Non-global solutions for a class of fourth-order wave equations

Tarek Saanouni

Department of Mathematics, University of Tunis El Manar, Faculty of Science of Tunis, LR03ES04 partial differential Equations and applications, 2092 Tunis, Tunisia
Qassim University, Faculty of Science and Literature of Uqlat Assgoor, Al-Qaseem, Saudi Arabia
E-mail: tarek.saanouni@ipeiem.rnu.tn

Abstract

Adopting the so-called concavity method, we establish a finite time blow-up result for a class of fourth-order non-linear wave equations with positive energy.

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1 Introduction

This paper studies non-existence of global solutions to the semi-linear fourth-order wave equation

$$\begin{cases} \ddot{u} + \Delta^2 u + u = f(u)\,; \\ (u, \dot{u})_{|t=0} = (u_0, u_1)\,, \end{cases} \tag{1.1}$$

where u is a real valued function of the variable $(t, x) \in [0, T) \times \mathbb{R}^N$ and the source term f is a real function to be fixed later.

Equations of the fourth-order appear in problems of solid mechanics, in the theory of thin plates and beams, elastic rods, and shallow water waves [7]. Moreover, in one space dimension, such equations describe a number of physical and biological phenomena, such as the analysis of elasto-plastic microstructure models for longitudinal motion of an elasto-plastic bar [2].

The semi-linear fourth-order wave equation with a pure power non-linearity

$$\ddot{u} + \Delta^2 u \pm u|u|^{p-1} = 0, \quad u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}, \quad p > 1,$$

is invariant under the scaling

$$u_\lambda(t, x) := \lambda^{\frac{4}{p-1}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

The homogeneous Sobolev $\|\,.\,\|_{\dot{H}^{s_c}}$ norm, for $s_c := \frac{N}{2} - \frac{4}{p-1}$, is invariant under the dilatation $u \mapsto u_\lambda$. The energy critical case $s_c = 2$, corresponds to the critical exponent $p_c := \frac{N+4}{N-4}$, $N \geq 5$. Local well-posedness holds in the energy space for $1 < p \leq p_c$ and $N \geq 5$ or $1 < p < \infty$ and $N \in [1, 4]$. Moreover, global well-posedness hold in the defocusing case [3]. Scattering for $1 + \frac{8}{N} \leq p < p_c$ was established by Levandosky [4], Levandosky and Strauss [5] and Pausader [8]. See [10] for an exponential source term in four space dimensions. Few results deal with finite time

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blow-up of solutions to fourth-order wave equations [11, 14].

It is the aim of this manuscript to give some sufficient conditions on the data which give finite time blowing-up solutions to (1.1). This extends some known results about non-global solutions to non-linear wave equations [12, 13, 9].

The rest of the paper is organized as follows. The next section contains some technical tools needed in the sequel and the main results. The two last sections are devoted to proving the main results.

In the rest of this note, for simplicity, denote the Lebesque space $L^r := L^r(\mathbb{R}^N)$ and the Sobolev space $H^2 := H^2(\mathbb{R}^N)$. Denote also $\|\,.\,\|_{L^2}$ by $\|\,.\,\|$ and

$$\|\,.\,\|_{H^2} := \Big(\|\,.\,\|^2 + \|\Delta\,.\,\|^2\Big)^{\frac{1}{2}}.$$

Moreover, for any $u \in H^2$, define the quantities

$$G(t) := \|u(t)\|^2 \quad \text{and} \quad I(t) := \|u(t)\|_{H^2}^2 - \int_{\mathbb{R}^N} u(t,x)f(u(t,x))\,dx.$$

2 Preliminary and main results

2.1 Preliminary

The Cauchy problem (1.1) is locally well-posed in the energy space [3, 10].

Proposition 2.1. Let $(u_0, u_1) \in H^2 \times L^2$ and $f \in \mathcal{C}^1(\mathbb{R})$. Assume that one of the next conditions holds

1/ $N = 4$ and f satisfies the following exponential growth condition

$$\begin{cases} f(0) = f'(0) \qquad = 0\,; \\ \forall \alpha > 0, \exists \Re_\alpha > 0 \quad \text{s. th} \quad |f(u) - f(v)|^2 \leq \Re_\alpha |u-v|^2 \Big(e^{\alpha u^2} - 1 + e^{\alpha v^2} - 1\Big). \end{cases} \tag{2.1}$$

2/ $N \geq 5$ and $f(u) = u^p$ for some $1 < p \leq \frac{N+4}{N-4}$.

Then, the semi-linear wave problem (1.1) has a unique maximal solution u in the class

$$C_{T^*}(H^2) \cap C^1_{T^*}(L^2).$$

Moreover, the solution satisfies conservation of the energy

$$E(t) := E(u(t)) := \frac{1}{2}\Big(\|u(t)\|_{H^2}^2 + \|\dot{u}(t)\|^2\Big) - \int_{\mathbb{R}^N} F(u(t,x))\,dx = E(0),$$

where F is the primitive of f vanishing on zero.

Remark 2.2. In the particular case $N = 4$, the critical exponent $p_c := \frac{N+4}{N-4}$ is not defined. By use of the Sobolev injection

$$H^1(\mathbb{R}^2) \hookrightarrow L^r(\mathbb{R}^2), \quad \text{for all} \quad 2 \leq r < \infty,$$

and Moser-Trudinger inequalities [1], it is proved that every polynomial source term is energy sub-critical and a critical non-linearity is of exponential growth at infinity [10].

Throughout this manuscript, consider two real numbers $-2 < a < -1$ and $0 < \varepsilon < 1$ satisfying

$$a < -\frac{1}{2}\Big(1 + \frac{e^{\varepsilon - 1}}{\varepsilon}\Big). \tag{2.2}$$

Remarks 2.3. Note that,

1/ the statement of changes of $\varepsilon \mapsto -\frac{1}{2}\Big(1 + \frac{e^{\varepsilon-1}}{\varepsilon}\Big)$ shows that $\mathcal{A}_a := \Big\{\varepsilon \in (0,1)$ such that $a < -\frac{1}{2}\Big(1 + \frac{e^{\varepsilon-1}}{\varepsilon}\Big)\Big\}$ is an infinite set if $a < -1$;

2/ here and hereafter, if $N = 4$, the non-linearity and it's primitive vanishing on zero are

$$f_a(u) := e^u - 1 + au \quad \text{and} \quad F_a(u) := \int_0^u f_a(s)\, ds = e^u - 1 - u + \frac{a}{2}u^2.$$

The following property about the non-linearity will be useful in the sequel.

Lemma 2.4. Take $-2 < a < -1$ and $\varepsilon \in \mathcal{A}_a$. Then,

1/ $b := b(a, \varepsilon) := \inf_{x\in\mathbb{R}} \Big(xf_a(x) - (2 + \varepsilon)F_a(x)\Big) \in \mathbb{R}$;

2/ the next inequality holds

$$xf_a(x) \geq (2 + \varepsilon)F_a(x) + (a + 1)\frac{\varepsilon}{2}x^2, \quad \text{for any} \quad x \in \mathbb{R}. \tag{2.3}$$

Proof. 1/ It is sufficient to write $\lim_{x\to\pm\infty} \Big(xf_a(x) - (2 + \varepsilon)F_a(x)\Big) = +\infty$;

2/ define the function

$$\begin{aligned} \varphi(x) := \varphi_{\varepsilon,a}(x) \quad &:= \quad -(2 + \varepsilon)F_a(x) + xf_a(x) - (a + 1)\frac{\varepsilon}{2}x^2 \\ &= \quad x(e^x - 1 + ax) - (2 + \varepsilon)(e^x - 1 - x + \frac{a}{2}x^2) - (a + 1)\frac{\varepsilon}{2}x^2 \\ &= \quad (x - 2 - \varepsilon)e^x + (2 + \varepsilon) + (1 + \varepsilon)x - \frac{\varepsilon}{2}(1 + 2a)x^2. \end{aligned}$$

Thus,

$$\varphi'(x) = (x - 1 - \varepsilon)e^x + (1 + \varepsilon) - \varepsilon(1 + 2a)x\,;$$
$$\varphi''(x) = (x - \varepsilon)e^x - \varepsilon(1 + 2a), \quad \varphi'''(x) = (x + 1 - \varepsilon)e^x.$$

Using (2.2), a statement of changes shows that $\varphi \geq 0$.

Q.E.D.

Recall a standard result about non-global solutions to an ordinary differential inequality.

Lemma 2.5. Let $\varepsilon > 0$. There is no real function $H \in C^2(\mathbb{R}_+)$ satisfying

$$H(0) > 0,\ H'(0) > 0 \quad \text{and} \quad HH'' - (1+\varepsilon)(H')^2 \geq 0 \quad \text{on} \quad \mathbb{R}_+.$$

Proof. Assume with contradiction, the existence of such a function. Then $(H^{-(1+\varepsilon)}H')' \geq 0$ and

$$\frac{H'}{H^{1+\varepsilon}} \geq \frac{H'(0)}{H(0)} > 0.$$

This is a Riccati inequality with blow-up time $T < \frac{1}{\varepsilon}\frac{H(0)}{H'(0)}$. This contradiction achieves the proof.

Q.E.D.

2.2 Main results

This subsection contains two Theorems about non-global solutions to (1.1) under some sufficient conditions on the data.

Theorem 2.6. Take $N \geq 5$, $f(u) = u^p$ for some $1 < 1+\varepsilon < p \leq \frac{N+4}{N-4}$, $E_0 > 0$ and $(u_0, u_1) \in H^2 \times L^2$. Assume that

$$E(0) = E_0, \quad I(0) < 0 \quad \text{and} \quad G'(0) > \frac{4(2+\varepsilon)}{\varepsilon}E_0.$$

Then, the maximal solution $u \in C_{T^*}(H^2) \cap C^1_{T^*}(L^2)$ to (1.1), blows-up in a finite time. Precisely,

$$T^* < \infty \quad \text{and} \quad \limsup_{t \to T^*} \|u(t)\| = \infty.$$

Now, consider non-global solutions to (1.1) in four space dimensions.

Theorem 2.7. Take $N = 4$, $-2 < a < -1$, $f := f_a$, $\varepsilon \in \mathcal{A}_a$, $E_0 > 0$ and $(u_0, u_1) \in H^2 \times L^2$. Assume that

$$E(0) = E_0, \quad I(0) < 0 \quad \text{and} \quad G'(0) > \frac{2(2+\varepsilon)}{\varepsilon(2+a)}E_0.$$

Then, the maximal solution $u \in C_{T^*}(H^2) \cap C^1_{T^*}(L^2)$ to (1.1), blows-up in a finite time. Precisely,

$$T^* < \infty \quad \text{and} \quad \limsup_{t \to T^*} \|u(t)\| = \infty.$$

3 Proof of Theorem 2.6

The proof is based on the following auxiliary result.

Lemma 3.1. Take $N \geq 5$ and $f(u) = u^p$ for some $1 < 1+\varepsilon < p \leq \frac{N+4}{N-4}$. Let $(u_0, u_1) \in H^2 \times L^2$ such that $E(0) = E_0 > 0$, $I(0) < 0$ and $G'(0) > \frac{4(2+\varepsilon)}{\varepsilon}E_0$. Then, $I < 0$ and $G' > \frac{4(2+\varepsilon)}{\varepsilon}E$ on $[0, T^*)$.

Proof. Compute using the equation (1.1), $G' = 2\int_{\mathbb{R}^N} u(x)\dot{u}(x)\,dx$ and $\frac{1}{2}G'' = \|\dot{u}\|^2 - I \geq -I$. Assume that I is not always negative and define

$$t := \min\Big\{s \in (0, T^*) \quad \text{such that} \quad I(s) = 0\Big\}.$$

Then G' is increasing on $[0,t]$ and

$$G' \geq G'(0) > \frac{4(2+\varepsilon)}{\varepsilon} E_0 \quad \text{on} \quad [0,t]. \tag{3.1}$$

Since $I(t) = 0$, yields

$$\begin{aligned} 2E &= \|u(t)\|_{H^2}^2 + \|\dot{u}(t)\|^2 - \frac{2}{1+p}\int_{\mathbb{R}^N} |u(t,x)|^{1+p}\,dx \\ &= \|u(t)\|_{H^2}^2 + \|\dot{u}(t)\|^2 - \frac{2}{1+p}\Big(\|u(t)\|_{H^2}^2 - I(t)\Big) \\ &= \frac{p-1}{1+p}\|u(t)\|_{H^2}^2 + \|\dot{u}(t)\|^2. \end{aligned}$$

Then, thanks to Cauchy-Schwarz inequality, since $p > 1 + \varepsilon$, we get

$$\begin{aligned} 2E &\geq \|\dot{u}(t)\|^2 + \frac{\varepsilon}{2+\varepsilon}\|u(t)\|_{H^2}^2 \\ &\geq \frac{\varepsilon}{2+\varepsilon}\Big(\|\dot{u}(t)\|^2 + \|u(t)\|^2\Big) \\ &\geq \frac{\varepsilon}{2+\varepsilon}G'(t). \end{aligned}$$

This contradicts (4.1) and finishes the proof. Q.E.D.

Now, return to the proof of Theorem 2.6. With contradiction, assume that u is global. Compute, using Cauchy-Schwarz inequality

$$(G')^2 = 4\|u\dot{u}\|_1^2 \leq 4\|u\|^2\|\dot{u}\|^2 \leq 4G\|\dot{u}\|^2.$$

For $\lambda \in \mathbb{R}$, define the real function

$$\begin{aligned} h_\lambda &:= GG'' - \frac{3+\lambda}{4}(G')^2 \\ &\geq G\Big(G'' - (3+\lambda)\|\dot{u}\|^2\Big) \\ &\geq -G\Big(2I + (1+\lambda)\|\dot{u}\|^2\Big). \end{aligned}$$

Now, take the case $f(u) = u^p$. Since $p > 1 + \varepsilon$, write

$$\begin{aligned} 2E &= \|u\|_{H^2}^2 + \|\dot{u}\|^2 - \frac{2}{1+p}\int_{\mathbb{R}^N} |u|^{1+p}\,dx \\ &= \|u\|_{H^2}^2 + \|\dot{u}\|^2 - \frac{2}{1+p}\Big(\|u\|_{H^2}^2 - I\Big) \\ &\geq \|u\|_{H^2}^2 + \|\dot{u}\|^2 - \frac{2}{2+\varepsilon}\Big(\|u\|_{H^2}^2 - I\Big). \end{aligned}$$

Then,

$$2(2+\varepsilon)E \geq (2+\varepsilon)\|\dot{u}\|^2 + \varepsilon\|u\|_{H^2}^2 + 2I.$$

Thus,

$$\begin{aligned} h_\lambda &\geq -G\Big(2I + (1+\lambda)\|\dot{u}\|^2\Big)\\ &\geq G\Big(-2(2+\varepsilon)E + (2+\varepsilon)\|\dot{u}\|^2 + \varepsilon\|u\|_{H^2}^2 - (1+\lambda)\|\dot{u}\|^2\Big)\\ &\geq G\Big(-2(2+\varepsilon)E + (1+\varepsilon-\lambda)\|\dot{u}\|^2 + \varepsilon\|\Delta u\|^2 + \varepsilon\|u\|^2\Big). \end{aligned}$$

Using the previous Lemma and Cauchy-Schwarz inequality, yields

$$\begin{aligned} h_{1+\frac{\varepsilon}{2}} &\geq G\Big(-2(2+\varepsilon)E + \frac{\varepsilon}{2}\|\dot{u}\|^2 + \varepsilon\|\Delta u\|^2 + \varepsilon\|u\|^2\Big)\\ &\geq G\Big(-2(2+\varepsilon)E + \frac{\varepsilon}{2}\|\dot{u}\|^2 + \frac{\varepsilon}{2}\|u\|^2\Big)\\ &\geq \frac{\varepsilon}{2}G\Big(-G' + \|\dot{u}\|^2 + \|u\|^2\Big)\\ &\geq 0. \end{aligned}$$

Finally,

$$GG'' - (1+\frac{\varepsilon}{8})(G')^2 \geq 0.$$

The first part of Theorem 2.6 is proved thanks to Lemma 2.5.

4 Proof of Theorem 2.7

In this section, take $N = 4$, $-2 < a < -1$, $f := f_a$ and $\varepsilon \in \mathcal{A}_a$. The proof is based on the following intermediate result.

Lemma 4.1. Let $(u_0, u_1) \in H^2 \times L^2$ such that $E(0) = E_0 > 0$, $I(0) < 0$ and $G'(0) > \frac{2(2+\varepsilon)}{\varepsilon(2+a)}E_0$. Then, $I < 0$ and $G' > \frac{2(2+\varepsilon)}{\varepsilon(2+a)}E$ on $[0, T^*)$.

Proof. Compute using the equation (1.1), $G' = 2\int_{\mathbb{R}^N} u(x)\dot{u}(x)\,dx$ and $\frac{1}{2}G'' = \|\dot{u}\|^2 - I \geq -I$. Assume that I is not always negative and define

$$t := \min\Big\{s \in (0, T^*) \quad \text{such that} \quad I(s) = 0\Big\}.$$

Then G' is increasing on $[0, t]$ and

$$G' \geq G'(0) > \frac{2(2+\varepsilon)}{\varepsilon(2+a)}E_0 \quad \text{on} \quad [0, t]. \tag{4.1}$$

Since $I(t) = 0$ and using (2.3), yields

$$\begin{aligned} 2E &= \|u(t)\|_{H^2}^2 + \|\dot{u}(t)\|^2 - 2\int_{\mathbb{R}^2} F_a(u(t,x))\,dx\\ &\geq \|u(t)\|_{H^2}^2 + \|\dot{u}(t)\|^2 - \frac{2}{2+\varepsilon}\int_{\mathbb{R}^2}\Big(u(t,x)f_a(u(t,x))dx - \frac{\varepsilon}{2}(1+a)u^2(t,x)\Big)\,dx\\ &\geq \|u(t)\|_{H^2}^2 + \|\dot{u}(t)\|^2 - \frac{2}{2+\varepsilon}(\|u(t)\|_{H^2}^2 - \frac{\varepsilon}{2}(1+a)\|u(t)\|^2). \end{aligned}$$

Then, thanks to Cauchy-Schwarz inequality, one gets

$$\begin{aligned} 2E &\geq \|\dot{u}(t)\|^2 + \frac{\varepsilon}{2+\varepsilon}\|\Delta u(t)\|^2 + \frac{(2+a)\varepsilon}{2+\varepsilon}\|u(t)\|^2 \\ &\geq \frac{\varepsilon(2+a)}{2+\varepsilon}\Big(\|u(t)\|^2 + \|\dot{u}(t)\|^2 + \|\Delta u(t)\|^2\Big) \\ &\geq \frac{\varepsilon(2+a)}{2+\varepsilon}G'(t). \end{aligned}$$

This contradicts (4.1) and finishes the proof. Q.E.D.

Now, return to the proof of Theorem 2.7. Taking account of (2.3), write

$$\begin{aligned} 2E &= \|u\|_{H^2}^2 + \|\dot{u}\|^2 - 2\int_{\mathbb{R}^2} F_a(u)\,dx \\ &\geq \|u\|_{H^2}^2 + \|\dot{u}\|^2 - \frac{2}{2+\varepsilon}\int_{\mathbb{R}^2}\Big(uf_a(u)dx - \frac{\varepsilon}{2}(1+a)u^2\Big)\,dx \\ &\geq \|u\|_{H^2}^2 + \|\dot{u}\|^2 - \frac{2}{2+\varepsilon}\Big(\|u\|_{H^1}^2 - I - \frac{\varepsilon}{2}(1+a)\|u\|^2\Big). \end{aligned}$$

Then,

$$2(2+\varepsilon)E \geq (2+\varepsilon)\|\dot{u}\|^2 + \varepsilon\|\Delta u\|^2 + \varepsilon(2+a)\|u\|^2 + 2I.$$

Thus,

$$\begin{aligned} h_\lambda &\geq -G\Big(2I + (1+\lambda)\|\dot{u}\|^2\Big) \\ &\geq G\Big(-2(2+\varepsilon)E + (2+\varepsilon)\|\dot{u}\|^2 + \varepsilon\|\Delta u\|^2 + \varepsilon(2+a)\|u\|^2 - (1+\lambda)\|\dot{u}\|^2\Big) \\ &\geq G\Big(-2(2+\varepsilon)E + (1+\varepsilon-\lambda)\|\dot{u}\|^2 + \varepsilon\|\Delta u\|^2 + \varepsilon(2+a)\|u\|^2\Big). \end{aligned}$$

Using the previous Lemma, one gets $G' > \frac{2(2+\varepsilon)}{\varepsilon(2+a)}E$, so

$$\begin{aligned} h_{1-\varepsilon(1+a)} &\geq G\Big((1+\varepsilon-(1-\varepsilon(1+a)))\|\dot{u}\|^2 + \varepsilon(2+a)\|u\|^2 - 2(2+\varepsilon)E\Big) \\ &\geq \varepsilon(2+a)G\Big(\|\dot{u}\|^2 + \|u\|^2 - G'\Big) \\ &> 0. \end{aligned}$$

Finally,

$$GG'' - (1 - \frac{\varepsilon(1+a)}{4})(G')^2 \geq 0.$$

The proof follows by Lemma 2.5 and the fact that $a < -1$.

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