mean L, two Seiffert means P and T, and the Neuman-Sándor mean M by introducing the the p-version of the Schwab-Borchardt mean SB_p as follows

$$L_p=L_p(a,b)=SB_p(A_{p/2},G)=\frac{A_{p/2}v_p}{\operatorname{arctanh}_p(v_p)},$$

$$P_p=P_p(a,b)=SB_p(G,A_{p/2})=\frac{A_{p/2}v_p}{\arcsin_p(v_p)},$$

$$T_p=T_p(a,b)=SB_p(A_{p/2},A_p)=\frac{A_{p/2}v_p}{\arctan_p(v_p)},$$

$$M_p=M_p(a,b)=SB_p(A_p,A_{p/2})=\frac{A_{p/2}v_p}{\operatorname{arcsinh}_p(v_p)},$$

where

$$SB_p(a,b)=b\,F\left(\frac{1}{p},\frac{1}{p};1+\frac{1}{p},1-\left(\frac{a}{b}\right)^p\right)^{-1},$$

$$v_p=\frac{|x^{p/2}-y^{p/2}|}{x^{p/2}+y^{p/2}},$$

and $A_p=A_p(a,b)$ is a power mean of order p.

Motivated by the work of Neuman [26, 27], Bhatia and Li [8], here we give a natural and new generalization of L, P, T and M by utilizing the generalized trigonometric and generalized hyperbolic functions as follows.

Generalization of means. For $p \geq 2$ and $a > b > 0$, we define

$$\begin{aligned}
\tilde{P}_p = \tilde{P}_p(a,b) = & \frac{a-b}{2\mathrm{arcsin}_p\left(\frac{a-b}{a+b}\right)} = \frac{x}{\mathrm{arcsin}_p(x)}A, \\
\tilde{T}_p = \tilde{T}_p(a,b) = & \frac{a-b}{2\mathrm{arctan}_p\left(\frac{a-b}{a+b}\right)} = \frac{x}{\mathrm{arctan}_p(x)}A, \\
\tilde{L}_p = \tilde{L}_p(a,b) = & \frac{a-b}{2\mathrm{artanh}_p\left(\frac{a-b}{a+b}\right)} = \frac{x}{\mathrm{artanh}_p(x)}A, \\
\tilde{M}_p = \tilde{M}_p(a,b) = & \frac{a-b}{2\mathrm{arsinh}_p\left(\frac{a-b}{a+b}\right)} = \frac{x}{\mathrm{arsinh}_p(x)}A,
\end{aligned} \tag{2.1}$$

where $x = (a-b)/(a+b)$. By utilizing [5, Lemma 1], the above functions can be expressed in terms of hypergeometric functions as follows,

$$\begin{aligned}
\tilde{P}_p &= \frac{A}{F\left(\frac{1}{p}, \frac{1}{p}; 1+\frac{1}{p}; x^p\right)}, \\
\tilde{T}_p &= \frac{A \cdot (1+x^p)^{1/p}}{F\left(\frac{1}{p}, \frac{1}{p}; 1+\frac{1}{p}; \frac{x^p}{1+x^p}\right)}, \\
\tilde{L}_p &= \frac{A}{F\left(1, \frac{1}{p}; 1+\frac{1}{p}; x^p\right)}, \\
\tilde{M}_p &= \frac{A \cdot (1+x^p)^{1/p}}{F\left(1, \frac{1}{p}; 1+\frac{1}{p}; \frac{x^p}{1+x^p}\right)}.
\end{aligned}$$

where $x = (a-b)/(a+b)$ and $a > b > 0$. Now we are in the position to state our main result. Our main result reads as follows.

Theorem 2.1. For $p \geq 2$ and $a > b > 0$, the functions $\tilde{P}_p$, $\tilde{T}_p$, $\tilde{L}_p$ and $\tilde{M}_p$ define a mean of two variables a and b.

Theorem 2.2. For $x \in (0,1)$ and $1 < p < q$,

1. the function $f_1(x) = \mathrm{arcsin}_p(x)/\mathrm{arcsin}_q(x)$ is strictly increasing,
2. the function $f_2(x) = \mathrm{arcsinh}_p(x)/\mathrm{arcsinh}_q(x)$ is strictly decreasing,
3. the function $f_3(x) = \mathrm{arctanh}_p(x)/\mathrm{arctanh}_q(x)$ is strictly increasing,
4. the function $f_4(x) = \mathrm{arctan}_p(x)/\mathrm{arctan}_q(x)$ is strictly decreasing (increasing) for $x \in (x, x_0)$ $(x \in (x_0, 1))$, where x_0 is the unique solution in $(0,1)$ to the equation $qx^{q-p} + (q-p)x^q - p = 0$.

In particular, for $2 \leq p < q$ one has

1. $\dfrac{\pi_q}{\pi_p} < \dfrac{\tilde{P}_p}{\tilde{P}_q} < 1$,

2. $1 < \dfrac{\tilde{M}_p}{\tilde{M}_q} < \dfrac{c_q}{c_p}$,

3. $1 < \dfrac{\tilde{L}_p}{\tilde{L}_q} < \dfrac{q}{p}$,

4. $\dfrac{\tilde{T}_p}{\tilde{T}_q} < (>) \dfrac{b_q}{b_p}$ for $x \in (0, x_0)(x \in (x_0, 1))$, where $x = (a-b)/(a+b), (a > b > 0)$, a, b are the arguments of means, i.e. $\tilde{T}_r = \tilde{T}_r(a, b)$, and

$$b_p = \operatorname{arctan}_p(1) = \frac{1}{2p}\left(\psi\left(\frac{1+p}{2p}\right) - \psi\left(\frac{1}{2p}\right)\right) = 2^{-\frac{1}{p}} F\left(\frac{1}{p}, \frac{1}{p}; 1+\frac{1}{p}; \frac{1}{2}\right), \tag{2.2}$$

$$c_p = \operatorname{arcsinh}_p(1) = \left(\frac{1}{2}\right)^{\frac{1}{p}} F\left(1, \frac{1}{p}; 1+\frac{1}{p}, \frac{1}{2}\right).$$

Theorem 2.3. For $a > b > 0$ and $x = (a-b)/(a+b)$, the following inequalities hold true,

$$\tilde{P}_p \tilde{M}_p \leq (\tilde{P}_{2p})^2 \leq k(x,p) \tilde{P}_p \tilde{M}_p, \tag{2.3}$$

where

$$k(x,p) = \frac{((1+x^p)^{2/p} + (1-x^p)^{2/p})^2}{4(1-x^{2p})^{1/p}}.$$

Theorem 2.4. For $a > b > 0$ and $x = (a-b)/(a+b)$, the following inequalities

$$\frac{1}{\tilde{P}_p} + \frac{r}{\tilde{M}_p} \leq \frac{r+1}{\tilde{P}_{2p}}, \tag{2.4}$$

and

$$(\tilde{P}_{2p})^{2p}\left(\frac{1}{(\tilde{P}_p)^p} + \frac{1}{(\tilde{M}_p)^p}\right) \leq R(x,p), \tag{2.5}$$

hold true, where

$$r = r(p,x) = \frac{(1+x^p)}{(1-x^p)}^{1/(2p)},$$

$$R = R(x,p) = [(1-x^{2p})^{1/(2p)} + (1-x^{2p})^{-1/(2p)}]^{1/(2p)}/2^{2p-1}.$$

Theorem 2.5. For $a > b > 0$ and $x = (a-b)/(a+b)$, one has,

$$A\tilde{L}_{2p} \geq \tilde{T}_p \tilde{L}_p, \tag{2.6}$$

$$\frac{A^2}{\tilde{T}_p \tilde{L}_p} - \frac{A}{\tilde{L}_{2p}} \leq \frac{x^{2p}}{4(1-x^{2p})}. \tag{2.7}$$

Theorem 2.6. For $a > b > 0$ and $x = (a-b)/(a+b)$, one has,

$$\frac{4(1+x^p)(1+x^p/(p+1))}{(x^p+2)^2} \le \frac{\tilde{T}_p}{A} \le 1 + \frac{x^p}{1+p}, \tag{2.8}$$

and

$$\frac{4(1-x^p)(1-x^p/(1+p))}{(2-x^p)^2} \le \frac{\tilde{L}_p}{A} \le 1 - \frac{x^p}{1+p}. \tag{2.9}$$

Theorem 2.7. For $a > b > 0$ and $x = (a-b)/(a+b)$, one has,

$$\frac{p\,x}{B\,(1/p, 1+1/p)} \le \frac{A}{\tilde{P}_p} \le \frac{p\,x\,B(1/p, 1+1/p)(2-x^p)^2]}{4(1-x^p)B(1/p, 1+1/p)]} \tag{2.10}$$

and

$$\frac{x}{j(x,p)} \le \frac{A}{\tilde{M}_p} \le \frac{x(1+(1+x^p)^{1/p})}{4j(x,p)(1+x^p)^{1/p}}, \tag{2.11}$$

where $j(x,p) = \int_0^x (1+t^p)^{1/p} dt$.

The paper is organized as follows. In section 1, we give the definition of the special functions involved in our formulation. Section 2 is dedicated to the definition of new means and the statement of the main result. Section 3 consists of preliminary earlier and related results, which will be used in the proving procedures sequel. Section 4 gives the proof of the main result.

3 Preliminaries and lemmas

Lemma 3.1. [3, Theorem 2] For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, and be differentiable on (a,b). Let $g^{'}(x) \neq 0$ on (a,b). If $f^{'}(x)/g^{'}(x)$ is increasing (decreasing) on (a,b), then so are

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \textit{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f^{'}(x)/g^{'}(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

For the proof of the following lemma, see [6, Theorem 2.1].

Lemma 3.2. For $x \in (0,1)$,

1. the function $p \mapsto \text{arcsin}_p(x)$ and $p \mapsto \text{arctanh}_p(x)$ are strictly decreasing and log-convex on $(1,\infty)$. Moreover, $p \mapsto \text{arcsin}_p(x)$ is strictly geometrically convex on $(1,\infty)$.

2. The function $p \mapsto \text{arctan}_p(x)$ is strictly increasing and concave on $(1,\infty)$.

It is easy to observe that the function $p \mapsto \text{arcsinh}_p(x)$ is strictly decreasing on $(1,\infty)$.

Lemma 3.3. We have

1. The function $f(t) = (1+t^p)^{-1}$ is strictly decreasing for $t \in (0,1)$,

2. The function $g(t) = (1-t^p)^1$ is strictly increasing on $(0,1)$,

3. The function $h(t) = (1-t^p)^{-1/p}$ is strictly increasing on $(0,1)$,
4. The function $s(t) = (1+t^p)^{-1/p}$ is strictly decreasing on $(0,1)$.

Proof. These are immediate consequences of definitions. Q.E.D.

For easy reference we recall some well-known inequalities from the literature as follows.

Cauchy-Bouniakowski inequality. If $f, g : [a,b] \to \mathbb{R}$ are integrable, then

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b f(x)^2 dx \int_a^b g(x)^2 dx. \tag{3.1}$$

Pólya-Szegő inequality. If $f, g : [a,b] \to \mathbb{R}$ are integrable, and for all $x \in [a,b]$

$$0 < \alpha < f(x) < A, \quad 0 < \beta < g(x) < B,$$

then

$$\frac{\int_a^b f(x)^2 dx \int_a^b g(x)^2 dx}{\left(\int_a^b f(x)g(x)dx\right)^2} \leq K(\alpha, A, \beta, B), \tag{3.2}$$

where

$$K = K(\alpha, A, \beta, B) = \frac{1}{4}\left(\sqrt{\frac{AB}{\alpha\beta}} + \sqrt{\frac{\alpha\beta}{AB}}\right)^2.$$

Chebyshev's inequality. Let $f, g : [a,b] \to \mathbb{R}$ be integrable. If f and g have same type of monotonicity, then

$$\int_a^b f(x)dx \cdot \int_a^b g(x)dx \leq (b-a)\int_a^b f(x)g(x)dx. \tag{3.3}$$

If f and g have distinct type of monotonicity, then

$$\int_a^b f(x)dx \cdot \int_a^b g(x)dx \geq (b-a)\int_a^b f(x)g(x)dx. \tag{3.4}$$

Grüss inequality. If $f, g : [a,b] \to \mathbb{R}$ are integrable, and for all $x \in [a,b]$

$$0 < \alpha < f(x) < A, \quad 0 < \beta < g(x) < B,$$

then

$$\left|(b-a)\int_a^b f(x)g(x)dx - \int_a^b f(x)dx \cdot \int_a^b g(x)dx\right| \leq \frac{(b-a)^2}{4} \cdot (A-\alpha)(B-\beta). \tag{3.5}$$

Minkowski's inequality. Let $f, g : [a,b] \to \mathbb{R}$ be integrable and $f, g > 0$. Write

$$h_t(f) = \left(\int_a^b f(x)^t dx \right)^{1/t}.$$

Then one has

$$h_t(f+g) \le h_t(f) + h_t(g), \quad \text{for} \quad t \ge 1, \tag{3.6}$$

$$h_t(f+g) \ge h_t(f) + h_t(g), \quad \text{for} \quad t \le 1. \tag{3.7}$$

Diaz-Metcalf inequality. Let $f, g : [a,b] \to \mathbb{R}$ be integrable and suppose that there exist constants m and M such that

$$m \le g(x)/f(x) \le M.$$

Then one has

$$\int_a^b g^2 dx + m \cdot M \cdot \int_a^b f^2 dx \le (m+M) \cdot \int_a^b fg dx. \tag{3.8}$$

4 Proof of main result and corollaries

Proof of Theorem 2.1. It is enough to prove that for $p \ge 2$ the following inequalities

$$L \le \tilde{L}_p < \tilde{P}_p < A < \tilde{M}_p < \tilde{T}_p \le Q \tag{4.1}$$

hold true, where $Q = Q(a,b) = \sqrt{(a^2+b^2)/2}$ is root square mean. For $p > 1$ and $x \in (0,1)$, the following inequalities

$$\mathrm{arctan}_p(x) < \mathrm{arcsinh}_p(x) < \mathrm{arcsin}_p(x) < \mathrm{arctanh}_p(x)$$

(see [13, Lemma 9]) imply that

$$\tilde{L}_p < \tilde{P}_p < \tilde{M}_p < \tilde{T}_p. \tag{4.2}$$

It is sufficient to prove that $\tilde{L}_p$ and $\tilde{T}_p$ are means. Since $\tilde{P}_p < A$, and from the monotonicity of $x/\mathrm{arcsinh}_p(x)$ we get $M_p > A$, so (4.2) can be completed as:

$$\tilde{L}_p < \tilde{P}_p < A < \tilde{M}_p < \tilde{T}_p, \quad p > 1.$$

Clearly, $\mathrm{arctanh}_p(x) \le \mathrm{arctanh}(x)$. Thus $x/\mathrm{arctanh}_p(x) \ge x/\mathrm{arctanh}(x)$ for $x \in (0,1)$, implying that $\tilde{L}_p/A \ge L/A$, so $\tilde{L}_p \ge L$. Let $x = (a-b)/(a+b)$, then it is easy to see that one has the following identity

$$\frac{Q}{A} = \sqrt{1+x^2}.$$

The last inequality $\tilde{T}_p/A < Q/A = \sqrt{1+x^2}$ in (4.1) can be written as $x/\mathrm{arctan}_p(x) < \sqrt{1+x^2}$, or equivalently,

$$\int_0^x \frac{1}{1+t^p} dt > \frac{x}{\sqrt{1+x^2}}.$$

Since $1+t^p \le 1+t^2$ (by $p \ge 2$ and $t \in (0,1)$), we get $1/(1+t^p) \ge 1/(1+t^2)$, so $\mathrm{actan}_p(x) \ge \arctan(x)$. Thus

$$\frac{x}{\mathrm{arctan}_p}(x) \le \frac{x}{\arctan(x)} = \frac{T}{A} < \frac{Q}{A},$$

by the known inequality $T < Q$. Therefore, $\tilde{T}_p < Q$, since we have also $\tilde{T}_p \geq T$, one has $T \leq \tilde{T}_p < Q$, with equality only for $p = 2$. This completes the proof of (4.1). $\square$

Corollary 4.1. For $p \geq 2$, $x, y > 0$ with $x \neq y$, we have

$$\frac{x+y}{2\left(1-\alpha\log\left(1-\left(\frac{x-y}{x+y}\right)^p\right)\right)} < \tilde{L}_p(x,y) < \frac{x+y}{2\left(1-\beta\log\left(1-\left(\frac{x-y}{x+y}\right)^p\right)\right)}$$

and

$$\frac{x+y}{2\left(1+\alpha\log\left(1+\left(\frac{x-y}{x+y}\right)^p\right)\right)}u < \tilde{M}_p(x,y) < \frac{x+y}{2\left(1+\beta\log\left(1+\left(\frac{x-y}{x+y}\right)^p\right)\right)}u$$

where

$$\alpha = 1/p, \quad \beta = 1/(1+p), \quad u = \left(1+\left(\frac{x-y}{x+y}\right)^p\right)^{-1/p}$$

Proof. Proof follows easily from [13, Theorem 2]. Q.E.D.

The proof of following three corollaries follow easily from [5, Theorem 1], Lemma 3.2, and [6, Corollary 2.2], respectively.

Corollary 4.2. For $p \geq 2$, $x, y > 0$ with $x \neq y$, we have

$$\sqrt[p]{1-\left(\frac{x-y}{x+y}\right)^p}\,\tilde{P}_p(x,y) < \tilde{L}_p(x,y) < \frac{\tilde{P}_p(x,y)}{A(x,y)^{p-1}}.$$

Corollary 4.3. For $p \geq 3$, we have the following Turán type inequalities for the means $\tilde{P}_p$, $\tilde{L}_p$, $\tilde{T}_p$,

$$\tilde{P}_p^2 > \tilde{P}_{p-1}\tilde{P}_{p+1},$$

$$\tilde{L}_p^2 > \tilde{L}_{p-1}\tilde{L}_{p+1},$$

$$\tilde{T}_p^2 < \tilde{T}_{p-1}\tilde{T}_{p+1}.$$

It also follows from Lemma 3.2 that for $p, q \geq 2$, we have

$$\tilde{P}_{\sqrt{pq}} \geq \sqrt{\tilde{P}_p\tilde{P}_q},$$

where equality holds for $p = q$.

Corollary 4.4. One has

$$P < \tilde{P}_3^2/\tilde{P}_4 < \tilde{P}_3^2/\tilde{L}_4,$$

$$L < \tilde{L}_3^2/\tilde{L}_4 < \tilde{L}_3^2/L,$$

$$T > \tilde{T}_3^2/\tilde{T}_4 > \tilde{T}_3^2/T.$$

Corollary 4.5. For $p \geq 2$ and $a > b > 0$, we have

$$(2/\pi_p)A < \tilde{P}_p < P_{2p} < A.$$

Proof. It follows from Lemma 3.1 that the function $t/\mathrm{arcsin}_p(t)$ is decreasing in $t \in (0,1)$. By using l'Hôpital rule, we get $\lim_{t\to 0}(t/\mathrm{arcsin}_p(t)) = 1$ and $\lim_{t\to 1}(t/\mathrm{arcsin}_p(t)) = (p\sin(\pi/p)/\pi)$. This implies the first inequality, the second inequality follows from Lemma 3.2, and the proof of third inequality follows from first one. Q.E.D.

Theorem 4.6. For $p \geq$ and $a > b > 0$, we have

1. $\frac{b_p}{c_p}\tilde{T}_p < \tilde{M}_p < \tilde{T}_p$,
2. $\frac{c_p}{a_p}\tilde{M}_p < \tilde{P}_p < \tilde{M}_p$,
3. $\frac{b_p}{a_p}\tilde{T}_p < \tilde{P}_p < \tilde{T}_p$,

where $a_p = \pi/2$, and b_p and c_p are as defined in Theorem 2.2.

Proof. It is easy to see from the definition (1.2) and (2.1) that the following ratios of the means can be simplified as below:

$$f_1(z) = \frac{\int_0^z (1+t^p)^{-1}dt}{\int_0^z (1+t^p)^{-1/p}dt} = \frac{\tilde{M}_p}{\tilde{T}_p}, \quad f_2(z) = \frac{\int_0^z (1+t^p)^{-1/p}dt}{\int_0^z (1-t^p)^{-1/p}dt} = \frac{\tilde{P}_p}{\tilde{M}_p},$$

$$f_3(z) = \frac{\int_0^z (1+t^p)^{-1}dt}{\int_0^z (1-t^p)^{-1/p}dt} = \frac{\tilde{P}_p}{\tilde{T}_p}.$$

For the monotonicity of the functions f_i, $i = 1,2,3$, we use the result given by Cheeger et. al [15, p.42] that if $h_1, h_2 : \mathbb{R} \to [0,\infty)$ are the integrable functions, and h_1/h_2 is decreasing then the function

$$x \mapsto \frac{\int_0^x h_1(t)dt}{\int_0^x h_2(t)dt}$$

is also decreasing. Clearly, the functions f_i, $i = 1,2,3$ are decreasing, and the limiting values follows easily from the definitions. This completes the proof. Q.E.D.

Proof of Theorem 2.2. Let $f_1(x) = f(x)/g(x)$, where $f(x) = \mathrm{arcsin}_p(x)$, $g(x) = \mathrm{arcsin}_q(x)$, $x \in (0,1)$, and $1 < p < q$. Applying Lemma 3.1, for $a = 0$, one has that $f_1(x) = (f(x) - f(0))/(g(x) - g(0))$. Now, after simple computations, we get

$$\frac{f'(x)}{g'(x)} = \frac{(1-x^q)^{1/q}}{(1-x^p)^{1/p}} = h_1(x).$$

Since

$$h_1'(x) = \frac{(x^p - x^q)}{x(1-x^q)(1-x^p)} \cdot h_1(x) > 0,$$

we get that $h_1(x)$ is strictly increasing in $(0,1)$. This implies that $f_1(x)$ is strictly increasing, too. This fact implies the proof of part (1).

For the proof of (2), write $f_2(x) = f(x)/g(x) = (f(x)-f(0))/(g(x)-g(0))$, where $f(x) = \operatorname{arcsinh}_p(x)$ and $g(x) = \operatorname{arcsinh}_q(x)$. One has

$$\frac{f'(x)}{g'(x)} = \frac{(1+x^q)^{1/q}}{(1+x^p)^{1/p}} = h_2(x).$$

After simple computations, we obtain

$$(1+x^p)(1+x^q)\cdot h_2'(x) = (x^{q-1}-x^{p-1})h_2(x) < 0,$$

as $q-1>0$, $p-1>0$, $q-1>p-1$ and $x\in(0,1)$. This means that $h_2'(x)<0$, so $h_2(x)$ is strictly decreasing; implying that $f_2(x)$ is strictly decreasing.

For (3), let $f_3(x) = f(x)/g(x) = \operatorname{arctanh}_p(x)/\operatorname{arctanh}_q(x)$. One has $f'(x)/g'(x) = (1-x^q)/(1-x^p) = h_3(x)$. After simple computations, we see that

$$(1-x^p)^2\cdot h_3'(x) = x^{p-1}\cdot u_1(x),$$

where $u_1(x) = (q-p)x^q - qx^{q-p} + p$, here $u_1(0) = p > 0$ and $u_1(1) = 0$. On the other hand, $u_1'(x) = q(q-p)x^{q-1}(1-x^{-p})$. Since $1-x^{-p} = (x^p-1)/x^p < 0$ (by $p>0, x\in(0,1)$), we get that $u_1'(x) < 0$. Thus $u_1(x) > u_1(1) = 0$. Therefore, $h_3'(x) > 0$, so $h_3(x)$ is strictly increasing. This implies that $f_3(x)$ is strictly increasing, too.

For the proof of part (4), let $f_4(x) = f(x)/g(x) = \arctan_p(x)/\arctan_q(x)$. One has $f'(x)/g'(x) = (1+x^q)/(1+x^p) = h_4(x)$. After simple computations, we conclude that

$$(1+x^p)^2 h_4'(x) = x^{p-1}u_2(x),$$

where

$$u_2(x) = qx^{q-p} + (q-p)x^q - p.$$

Here $u_2(0) = -p$, and $u_2(1) = 2(q-p) > 0$, so $u_2(x)$ has at least a zero in $(0,1)$. We will show that, there is a single such zero. Indeed, one has

$$u_2'(x) = q(q-p)x^{q-p-1} + q(q-p)x^{q-1} > 0,$$

so $u_2(x)$ is strictly increasing in $(0,1)$. Let x_0 be the single zero of $u_2(x) = 0$. As $u_2(0) = -p$, clearly $u_2(x) < 0$ for $x\in(0,x_0)$ and similarly, $u_2(x) > 0$ for $x\in(x_0,1)$. As $h_4'(x) < 0$, resp. $h_4'(x) > 0$ in these intervals, the proof of (4) follows from the monotonicity of $h_4(x)$ and Lemma 3.1. $\square$

Proof of Theorem 2.3. Let $f(t) = \sqrt{F(t)}$ and $g(t) = \sqrt{G(t)}$ in Cauchy-Bouniakowski inequality (3.1), where $F(t)$, $G(t) > 0$. Put $[a,b] = [0,x]$, Then one gets the inequality:

$$\left(\int_0^x \sqrt{FG}dt\right)^2 \le \int_0^x F dt\cdot\int_0^x G dt. \tag{4.3}$$

With the same notations, from the Pólya-Szegö inequality (3.2) one gets:

$$k(x,p)\left(\int_0^x \sqrt{FG}dt\right)^2 \geq \int_0^x F dt \cdot \int_0^x G dt, \tag{4.4}$$

here $k(x,p)$ is as defined in Theorem 2.3. Let now $f(t) = (1-t^p)^{-1/p}$ and $g(t) = (1+t^p)^{-1/p}$. From (4.3) and (4.4) one obtains

$$\operatorname{arcsin}_{2p}(x)^2 \leq \operatorname{arcsin}_p(x)\operatorname{arcsinh}_p(x), \tag{4.5}$$

and

$$\operatorname{arcsin}_p(x)\operatorname{arcsinh}_p(x) \leq k(x,p)\operatorname{arcsin}_{2p}(x)^2, \tag{4.6}$$

respectively. By definition, inequality (4.5) and (4.6) imply the proof of left hand-side and right-hand side of (2.3), respectively. $\square$

Proof of Theorem 2.4. Apply the Diaz-Metcalf inequality (3.8) for $f(t) = \sqrt{F(t)}$, $g(t) = \sqrt{G(t)}$, $[a,b] = [0,x]$, yielding

$$\int_0^x G dt + M\cdot m\cdot \int_0^x F dt \leq (M+m)\cdot \int_0^x \sqrt{FG}dt.$$

Let $F(t) = (1+t^p)^{-1/p}$, $G(t) = (1-t^p)^{-1/p}$. Here $G(t)/F(t) = ((1+t^p)/(1-t^p))^{1/p}$, which is strictly increasing. Thus

$$m = 1 \leq \sqrt{F/G} \leq ((1+x^p)/(1-x^p))^{1/(2p)} = M.$$

One obtains

$$\operatorname{arcsin}_p(x) + M\cdot \operatorname{arcsinh}_p(x) \leq (M+1)\operatorname{arcsin}_{2p}(x), \tag{4.7}$$

this implies the proof of (2.4).

Let $[a,b] = [0,x]$ and $f(t) = (1+t^p)^{-1}$ and $g(t) = (1-t^p)^{-1}$. As $f(t)+g(t) = 2/(1-t^{2p})$, applying the Minkowski inequality (3.7) for $t = 1/p$, $p > 1$, we get

$$\operatorname{arcsin}_p(x)^p + \operatorname{arcsinh}_p(x)^p \leq 2\left(\int_0^x A^2 dt\right)^p, \tag{4.8}$$

where $A(t) = 1/(1-t^{2p})^{1/(2p)}$. Clearly, $\int_0^x A(t)dt = \operatorname{arcsin}_{2p}(x)$, so for obtaining an upper bound for $\int_0^x A(t)^2 dt$, we apply the Pólya-Szegő inequality for $f(t) = 1/(1-t^{2p})^{1/p}$ and $g(t) = 1$. Since in this case one has $1 \leq f(t) \leq 1/(1-x^{2p})^{1/p}$, we get from (3.2)

$$\int_0^x A(t)^2 dt \leq \operatorname{arcsin}_{2p}(x)^2 R(x,p),$$

By using (4.8), finally we get

$$\frac{x^p(\operatorname{arcsin}_p(x)^p + \operatorname{arcsinh}_p(x)^p)}{\operatorname{arcsin}_{2p}(x)^{2p}} \leq R(x,p), \tag{4.9}$$

this implies inequality (2.5). ☐

Proof of Theorem 2.5. Apply the Chebyshev inequality (3.4) for the functions (1) and (2) of Lemma 3.3, which are of different type of monotonicity. One obtains the following inequality

$$x \cdot \text{arctanh}_{2p}(x) \geq \arctan_p(x)\text{arctanh}_p(x). \tag{4.10}$$

This implies (2.6). The proof of (2.7) follows if we apply the Grüss inequality for the same functions as above and utilize relation (4.10). ☐

Proof of Theorem 2.6. Applying Cauchy-Bouniakowski inequality (3.1) for $f(t) = \sqrt{F(t)}$, and $g(t) = 1/\sqrt{F(t)}$, we get the following inequality

$$\int_a^b F dt \cdot \int_a^b 1/F dt \geq (b-a)^2. \tag{4.11}$$

Applying the same notations as above for the Pólya-Szegő inequality (3.2), one obtains the inequality (called also as Schweizer inequality). Suppose that $0 < \alpha < F(t) < A$. Then

$$\int_a^b F dt \cdot \int_a^b 1/F dt \geq \frac{(b-a)^2(\alpha + A)^2}{4\alpha A}. \tag{4.12}$$

Let now $F(t) = 1 + t^p$, with $t \in [a, b] = [0, x]$ in (4.11) and (4.12). As $\alpha = 1$, $A = 1 + x^p$, one obtains the following double inequality

$$\frac{4(1+x^p)(1+x^p/(p+1)}{(x^p+2)^2} \leq \frac{x}{\arctan_p(x)} \leq 1 + \frac{x^p}{1+p} \tag{4.13}$$

and

$$\frac{4(1-x^p)(1-x^p/(1+p)}{(2-x^p)^2} \leq \frac{x}{\text{arctanh}_p(x)} \leq 1 - \frac{x^p}{1+p}. \tag{4.14}$$

The right side of (4.13) and (4.14) are obtained from (4.11), while the left side of (4.13) and (4.14) are obtained from (4.12). This completes the proof of Theorem 2.6. ☐

Proof of Theorem 2.7. Letting $t^p = u$ one has $dt = (1/p)u^{1/p-1}du$, and applying inequality (4.11) and (4.12) for $F(t) = (1-t^p)^{1/p}$ we get

$$\int_0^x F dt = (1/p) \int_0^{x^p} u^{1/p-1}(1-u)^{1/p} du.$$

As

$$\int_a^b u^{a-1}(1-u)^{b-1} du = B(a,b)B(a,b:x),$$

where $B(a, b)$ is the beta function, and $B(a, b : x)$ is the incomplete beta function, we get

$$\int_0^x (1-t^p)^{1/p} dt = (1/p) \cdot B(1/p, 1+1/p)B(1/p, 1+1/p : x^p) \tag{4.15}$$

Applying (4.15), and utilizing (4.11) and (4.12), we get the following double inequality

$$\frac{p\,x^2}{B\,(1/p,1+1/p)} \le \operatorname{arcsin}_p(x) \le \frac{p\,x^2\,B(1/p,1+1/p)(2-x^p)^2]}{4(1-x^p)B(1/p,1+1/p)]}.$$

This implies (2.10). For the proof of (2.11), we apply (4.11) and (4.12) for $F(t)=(1+t^p)^{1/p}$ and get the following double inequality

$$\frac{x^2}{j(x,p)} \le \operatorname{arcsinh}_p(x) \le \frac{x^2(1+(1+x^p)^{1/p})}{4j(x,p)(1+x^p)^{1/p}}.$$

This completes the proof of Theorem 2.7. $\square$

We finish this paper by giving the following remark.

Remark 4.7. In [26], Neuman studied the p-version of Schwab-Borchardt mean S_p, $p>1$, which was expressed in terms of hypergeometric function as follows,

$$\frac{y}{S_p(x,y)} = F\left(\frac{1}{p},\frac{1}{p},1+\frac{1}{p},1-\left(\frac{x}{y}\right)^p\right), \tag{4.16}$$

([27, (9)]). Here we give a proof, which leads us to formula [26, (22)]. For $x>y>0$, $p>1$, (4.16) can be written as

$$\frac{y}{S_p(x,y)} = F\left(\frac{1}{p},\frac{1}{p},1+\frac{1}{p},-w^p\right)$$

where $w=\left(\frac{x^p-y^p}{y^p}\right)^{1/p}$. By applying the following transformation formula (see [1, 15.3.5])

$$F(a,b;c;z)=(1-z)^{-b}F\left(b,c-a;c;-\frac{z}{1-z}\right),$$

[5, Lemma 1] and the identity $\operatorname{arcsinh}_p(\sqrt[p]{t^p-1})=\operatorname{arccosh}_p(t)$ we get

$$\begin{aligned}
\frac{y}{S_p(x,y)} &= \frac{w}{w(1+w^p)^{1/p}}F\left(\frac{1}{p},\frac{1}{p},1+\frac{1}{p},\frac{w^p}{1+w^p}\right)\\
&= \frac{\operatorname{arcsinh}_p(w)}{w} = \frac{\operatorname{arcsinh}_p\left(\frac{x^p-y^p}{y^p}\right)^{1/p}}{\left(\frac{x^p-y^p}{y^p}\right)^{1/p}}\\
&= \frac{y\operatorname{arcsinh}_p\sqrt[p]{(x/y)^p-1}}{(x^p-y^p)^{1/p}} = \frac{y\operatorname{arccosh}_p(x/y)}{\sqrt[p]{(x^p-y^p}}.
\end{aligned}$$

The case when $0<x<y$ follows similarly.

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