Some estimations of summation-integral-type operators

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Abstract

In this paper, the Szász-Mirakjan-Kantorovich type operators are studied with the help of direct result and weighted approximation properties. Some basic lemmas, theorems are given and proved, at last, we discuss the rate of convergence and a comparison takes place with the Szász-Mirakjan-Kantorovich operators by graphical representations.

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1 Introduction

In 1912, Bernstein introduced the following operators, known as Bernstein polynomials or Bernstein operators

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} (1-x)^{n-k} x^k f\left(\frac{k}{n}\right) \tag{1.1}$$

In 1930, L. V. Kantorovich [19] introduced operators by an extension of Bernstein's operators in integral space by considering the operators $Q_n : L_1([0,1]) \to C_1([0,1])$, for any function $f \in L_1([0,1])$ by

$$Q_n(f;x) = (n+1) \sum_{k=0}^{n} q_{n,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} q_{n,k}(t) f(t)\, dt, \quad x \in [0,1], \tag{1.2}$$

where $q_{n,k}(x) = \left(\begin{array}{c} n \\ k \end{array} \right) x^k (1-x)^{n-k}$ are Bernstein's polynomials.

After long time, between 1941 and 1950, the two authors mentioned ([16], [27]) and Favard [17] independently introduced so called Szász-Mirakjan operators, defined by

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right),$$

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where $f \in C[0,\infty)$ and $0 \leq x < \infty$.

In ([28], [44]), various approximation properties of the classical Szász-Mirakjan operators are studied. In 2007, Duman, Özarslan [25] studied the modified Szász-Mirakjan-type operators and got better error estimation than classical one. The properties like global approximation in polynomial and exponential weight spaces, uniform approximation, simultaneous approximation of modification and generalization of the Szász-Mirakjan-type are studied in ([32, 34, 20, 21, 11, 10, 33, 42, 44, 13]).

After two decades, to study the approximation problems for generalization of the transform of the Szász-Mirakjan operators $S_n(f;x)$ when $f(x)$ is an integrable function, Butzer [29] introduced an integral modification of the $S_n(f;x)$ by considering Lebesgue integrable function $f : [0,c] \to \mathbb{R}$, $c > 0$ and $F(x) = O(x^h)$ as $x \to \infty$ for some k, where $F(x) = \int\limits_0^x f(t)\, dt$, then we have defined operators as:

$$K_n(f;x) = ne^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt, \tag{1.3}$$

The operators $K_n(f;x)$ were named in [44] as to Szász-Mirakjan-Kantorovich operators by analogy with the Kantorovich operators that constitute a similar integral modification of Bernstein operators (see, e.g. [[14], pp. 333-335]).

On other hand many researchers ([36], [26], [24], [22], [39], [40], [41], [2], [1], [31], [3]) studied many properties and some generalization are also regarding Kantorovich variant.

In 1957, it had been defined a new type of operators by V. A. Baskakov [8], so called the Baskakov operators and the Kantorovich variant [46] of Baskakov operators is as:

$$BK_n(f;n) = n \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \int\limits_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt, \quad x \geq 0,\ n \in \mathbb{N}. \tag{1.4}$$

In regard of the Kantorovich variant, there are many research papers which generalize the behavior and properties of the corresponding operators, some of them are ([35, 18, 9, 12]). In [9], the authors proposed integral modification of the generalization Baskakov operators constructed by Miheşan [38]. In 2013, Altomare et al. [15], constructed the Kantorovich variant of the Szász-Mirakjan operators defined as:

$$D_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \left\{ \frac{n}{a_n - b_n} \int\limits_{\frac{k+a_n}{n}}^{\frac{k+b_n}{n}} f(t)\, dt \right\}, \quad x \geq 0, \quad n \in \mathbb{N}, \tag{1.5}$$

where (a_n) and (b_n) are sequence of real numbers such that $0 \leq a_n < b_n \leq 1$. In same direction, to study the approximation problems, Erençin and B$\ddot{u}$y$\ddot{u}$kdurakoğlu [6], presented the modified

Baskakov-Kantorovich operators as:

$$E_n(f;x) = e^{\frac{-a_n x}{1+x}} \sum_{k=0}^{\infty} \frac{P_k(n,a_n)}{k!} \frac{x^k}{(1+x)^{n+k}} \left\{ \frac{b_n}{d_n - c_n} \int_{\frac{k+d_n}{n}}^{\frac{k+a_n}{n}} f(t)\, dt \right\}, \quad x \geq 0, \quad n \in \mathbb{N}, \tag{1.6}$$

where $P_k(n,a_n) = \sum\limits_{i=0}^{k} \binom{k}{i} (n)_i a_n^{k-i}$, with $(n)_0 = 1, (n)_i = n(n+1)\cdots(n+i-1),\ i \geq 1$ and $a_n,\ b_n,\ c_n$ are the real sequences related by

$$\begin{aligned} &1. \quad a_n \geq 0, \quad b_n \geq 1, \quad 0 \leq c_n < d_n \leq 1, \\ &2. \quad \lim_{n\to\infty} \frac{n}{b_n} = 1, \quad \lim_{n\to\infty} \frac{a_n}{b_n} = 0. \end{aligned}$$

The authors gave direct results and computed the order of approximation by the weighted modulus of continuity for above operators (1.6).

Recently, Mishra and Yadav [43], constructed new Szász-Mirakjan-type operators which preserve $a^x,\ \ a > 1$ fixed for all $x \geq 0$, by choosing a continuous sequence of functions $s_n(x)$ defined below:

$$s_n(x) = \frac{x \log a}{(-1 + a^{\frac{1}{n}})n}, \qquad \forall\ \ x \geq 0, \quad a > 1,$$

in the operators, defined below

$$L_n^*(f; s_n(x)) = e^{-ns_n(x)} \sum_{k=0}^{\infty} \frac{(ns_n(x))^k}{k!} f\left(\frac{k}{n}\right), \qquad f \in C[0,\infty), \tag{1.7}$$

then constructed new operators of the operators (1.7) are as:

$$S_{n,a}^*(f;x) = \sum_{k=0}^{\infty} a^{\left(\frac{-x}{-1+a^{\frac{1}{n}}}\right)} \frac{(x \log a)^k}{(-1 + a^{\frac{1}{n}})^k k!} f\left(\frac{k}{n}\right). \tag{1.8}$$

Many properties are studied including convergence, Voronovskaya-type theorem etc. of the operators (1.8) and having better rate of convergence than classical Szász-Mirakjan operators under certain conditions such as generalized convexity and also by graphics. Inspired by this work related to Kantorovich type operators in the space of integral functions, we considered the Kantorovich version of the operators (1.8) as follows:

$$\tilde{S}_{n,a}^*(f;x) = n \sum_{k=0}^{\infty} a^{\left(\frac{-x}{-1+a^{\frac{1}{n}}}\right)} \frac{(x \log a)^k}{(-1 + a^{\frac{1}{n}})^k k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt, \tag{1.9}$$

defined in paper [43]. Here we get the following identities. Let us define function $e_i = x^i$, where $i = 0, 1, 2, 3$, then we have following lemma.

Lemma 1.1. *For each $x \in [0,\infty)$ and $a>1$ fixed, we have*

1. $\tilde{S}^*_{n,a}(e_0;x) = 1,$
2. $\tilde{S}^*_{n,a}(e_1;x) = \dfrac{1}{2n} + \dfrac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n},$
3. $\tilde{S}^*_{n,a}(e_2;x) = \dfrac{1}{3n^2} + \dfrac{2x\log a}{\left(-1+a^{\frac{1}{n}}\right)n^2} + \dfrac{x^2(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^2},$
4. $\tilde{S}^*_{n,a}(e_3;x) = \dfrac{1}{4n^3} + \dfrac{7}{2}\dfrac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n^3} + \dfrac{9}{2}\dfrac{x^2(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^3} + \dfrac{x^3(\log a)^3}{\left(-1+a^{\frac{1}{n}}\right)^3 n^3}.$

Proof. Since we have $x \in [0,\infty)$ and $a>1$ fixed,

1.
$$\begin{aligned}\tilde{S}^*_{n,a}(e_0;x) &= n\sum_{k=0}^{\infty} a^{\left(\frac{-x}{-1+a^{\frac{1}{n}}}\right)} \frac{(x\log a)^k}{(-1+a^{\frac{1}{n}})^k k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} 1\,dt \\ &= n\sum_{k=0}^{\infty} a^{\left(\frac{-x}{-1+a^{\frac{1}{n}}}\right)} \frac{(x\log a)^k}{(-1+a^{\frac{1}{n}})^k k!}\frac{1}{n} \\ &= \left(a^{\frac{-x}{-1+a^{\frac{1}{n}}}}\right)\left(a^{\frac{x}{-1+a^{\frac{1}{n}}}}\right) \\ &= 1.\end{aligned}$$

2.
$$\begin{aligned}\tilde{S}^*_{n,a}(e_1;x) &= n\sum_{k=0}^{\infty} a^{\left(\frac{-x}{-1+a^{\frac{1}{n}}}\right)} \frac{(x\log a)^k}{(-1+a^{\frac{1}{n}})^k k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} t\,dt \\ &= n\sum_{k=0}^{\infty} a^{\left(\frac{-x}{-1+a^{\frac{1}{n}}}\right)} \frac{(x\log a)^k}{(-1+a^{\frac{1}{n}})^k k!}\frac{1}{2}\left(\frac{2k}{n^2}+\frac{1}{n^2}\right) \\ &= \frac{1}{n}\sum_{k=0}^{\infty} a^{\left(\frac{-x}{-1+a^{\frac{1}{n}}}\right)} \frac{(x\log a)^k}{(-1+a^{\frac{1}{n}})^k k!}k + \frac{1}{2n}\sum_{k=0}^{\infty} a^{\left(\frac{-x}{-1+a^{\frac{1}{n}}}\right)} \frac{(x\log a)^k}{(-1+a^{\frac{1}{n}})^k k!} \\ &= \frac{x\log a}{(-1+a^{\frac{1}{n}})n} + \frac{1}{2n}.\end{aligned}$$

Similarly, we can prove other equalities. Q.E.D.

2 Preliminary results

This section contains basic lemmas to prove our main theorem.
Let us define a function $\xi_x(t)$ by $\xi_x(t) = (t-x)$ then by Lemma 1.1, we can get following results.

Lemma 2.1. *For every $x \geq 0$, we have*

1. $\tilde{S}^*_{n,a}(\xi_x(t);x) = -\dfrac{(-1+2nx)}{2n} + \dfrac{x\log a}{n(-1+a^{\frac{1}{n}})}$,
2. $\tilde{S}^*_{n,a}(\xi_x^2(t);x) = \dfrac{(1-3nx+3n^2x^2)}{3n^2} - \dfrac{2(-1+a^{\frac{1}{n}})(-1+nx)x\log a}{\left(-1+a^{\frac{1}{n}}\right)^2 n^2} + \dfrac{x^2(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^2}$,
3. $$\tilde{S}^*_{n,a}(\xi_x^3(t);x) = -\frac{(-1+4nx-6n^2x^2+4n^3x^3)}{4n^3} + \frac{x(7-12nx+6n^2x^2)\log a}{2\left(-1+a^{\frac{1}{n}}\right)n^3} - \frac{3x^2(-3+2nx)(\log a)^2+4x^3(\log a)^3}{2\left(-1+a^{\frac{1}{n}}\right)^2 n^3},$$
4. $$\begin{aligned}\tilde{S}^*_{n,a}(\xi_x^4(t);x) = \frac{1}{5\left(-1+a^{\frac{1}{n}}\right)^4 n^4}\Big(&\left(-1+a^{\frac{1}{n}}\right)^4(1-5nx+10n^2x^2-10n^3x^3+5n^4x^4)\\ &-10\left(-1+a^{\frac{1}{n}}\right)^3 x(-3+7nx-6n^2x^2+2n^3x^3)\log a\\ &+15\left(-1+a^{\frac{1}{n}}\right)^2 x^2(5-6nx+2n^2x^2)(\log a)^2\\ &-20\left(-1+a^{\frac{1}{n}}\right)x^3(-2+nx)(\log a)^3+5x^4(\log a)^4\Big).\end{aligned}$$

Proof. 1. Using the linearity property and by Lemma 1.1, we can get the proof.

$$\begin{aligned}
1.\ \tilde{S}^*_{n,a}(\xi_x(t);x) &= n\sum_{k=0}^{\infty} a^{\left(\frac{-x}{-1+a^{\frac{1}{n}}}\right)}\frac{(x\log a)^k}{(-1+a^{\frac{1}{n}})^k k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}(t-x)\,dt\\
&= n\sum_{k=0}^{\infty} a^{\left(\frac{-x}{-1+a^{\frac{1}{n}}}\right)}\frac{(x\log a)^k}{(-1+a^{\frac{1}{n}})^k k!}\left\{\frac{1}{2}\left(\frac{2k}{n^2}+\frac{1}{n^2}\right)-\frac{x}{n}\right\}\\
&= n\sum_{k=0}^{\infty} a^{\left(\frac{-x}{-1+a^{\frac{1}{n}}}\right)}\frac{(x\log a)^k}{(-1+a^{\frac{1}{n}})^k k!}\frac{1}{2}\left(\frac{2k}{n^2}+\frac{1}{n^2}\right)\\
&- n\sum_{k=0}^{\infty} a^{\left(\frac{-x}{-1+a^{\frac{1}{n}}}\right)}\frac{(x\log a)^k}{(-1+a^{\frac{1}{n}})^k k!}\frac{x}{n}\\
&= \frac{x\log a}{(-1+a^{\frac{1}{n}})n}+\frac{1}{2n}-\frac{x}{n}\\
&= -\frac{(-1+2nx)}{2n}+\frac{x\log a}{n(-1+a^{\frac{1}{n}})}.
\end{aligned}$$

Similarly, we can prove other equalities with the help of Lemma 1.1. Q.E.D.

Lemma 2.2. *For the given operator $\tilde{S}^{*}_{n,a}f$ defined by (1.9), we have*

$$\tilde{S}^{*}_{n,a}(\xi_x^4(t);x) \leq A_a(n)\ (x^4+x^3+x^2+x+1), \tag{2.1}$$

where $A_a(n)=\max\{C_1(n),\ C_2(n),\ C_3(n),\ C_4(n),\ C_5(n)\}$ *with*

$$\begin{aligned}
C_1(n) &= \left|1-\frac{4\log a}{n\left(-1+a^{\frac{1}{n}}\right)}+\frac{6(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^2}-\frac{4(\log a)^3}{\left(-1+a^{\frac{1}{n}}\right)^3 n^3}+\frac{(\log a)^4}{\left(-1+a^{\frac{1}{n}}\right)^4 n^4}\right|,\\
C_2(n) &= \frac{2}{n}+\frac{12\log a}{\left(-1+a^{\frac{1}{n}}\right)n^2}+\frac{18(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^3}+\frac{8(\log a)^3}{\left(-1+a^{\frac{1}{n}}\right)^3 n^4},\\
C_3(n) &= \frac{2}{n^2}+\frac{14\log a}{\left(-1+a^{\frac{1}{n}}\right)n^3}+\frac{15(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^4},\\
C_4(n) &= \frac{1}{n^3}+\frac{6\log a}{\left(-1+a^{\frac{1}{n}}\right)n^4},\\
C_5(n) &= \frac{1}{5n^2}.
\end{aligned}$$

Proof. From the above Lemma 2.1, we can write

$$\begin{aligned}
\tilde{S}^{*}_{n,a}(\xi_x^4(t);x) &= x^4\left(1-\frac{4\log a}{n\left(-1+a^{\frac{1}{n}}\right)}+\frac{6(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^2}-\frac{4(\log a)^3}{\left(-1+a^{\frac{1}{n}}\right)^3 n^3}+\frac{(\log a)^4}{\left(-1+a^{\frac{1}{n}}\right)^4 n^4}\right)\\
&\quad+x^3\left(\frac{-2}{n}+\frac{12\log a}{\left(-1+a^{\frac{1}{n}}\right)n^2}-\frac{18(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^3}+\frac{8(\log a)^3}{\left(-1+a^{\frac{1}{n}}\right)^3 n^4}\right)\\
&\quad+x^2\left(\frac{2}{n^2}-\frac{14\log a}{\left(-1+a^{\frac{1}{n}}\right)n^3}+\frac{15(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^4}\right)\\
&\quad+x\left(\frac{-1}{n^3}+\frac{6\log a}{\left(-1+a^{\frac{1}{n}}\right)n^4}\right)+\frac{1}{5n^2}\\
&\leq x^4\left(\left|1-\frac{4\log a}{n\left(-1+a^{\frac{1}{n}}\right)}+\frac{6(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^2}-\frac{4(\log a)^3}{\left(-1+a^{\frac{1}{n}}\right)^3 n^3}+\frac{(\log a)^4}{\left(-1+a^{\frac{1}{n}}\right)^4 n^4}\right|\right)\\
&\quad+x^3\left(\frac{2}{n}+\frac{12\log a}{\left(-1+a^{\frac{1}{n}}\right)n^2}+\frac{18(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^3}+\frac{8(\log a)^3}{\left(-1+a^{\frac{1}{n}}\right)^3 n^4}\right)
\end{aligned}$$

$$
\begin{aligned}
&+x^2\left(\frac{2}{n^2}+\frac{14\log a}{\left(-1+a^{\frac{1}{n}}\right)n^3}+\frac{15(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^4}\right)\\
&+x\left(\frac{1}{n^3}+\frac{6\log a}{\left(-1+a^{\frac{1}{n}}\right)n^4}\right)+\frac{1}{5n^2}\\
&= C_1(n)x^4+C_2(n)x^3+C_3(n)x^2+C_4(n)x+C_5(n),
\end{aligned}
$$

Since each $C_i(n), \quad i=1,2,3,4,5$ converges, so one has

$$
\tilde{S}^*_{n,a}(\xi_x^4(t);x) \quad \leq \quad A_a(n)(x^4+x^3+x^2+x+1).
$$

Q.E.D.

3 Direct results

Consider $C_B[0,\infty)$, the space of all continuous and bounded functions defined on the interval $[0,\infty)$, endowed with the supremum norm:

$$
\|f\| = \sup_{0\leq x<\infty} |f(x)|,
$$

and also Peetre's $K-$functional is defined as:

$$
K_2(f;\delta) = \inf_{g\in C_B^2}\{\|f-g\|+\delta\|g''\|\}, \quad \text{for any } \delta>0,
$$

where $C_B^2[0,\infty) = \{g\in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By De Vore and Lorentz ([30]), for an absolute value C_1' (independent of $\delta>0$ and f), it exist as given below:

$$
K_2(f;\delta) \leq C_1'\omega_2(f;\sqrt{\delta}), \tag{3.1}
$$

where $\omega_2(f;\delta) = \sup\limits_{\substack{0\leq h\leq\delta\\0\leq x<\infty}} |f(x+2h)-2f(x+h)+f(x)|$, is the second order modulus of smoothness of $f\in C_B[0,\infty)$ and for $f\in C_B[0,\infty)$, the modulus of continuity is given by:

$$
\omega(f;\delta) = \sup_{\substack{0\leq h\leq\delta\\0\leq x<\infty}} |f(x+h)-f(x)|.
$$

Theorem 3.1. *Let $f\in C_B[0,\infty)$. For each $x\geq 0$, there exists a positive constant D such that*

$$
|\tilde{S}^*_{n,a}(f;x)-f(x)| \leq D\ \omega_2(f;\sqrt{\delta_n}) + \omega\left(f;\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}\right),
$$

where $\delta_n = \left\{\tilde{\tilde{S}}^*_{n,a}((t-x)^2;x)+\left(\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}-x\right)^2\right\}.$

Proof. Let us define the auxiliary operators

$$\tilde{\tilde{S}}^{*}_{n,a}(f;x)=\tilde{S}^{*}_{n,a}(f;x)-f\left(\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}\right)+f(x), \tag{3.2}$$

for every $x\in[0,\infty)$ and the operator (3.2) preserves $e_0,\ e_1$, while $\tilde{\tilde{S}}^{*}_{n,a}(\xi_x(t);x)=0$, as using the Lemma 1.1, we have

$$\begin{aligned}\tilde{\tilde{S}}^{*}_{n,a}(\xi_x(t);x) &= \tilde{S}^{*}_{n,a}(\xi_x(t);x)-\left(\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}-x\right)\\ &= \tilde{S}^{*}_{n,a}(t;x)-x\tilde{S}^{*}_{n,a}(1;x)-\left(\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}-x\right)\\ &= 0.\end{aligned}$$

Let $g\in C^2_B[0,\infty)$ for all $x\in[0,\infty)$. By Taylor's expansion we have

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-y)g''(y)\ dy.$$

Applying the operators $\tilde{\tilde{S}}^{*}_{n,a}$ on both sides, we get

$$\begin{aligned}\tilde{\tilde{S}}^{*}_{n,a}(g;x)-g(x) &= g'(x)\tilde{\tilde{S}}^{*}_{n,a}(t-x;x)+\tilde{\tilde{S}}^{*}_{n,a}\left(\int_x^t(t-y)g''(y)\ dy\right)\\ &= \tilde{\tilde{S}}^{*}_{n,a}\left(\int_x^t(t-y)g''(y)\ dy\right)\\ &= \tilde{S}^{*}_{n,a}\left(\int_x^t(t-y)g''(y)\ dy\right)-\int_x^{\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}}\left(\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}-y\right)g''(y)\ dy.\end{aligned}$$

Using the inequalities

$$\left|\int_x^t(t-y)g''(y)\ dy\right|\le(t-x)^2\|g''\|$$

and

$$\left|\int\limits_{x}^{\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}}\left(\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}-y\right)g''(y)\ dy\right| \leq \left(\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}-x\right)^2 \|g''\|,$$

we have

$$\begin{aligned} |\tilde{\tilde{S}}^*_{n,a}(g;x)-g(x)| &\leq \left\{\tilde{\tilde{S}}^*_{n,a}((t-x)^2;x)+\left(\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}-x\right)^2\right\}\|g''\| \quad (3.3)\\ &= \delta_n\|g''\|, \quad (3.4)\end{aligned}$$

where

$$\delta_n = \left\{\tilde{\tilde{S}}^*_{n,a}((t-x)^2;x)+\left(\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}-x\right)^2\right\}.$$

Also $|\tilde{S}^*_{n,a}(f;x)| \leq \|f\|$, using this property, next we have

$$\begin{aligned} |\tilde{S}^*_{n,a}(f;x)-f(x)| &\leq |\tilde{\tilde{S}}^*_{n,a}(f-g;x)-(f-g)(x)|+|\tilde{\tilde{S}}^*_{n,a}(g;x)-g(x)|\\ &\quad +\left|f\left(\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}\right)-f(x)\right|\\ &\leq 4\|f-g\|+|\tilde{\tilde{S}}^*_{n,a}(g;x)-g(x)|+\left|f\left(\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}\right)-f(x)\right|,\end{aligned}$$

and using Equation (3.3), we reach on

$$|\tilde{S}^*_{n,a}(f;x)-f(x)| \leq 4\|f-g\|+\delta_n\|g''\|+w\left(f;\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}\right).$$

Taking the infimum for all $g \in C^2_B[0,\infty)$ on the right hand side and considering inequality (3.1), we obtain

$$\begin{aligned} |\tilde{S}^*_{n,a}(f;x)-f(x)| &\leq 4K_2(f;\frac{1}{4}\delta_n)+w\left(f;\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}\right)\\ &\leq 4C'_1\omega_2(f;\sqrt{\delta_n})+\omega\left(f;\frac{1}{2n}+\frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}\right)\end{aligned}$$

$$= D\omega_2(f;\sqrt{\delta_n}) + \omega\left(f; \frac{1}{2n} + \frac{x\log a}{\left(-1+a^{\frac{1}{n}}\right)n}\right).$$

Q.E.D.

Theorem 3.2. *Let $f \in C_B[0,\infty)$ and if $f \in Lip_M(\alpha)$, $\alpha \in (0,1]$ i.e.*

$$|f(t) - f(x)| \leq M|t-x|^{\alpha}, \qquad t, x \in [0,\infty),$$

then for each $x \geq 0$, we have

$$|\tilde{S}^*_{n,a}(f;x) - f(x)| \leq M\delta^{\frac{\alpha}{2}}_{n,a},$$

where M is any positive constant and $\delta_{n,a} = (\tilde{S}^*_{n,a}((t-x)^2;x))^{\frac{1}{2}}$.

Proof. Since we have $f \in C_B[0,\infty) \cap \mathrm{Lip}_M(\alpha)$, now

$$\begin{aligned}
|\tilde{S}^*_{n,a}(f;x) - f(x)| &\leq \tilde{S}^*_{n,a}(|f(t)-f(x)|;x) \\
&\leq M\tilde{S}^*_{n,a}(|t-x|^{\alpha};x) \\
&= M\left(n\sum_{k=0}^{\infty} a^{\left(\frac{-x}{-1+a^{\frac{1}{n}}}\right)} \frac{(x\log a)^k}{(-1+a^{\frac{1}{n}})^k k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|^{\alpha}\, dt,\right).
\end{aligned}$$

Now applying Hölder inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

$$\begin{aligned}
|\tilde{S}^*_{n,a}(f;x) - f(x)| &\leq M\left(n\sum_{k=0}^{\infty} a^{\left(\frac{-x}{-1+a^{\frac{1}{n}}}\right)} \frac{(x\log a)^k}{(-1+a^{\frac{1}{n}})^k k!} \left\{\int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^2\, dt,\right\}^{\frac{\alpha}{2}}\right) \\
&\leq M(\tilde{S}^*_{n,a}((t-x)^2;x))^{\frac{\alpha}{2}} \\
&= M\delta^{\frac{\alpha}{2}}_{n,a}.
\end{aligned}$$

Thus the proof is completed. Q.E.D.

4 Weighted approximation

Now we introduce the convergence properties of the given operators (1.9) via the Korovkin type theorem given by Gadjiev in [4] and [5]. Since the weighted function is $w(x) = 1 + \gamma^2(x)$, where $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is an unbounded strictly increasing continuous function such that there exists $M > 0$ and $\alpha \in (0,1]$ with the property

$$|x - y| \leq M|\gamma(x) - \gamma(y)|, \quad \forall\, x, y \geq 0. \tag{4.1}$$

Let $w(x) = 1 + x^2$ be a weighted function and let $B_w[0,\infty)$ the space defined by

$$B_w[0,\infty) = \{f : [0,\infty) \to \mathbb{R} \;\mid\; \|f\|_w = \sup_{x\geq 0} \tfrac{f(x)}{w(x)} < +\infty\}.$$

Also define the spaces

$$\begin{aligned} C_w[0,\infty) &= \{f \in B_w[0,\infty), \quad f \quad \text{is continuous}\}, \\ C_w^k[0,\infty) &= \{f \in C_w[0,\infty), \quad \lim_{x\to\infty} \frac{|f(x)|}{w(x)} = k_f < +\infty\}. \end{aligned}$$

Theorem 4.1. *Let $\tilde{S}^*_{n,a}f$ be a linear positive operator defined by (1.9), then for each $f \in C_w^k[0,\infty)$, we have*

$$\lim_{n\to\infty} \|\tilde{S}^*_{n,a}(f;x) - f(x)\|_w = 0.$$

Proof. It is sufficient to prove the above theorem, if $\lim_{n\to\infty} \|\tilde{S}^*_{n,a}(e_i;x) - e_i\|_w = 0$ holds, for $i = 0,1,2$. By using Lemma 1.1, it is obvious that

$$\|\tilde{S}^*_{n,a}(e_0;x) - 1\|_w = 0.$$

Now we have

$$\begin{aligned} \|\tilde{S}^*_{n,a}(e_1;x) - e_1)\|_w &= \sup_{x\ge 0} \left(\frac{1}{2n} + \frac{x\log a}{\left(-1 + a^{\frac{1}{n}}\right) n} - x \right) \frac{1}{1+x^2} \\ &= \sup_{x\ge 0} \left(\frac{1}{2n}\frac{1}{1+x^2} + \frac{x\log a}{\left(-1 + a^{\frac{1}{n}}\right) n}\frac{1}{1+x^2} - \frac{x}{1+x^2} \right) \\ &= \sup_{x\ge 0} \left(\frac{1}{2n}\frac{1}{1+x^2} + \left(\frac{\log a}{\left(-1 + a^{\frac{1}{n}}\right) n} - 1 \right) \frac{x}{1+x^2} \right) \\ &\le \left(\frac{1}{2n} + \left(\frac{\log a}{\left(-1 + a^{\frac{1}{n}}\right) n} - 1 \right) \right) \end{aligned}$$

i.e.,

$$\lim_{n\to\infty} \|\tilde{S}^*_{n,a}(e_1;x) - e_1\|_w = 0. \tag{4.2}$$

Similarly,

$$\begin{aligned} \|\tilde{S}^*_{n,a}(e_2;x) - e_2\|_w &= \sup_{x\ge 0} \left(\frac{1}{3n^2} + \frac{2x\log a}{\left(-1 + a^{\frac{1}{n}}\right) n^2} + \frac{x^2(\log a)^2}{\left(-1 + a^{\frac{1}{n}}\right)^2 n^2} - x^2 \right) \frac{1}{1+x^2} \\ &= \sup_{x\ge 0} \left(\frac{1}{3n^2}\frac{1}{1+x^2} + \frac{2\log a}{\left(-1 + a^{\frac{1}{n}}\right) n^2}\frac{1}{1+x^2} + \left(\frac{(\log a)^2}{\left(-1 + a^{\frac{1}{n}}\right)^2 n^2} - 1 \right) \frac{x^2}{1+x^2} \right) \end{aligned}$$

$$\leq \left(\frac{1}{3n^2} + \frac{2\log a}{\left(-1+a^{\frac{1}{n}}\right)n^2} + \left(\frac{(\log a)^2}{\left(-1+a^{\frac{1}{n}}\right)^2 n^2} - 1\right)\right).$$

i.e., we have

$$\lim_{n\to\infty} \|\tilde{S}^*_{n,a}(e_2;x) - x^2\|_w = 0. \tag{4.3}$$

Thus the proof is completed. Q.E.D.

For computing the order of approximation of the operators (1.9) in term of the weighted modulus of continuity $\Omega_2(f;\delta)$ (see [7],[23]) defined by

$$\Omega_2(f;\delta) = \sup_{0\leq x<\infty,\, 0<h\leq\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^2}, \qquad f\in C_w^k[0,\infty),$$

we use some elementary properties of $\Omega_2(f;\delta)$ given below:

Lemma 4.2. *Let $f\in C_w^k[0,\infty)$, then*

1. *$\Omega_2(f;\delta)$ is monotone increasing function of δ,*

2. *$\lim_{\delta\to 0^+} \Omega_2(f;\delta) = 0$,*

3. *for each $\lambda\in[0,\infty)$, $\quad \Omega_2(f;\lambda\delta) \leq (\lambda+1)\Omega_2(f;\delta)$.*

Theorem 4.3. *Let $\{\tilde{S}^*_{n,a}\}$ be the sequence of linear positive operators defined by (1.9). Then for each $f\in C_w^k[0,\infty)$, we have*

$$\frac{|\tilde{S}^*_{n,a}(f;x) - f(x)|}{(1+x^2)^3} \leq C\ \Omega_2(f;(A_a(n))^{\frac{1}{4}}),$$

where $A_a(n)$ is maximum of the $C_1(n)$ and $C_2(n)$ and is defined in Lemma 2.2.

Proof. For any $x\geq 0$, $t\geq 0$ then by definition of $\Omega_2(f;\delta)$ and using the property (3) of above Lemma 4.2 we have,

$$|f(x)-f(t)| \leq (1+x^2)(1+|t-x|^2)\Omega_2(f;|t-x|)$$

as well as,

$$|f(x)-f(t)| \leq 2(1+x^2)(1+(t-x)^2)\ \Omega_2(f;\delta_n)\left(1+\frac{|t-x|}{\delta_n}\right) \tag{4.4}$$

Now using the monotonicity of the operators $\tilde{S}^*_{n,a}(f;x)$ and following inequality (see [23])

$$\left(1+\frac{|t-x|}{\delta_n}\right)(1+(t-x)^2) \leq 2(1+\delta_n^2)\left(1+\frac{(t-x)^4}{\delta_n^4}\right), \tag{4.5}$$

one can get,

$$\begin{aligned}
|\tilde{S}^*_{n,a}(f;x) - f(x)| &\leq 2(1+x^2)\ \tilde{S}^*_{n,a}\left((1+(t-x)^2)\left(1+\frac{|t-x|}{\delta_n}\right);x\right)\ \Omega_2(f;\delta_n) \\
&\leq 4(1+\delta^2)(1+x^2)\ \tilde{S}^*_{n,a}\left(1+\frac{(t-x)^4}{\delta_n^4};x\right)\ \Omega_2(f;\delta_n) \\
&= 4(1+\delta^2)(1+x^2)\left(1+\frac{1}{\delta_n^4}\ \tilde{S}^*_{n,a}((t-x)^4);x)\right)\Omega_2(f;\delta_n) \\
&\leq C(1+x^2)\left(1+\frac{1}{\delta_n^4}\ \tilde{S}^*_{n,a}((t-x)^4);x)\right)\Omega_2(f;\delta_n).
\end{aligned}$$

With the help of Lemma (2.1), we obtain

$$\begin{aligned}
|\tilde{S}^*_{n,a}(f;x) - f(x)| &\leq C(1+x^2)\left(1+\frac{A_a(n)(x^4+x^3+x^2+x+1)}{\delta_n^4}\right)\Omega_2(f;\delta_n) \\
&\leq C(1+x^2)\Omega_2(f;(A_a(n))^{\frac{1}{4}})(x^4+x^3+x^2+x+1) \\
&\leq C(1+x^2)^3\ \Omega_2(f;(A_a(n))^{\frac{1}{4}}).
\end{aligned}$$

Thus the proof is completed. Q.E.D.

5 Graphical representation and comparison to the Szász-Mirakjan-Kantorovich operators

In this section, we shall analyze the graphics, conclude better rate of convergence.
By Figure 1, we can observe that the rate of convergence is good, but by comparing to the Szász-Mirakjan-Kantorovich operators (1.3), we can see that the defined operators (1.9) have better rate of convergence, for large value of n both converge to the given function and said operators (1.9) have also better accuracy can be seen in last one graphic of same figure.

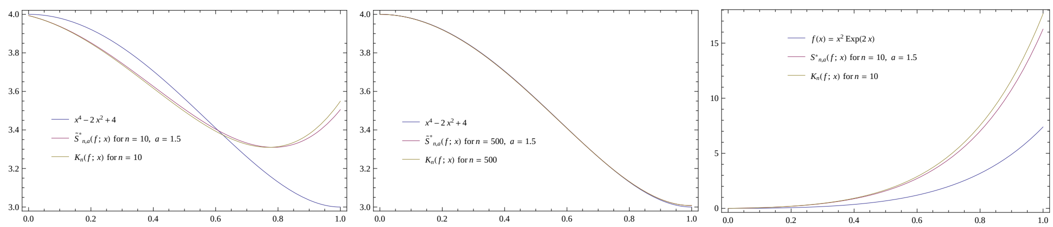

FIGURE 1. Convergence of the operators $\tilde{S}^*_{n,a}(f;x)$ and $K_n(f;x)$

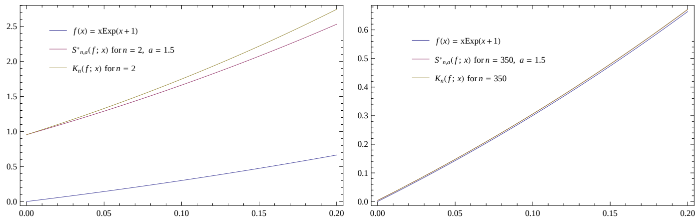

FIGURE 2. Convergence of the operators $\tilde{S}^*_{n,a}(f;x)$ and $K_n(f;x)$

Same as in Figure 2, the convergence behavior can be seen and is as good as we want. If we turn for comparison with Szász-Mirakjan-Kantorovich operators $K_n(f;x)$, above figure represents the better rate of convergence.

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