The transmuted Gompertz-G family of distributions: properties and applications

Hesham Reyad[1], Farrukh Jamal[2], Soha Othman[3] and G.G. Hamedani[4]

[1]Department of Management Information Systems and Production Management, Qassim University, KSA
[2]Department of Statistics, Govt.S.A. Postgraduate College, Dera Nawab Sahib, Pakistan
[3]Department of Applied Statistics, Cairo University, Egypt
[4]Department of Mathematics, Statistics and Computer Science, Marquette University, USA

E-mail: hesham_reyad@yahoo.com, Drfarrukh1982@gmail.com, soha_othman@yahoo.com, gholamhoss.hamedani@marquette.edu

Abstract

We introduce and study a new class of continuous distributions called the transmuted Gompertz-G family which extends the Gompertz class proposed by Alizadeh et al. (2016a). Explicit expressions for the ordinary and incomplete moments, generating function, probability weighted moment, Lorenz and Bonferroni curves, order statistics, Rényi and Shanon entropies, stress strength model moment of residual and reversed residual life and characterizations for the new family are investigated. We discuss the maximum likelihood estimates for the model parameters. The performance of the new family is assesed by means of two applications

2010 Mathematics Subject Classification. **62Exx**. 62Fxx, 62H05
Keywords. Rényi and Shanon entropies, Gompertz-G family, maximum likelihood estimates, order statistic, stress strength model, transmuted-G family.

1 Introduction

In recent years, statisticians are concerned with obtaining new classes of univarite continuous distributions by adding new shape parameter (s) to the parent distribution. These new generated families are more flexiable than old known distributions in fitting data corresponding to different areas such as medicine, finance and economics. Some of the well-known generated families are the following: exponentiated-G by Gupta et al. (1998), beta-G by Eugene et al. (2002), Kumaraswamy-G by Cordeiro and de Castro (2011), logistic-G by Torabi and Montazari (2014), Lomax-G by Cordeiro et al. (2014), odd-Burr generalized-G by Alizadeh et al. (2016b), beta Weibull-G by Yousof et al. (2017b), Type I general exponential class of distributions by Hamedani et al. (2017), odd power Cauchy-G by Alizadeh et al. (2018) and odd Burr-G Poisson family by Nasir et al. (2018), among others.

Shaw and Buckley (2007) introduced the transmuted-G (T-G) family of distributions with cdf and pdf given by

$$F(x;\varphi) = H(x;\varphi)\left[1+\lambda-\lambda H(x;\varphi)\right], \qquad x \in R, \tag{1}$$

and

$$f(x;\varphi) = h(x;\varphi)\left[1+\lambda-2\lambda H(x;\varphi)\right], \qquad x \in R. \tag{2}$$

respectively, where, $|\lambda| \leq 1$, is a shape parameter, $H(x;\varphi)$ and $h(x;\varphi)$ are the distribution function (cdf) and density function (pdf) respectively of a baseline model with parameter vector φ. Moreover,

Tbilisi Mathematical Journal 11(3) (2018), pp. 47–67.
Tbilisi Centre for Mathematical Sciences.

Received by the editors: 04 February 2018.
Accepted for publication: 15 March 2018.

Alizadeh et al. (2016a) proposed the Gompertz-G (GO-G) family of distributions with cdf and pdf given by

$$F_1(x;\varphi) = 1 - e^{\frac{\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]}, \qquad \theta > 0,\ \alpha > 0,\ x \in R, \tag{3}$$

$$f_1(x;\varphi) = \theta\, g(x;\varphi)\, \bar{G}(x;\varphi)^{-\alpha-1}\, e^{\frac{\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]}, \qquad x \in R, \tag{4}$$

where, $\bar{G}(x;\ \varphi) = 1 - G(x;\ \varphi)$.
The goal of this paper is to introduce a new family of continuous distributions called the transmuted Gompertz-G family (TGO-G for short) by using the $F_1(x;\varphi)$ as the baseline distribution in the T-G class and study some of its mathematical properties. The cdf and pdf of the TGO-G family are given, respectively, by

$$F(x) = \left\{1 - e^{\frac{\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]}\right\}\left\{1 + \lambda e^{\frac{\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]}\right\}, \quad x \in R, \tag{5}$$

$$f(x) = \theta\, g(x;\varphi)\, \bar{G}(x;\varphi)^{-\alpha-1}\, e^{\frac{\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]}\left\{1 - \lambda + 2\lambda e^{\frac{\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]}\right\}, \quad x \in R. \tag{6}$$

Henceforth, a random variable with density (6) is denoted by $X \sim \text{TGO}\,(\theta, \alpha, \lambda, \varphi)$.

The remainder of this paper is orgainzed as follows. In Section 2, usefull expansion of TGO-G is discussed. In Section 3, some special models corresponding to TGO-G are introduced. In Section 4, some mathematical properties of the TGO-G are obtained. In Section 5, characterizations for the new family are presented. In Section 6,.the maximum likelihood estimates are obtained for the parameters of TGO-G. A simulation study is conducted in Section 7. In Section 8, two application for TGO-G are presented. Some concluding remarks are given in the last Section.

2 Linear representation

In this section, we introduce useful expansions for the TGO-G pdf and cdf.
The pdf in (6) can be written as

$$\begin{aligned} f(x) &= \theta\,(1-\lambda) g(x)\, \bar{G}(x)^{-\alpha-1}\, e^{\frac{\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]} + 2\theta\,\lambda g(x)\, \bar{G}(x)^{-\alpha-1}\, e^{\frac{2\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]} \\ &= \sum_{j=0}^{\infty} \frac{\theta^{j+1}\left(1-\lambda+2^j\lambda\right)}{\alpha^j\, j!} g(x)\, \bar{G}(x)^{-\alpha-1}\left\{1-\bar{G}(x)^{-\alpha}\right\}^j \\ &= \sum_{i=0}^{\infty}\sum_{j=i}^{\infty}\sum_{h=0}^{\infty} \frac{(-1)^{i+h}\theta^{j+1}\left[1-\lambda\left(1-2^j\right)\right]}{\alpha^j\, j!}\binom{j}{i}\binom{-\alpha(i+1)-1}{h} g(x)\, G(x)^h \\ &= \sum_{i=0}^{\infty}\sum_{j=i}^{\infty}\sum_{h=0}^{\infty} a_{i,j,h}\, \pi_h(x) \end{aligned} \tag{7}$$

where, $a_{i,j,h} = \frac{(-1)^{i+h}\theta^{j+1}\left[1-\lambda\left(1-2^j\right)\right]}{(h+1)\alpha^j\, j!}\binom{j}{i}\binom{-\alpha(i+1)-1}{h}$ and $\pi_h(x) = (h+1)g(x)G(x)^h$ is the exponentiated-G density with power parameter $h+1$.

By integrating (7) with respect to x, we have

$$F(x) = \sum_{i=0}^{\infty}\sum_{j=i}^{\infty}\sum_{h=0}^{\infty} a_{i,j,h}\, \Pi_h(x) \tag{8}$$

where, $\Pi_h(x) = G(x)^h$.

3 The TGO-G sub-models

In this section, we introduce three special sub-models of the TGO-G family.

3.1 The TGO-Weibull (TGOW) model

The cdf and pdf of the Weibull distribution are the following $G(x) = e^{-ax^b}$ and $g(x) = abx^{b-1}e^{-ax^b}$ $x > 0$, $a, b > 0$, respectively. Then, the pdf and cdf of TGO-Weibull (TGOW) distribution are given, respectively, by

$$f(x) = \theta abx^{b-1}e^{\left[a\alpha x^b + \frac{\theta}{\alpha}\left(1-e^{a\alpha x^b}\right)\right]}\left\{1-\lambda+2\lambda e^{\frac{\theta}{\alpha}\left(1-e^{a\alpha x^b}\right)}\right\}, \qquad x > 0,$$

and

$$F(x) = \left\{1-e^{\frac{\theta}{\alpha}\left(1-e^{a\alpha x^b}\right)}\right\}\left\{1+\lambda e^{\frac{\theta}{\alpha}\left(1-e^{a\alpha x^b}\right)}\right\}, \qquad x \geq 0.$$

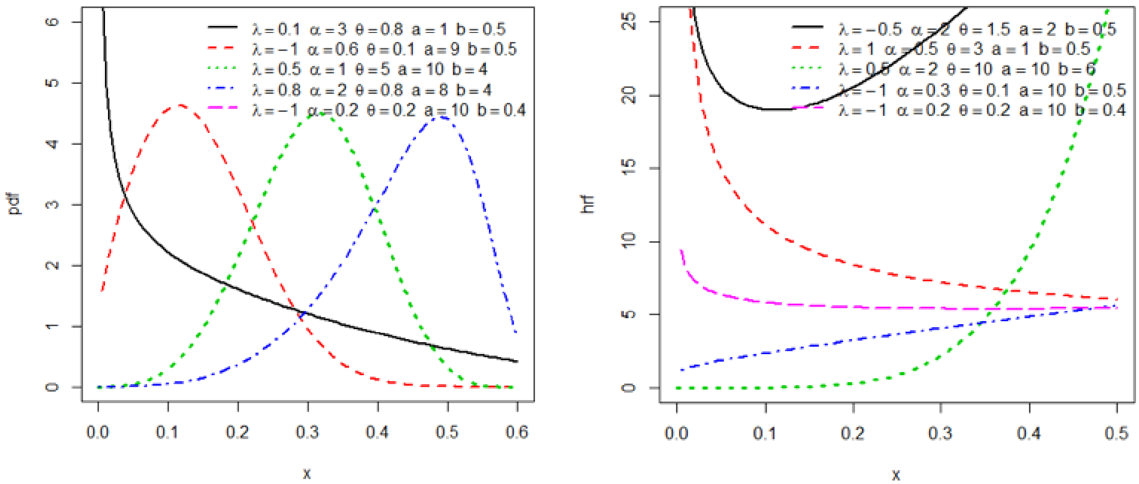

Figure 1: Plots of the TGOW pdf and hrf for selected parameters values.

3.2 The TGO-Lomax (TGOLx) model

Consider the cdf and pdf of the Lomax distribution $G(x) = 1-(1+bx)^{-a}$ and $g(x) = ab(1+bx)^{-(a+1)}, x > 0,\ \ a,\ b > 0$ respectively. Then, the pdf and cdf of TGO-Lomax (TGOLx) are given, respectively, by

$$f(x) = \theta ab(1+bx)^{a\alpha-1}e^{\frac{\theta}{\alpha}[1-(1+bx)^{a\alpha}]}\left\{1-\lambda+2\lambda e^{\frac{\theta}{\alpha}[1-(1+bx)^{a\alpha}]}\right\}, \quad x>0,$$

and

$$F(x) = \left\{1-e^{\frac{\theta}{\alpha}[1-(1+bx)^{a\alpha}]}\right\}\left\{1+\lambda e^{\frac{\theta}{\alpha}[1-(1+bx)^{a\alpha}]}\right\}, \qquad x\geq 0.$$

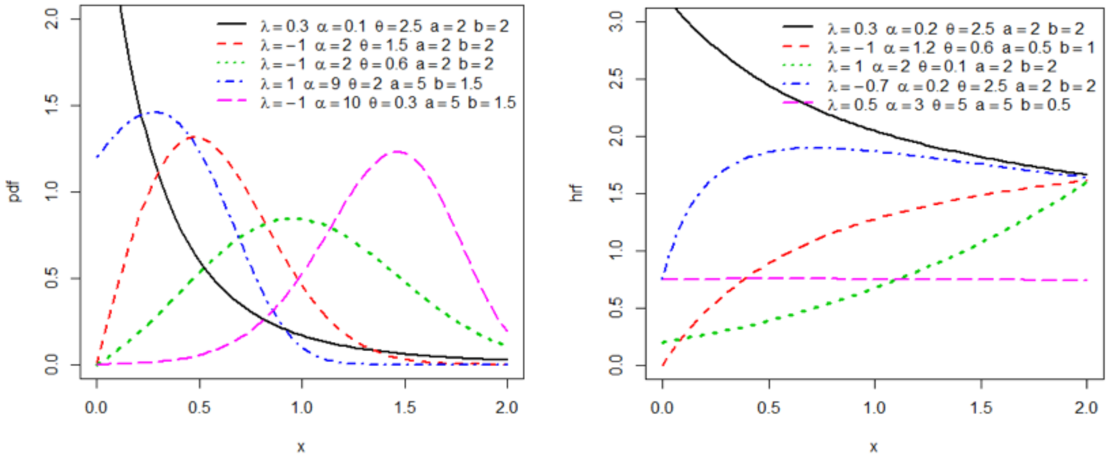

Figure 2: Plots of the TGOLx pdf and hrf for selected parameters values.

3.3 The TGO-Kumaraswamy (TGOKw) Model

The cdf and pdf of the Kumaraswamy distribution are $G(x) = 1-\left(1-x^b\right)^a$ and $g(x) = a\,b\,x^{b-1}\left(1-x^b\right)^{a-1}, 0<x<1,\ a,\ b>0$, respectively. Then, the pdf and cdf of TGO- Kumaraswamy (TGOKw) are given, respectively, by

$$f(x) = \theta abx^{b-1}\left(1-x^b\right)^{-(a\alpha+1)}e^{\frac{\theta}{\alpha}\left[1-(1-x^b)^{-a\alpha}\right]}\left\{1-\lambda+2\lambda e^{\frac{\theta}{\alpha}\left[1-(1-x^b)^{-a\alpha}\right]}\right\},\ x>0,$$

and

$$F(x) = \left\{1-e^{\frac{\theta}{\alpha}\left[1-(1-x^b)^{-a\alpha}\right]}\right\}\left\{1+\lambda e^{\frac{\theta}{\alpha}\left[1-(1-x^b)^{-a\alpha}\right]}\right\}, \qquad x\geq 0.$$

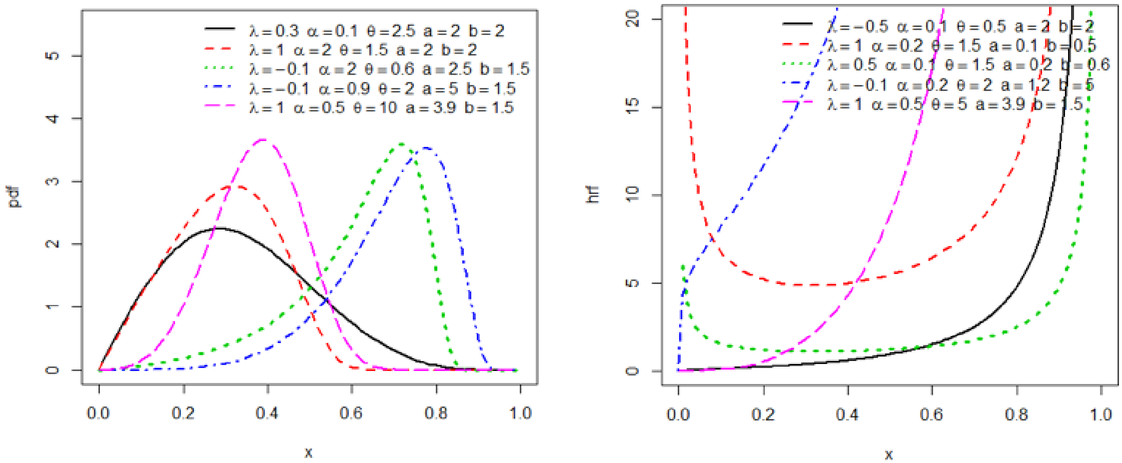

Figure 3: Plots of the TGOKw pdf and hrf for selected parameters values.

4 Mathematical properties

In this section, we discuss some of the mathematical properties of the TGO-G family such as; quantile function, ordinary and incomplete moments, generating function, probability weighted moment, Lorenz and Bonferroni curves, order statistics, Rényi and Shanon entropies, stress strength model and moment of residual and reversed residual life.

4.1 Quantile function

The quantile function of the TGO-G family say $Q(u) = F^{-1}(u)$ for $u \in (0,1)$,$\lambda \neq 0$,$\theta > 0$ and $\alpha > 0$ is the solution of the non-linear equation

$$Q(u) = G^{-1}\left(1 - \left\{1 - \frac{\alpha}{\theta}\log\left[\frac{(\lambda-1) - \sqrt{(\lambda-1)^2 - 4\lambda(u-1)}}{2\lambda}\right]\right\}^{-1/\alpha}\right). \tag{9}$$

4.2 The shape of the TGO-G family

We can describe mathematically the shape of the density and hazard functions of the TGO-G family. The critical points of the density function in (6) are the roots of the following equation:

$$\frac{g'(x)}{g(x)} + \frac{(\alpha+1)G'(x)}{\bar{G}(x)} - \frac{\theta G'(x)}{\bar{G}(x)^{\alpha+1}} - \frac{2\lambda\theta G'(x) e^{\frac{\theta}{\alpha}\left\{1-\bar{G}(x)^{-\alpha}\right\}}}{\bar{G}(x)^{\alpha+1}\left\{1-\lambda+2\lambda e^{\frac{\theta}{\alpha}\left\{1-\bar{G}(x)^{-\alpha}\right\}}\right\}} = 0.$$

Also, the critical points of the hazard function are the roots of the following equation:

$$\frac{g'(x)}{g(x)} + \frac{(\alpha+1)G'(x)}{\bar{G}(x)} - \frac{\theta G'(x)}{\bar{G}(x)^{\alpha+1}} - \frac{2\lambda\theta G'(x) e^{\frac{\theta}{\alpha}\left\{1-\bar{G}(x)^{-\alpha}\right\}}}{\bar{G}(x)^{\alpha+1}\left\{1-\lambda+2\lambda e^{\frac{\theta}{\alpha}\left\{1-\bar{G}(x)^{-\alpha}\right\}}\right\}}$$

$$+\frac{\theta G'(x)\bar{G}(x)^{-(\alpha+1)}e^{\frac{\theta}{\alpha}\left\{1-\bar{G}(x)^{-\alpha}\right\}}\left\{1-\lambda+2\lambda e^{\frac{\theta}{\alpha}\left\{1-\bar{G}(x)^{-\alpha}\right\}}\right\}}{1-\left\{1-e^{\frac{\theta}{\alpha}\left\{1-\bar{G}(x)^{-\alpha}\right\}}\right\}\left\{1+\lambda e^{\frac{\theta}{\alpha}\left\{1-\bar{G}(x)^{-\alpha}\right\}}\right\}}=0,$$

where,$g'(x_i,\varphi)=\partial g(x_i,\varphi)/\partial\varphi,\;\; G'(x_i,\varphi)=\partial G(x_i,\varphi)/\partial\varphi.$

4.3 Probability weighted moments

The PWM criteria can be applied for estimating the parameters of that distributions whose inverse does not have an explicit form. The$(r+s)$th PWM of X with TGO-G distribution, say $M_{r,s}$, is defined as

$$M_{r,s}=E\left(X^r F(x)^s\right)=\int_{-\infty}^{\infty} X^r F(x)^s f(x)dx, \tag{10}$$

Using (5) and (6), we can obtain

$$f(x)F(x)^s=\sum_{m=0}^{\infty}\eta_m\, g(x)\, G(x)^m, \tag{11}$$

where,

$$\eta_m=\sum_{j,\ell,r,k=0}^{\infty}\sum_{i=\ell}^{\infty}(-1)^{j+i+\ell+k}(r!)^{-1}\alpha^{-r}\theta^{r+1}\lambda^{\ell}(1+\lambda)^{s-j}\left[(1-\lambda)\left(i+\ell+1\right)^r+2\lambda\left(i+\ell+2\right)^r\right]$$

$$\times\binom{s}{j}\binom{s-j}{i}\binom{j}{\ell}\binom{r}{k}\binom{-\alpha(k+1)}{m}.$$

Inserting (11) in (10), we obtain

$$M_{r,s}=\sum_{m=0}^{\infty}\eta_m\, g(x)\,\psi_{r,m}, \tag{12}$$

where, $\psi_{r,m}=\int_{-\infty}^{\infty}x^r g(x)G(x)^m dx$ is the probability weighted moment of the parent distribution.

4.4 Ordinary, incomplete moments and generating function

Let X be a random variable with TGO-G distribution, then the ordinary moment, say μ_r', is given by

$$\begin{aligned}\mu_r'=E(X^r)&=\int_{-\infty}^{\infty}x^r f(x)\,dx\\ &=\sum_{i,h=0}^{\infty}\sum_{j=i}^{\infty}a^*_{i,j,h}\,\psi_{r,h},\end{aligned} \tag{13}$$

where, $a^*_{i,j,h}=(h+1)\;a_{i,j,h}$.

The nth central moment of the TGO-G distribution, say μ_n, can be obtained from

$$\mu_n=\sum_{r=0}^{n}\binom{n}{r}\left(-\mu_1'\right)^{n-r}\,E(x^r)$$

$$= \sum_{r=0}^{n} \sum_{i,h=0}^{\infty} \sum_{j=i}^{\infty} \binom{n}{r} (-\mu_1')^{n-r} a^{*}_{i,j,h} \psi_{r,h}, \tag{14}$$

where, $\psi_{r,h} = \int_{-\infty}^{\infty} x^r g(x) G(x)^h dx$ is the probability weighted moment of the baseline distribution. From (14), the measures of skewness and kurtosis of the TGO-G distribution can be obtained as

$$\text{Skewness}(X) = \frac{\mu\prime_3 - 3\mu\prime_2\mu\prime_1 + 2\mu\prime_1^3}{(\mu\prime_2 - \mu\prime_1^2)^{3/2}}, \tag{15}$$

and

$$\text{Kurtosis}(X) = \frac{\mu\prime_4 - 4\mu\prime_1\mu\prime_3 + 6\mu\prime_1^2\mu\prime_3 - 3\mu\prime_1^4}{\mu\prime_2 - \mu\prime_1^2}, \tag{16}$$

respectively. Figures (4) and (5) show the behavior of skewness and kurtosis of TGOLx distribution.

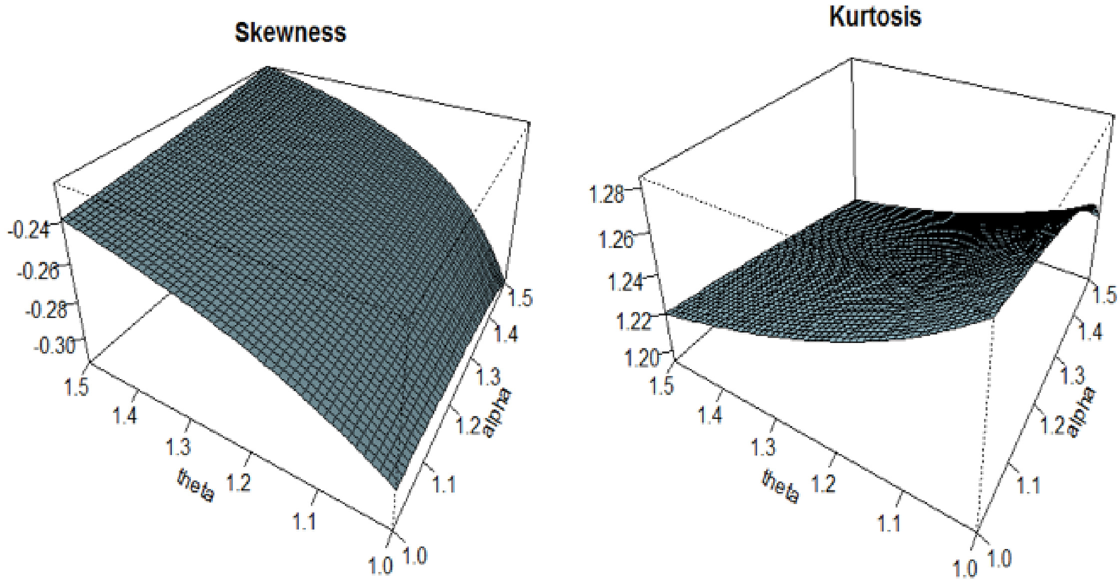

Figure 4: The skewness and kurtosis of TGOLx ($a = 0.5$, $b = 2$, $\lambda = 1$)distribution.

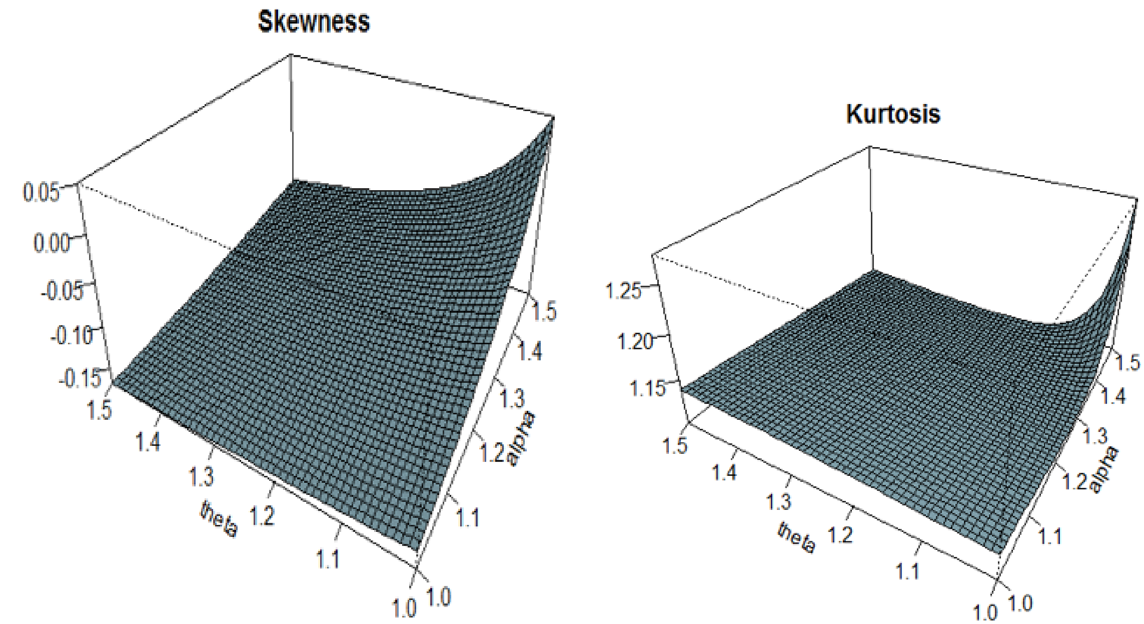

Figure 5: The skewness and kurtosis of TGOLx ($a = 1.5$, $b = 2$, $\lambda = 1$)distribution.

From Figures (4) and (5) we observe that α has significant control on skewness and kurtosis when a, b and λ are fixed.

The rth incomplete moment of the TGO-G distribution, denoted by $\varphi_w(t)$, is

$$\varphi_w(t) = \int_{-\infty}^{t} x^w f(x)\, dx$$

$$= \sum_{i,h=0}^{\infty} \sum_{j=i}^{\infty} a^*_{i,j,h}\, \kappa_{w,h}, \tag{17}$$

where, $\kappa_{w,h} = \int_{-\infty}^{t} x^w g(x)\ G(x)^h dx$.

There are two important applications related to the incomplete moments called, the Lorenz and Bonferroni curves that used in different fields such as demography and economics. The Lorenz and Bonferroni curves for the TGO-G family are given respectively as below

$$L_F(x) = \frac{\varphi_1(t)}{\mu'_1} = \frac{\sum_{i,h=0}^{\infty} \sum_{j=i}^{\infty} a^*_{i,j,h}\, \kappa_{1,h}}{\sum_{i,h=0}^{\infty} \sum_{j=i}^{\infty} a^*_{i,j,h}\, \psi_{1,h}}, \tag{18}$$

and

$$B\left(F(x)\right) = \frac{L_F(x)}{F(x)} = \frac{\sum_{i,h=0}^{\infty} \sum_{j=i}^{\infty} a^*_{i,j,h}\, \kappa_{1,h}}{F(x) \sum_{i,h=0}^{\infty} \sum_{j=i}^{\infty} a^*_{i,j,h}\, \psi_{1,h}}. \tag{19}$$

The moment and probability generating functions of the TGO-G distribution, denoted as $M_x(t)$ and $M_{[x]}(t)$ respectively can be obtained to be

$$M_x(t) = E(e^{tx}) = \sum_{r=0}^{n} \sum_{i,h=0}^{\infty} \sum_{j=i}^{\infty} \frac{t^r}{r!} a^*_{i,j,h}\, \psi_{r,h}, \tag{20}$$

and

$$M_{[x]}(t) = E(e^{tx}) = \sum_{r=0}^{n} \sum_{i,h=0}^{\infty} \sum_{j=i}^{\infty} \frac{(\ln)t^r}{r!} a^*_{i,j,h}\, \psi_{r,h}. \tag{21}$$

4.5 Order statistics

Let $X_{1:n} \leq X_{2:n}, ... \leq X_{n:n}$ be order statistics corresponding to a sample of size n from the TGO-G family. The pdf of $X_{k:n}$, the $k$thorder statistic, is given by

$$f_{X_{k:n}}(x) = \frac{1}{\beta(k, n-k+1)} \sum_{w=0}^{n-k} (-1)^w \begin{pmatrix} n-k \\ w \end{pmatrix} f(x) F(x)^{k+w-1}, \tag{22}$$

where, $\beta(.,.)$ is the beta function. From (5) and (6), we can obtain

$$f(x)F(x)^{k+w-1} = \sum_{m=0}^{\infty} \delta_m\, \pi_{m+1}(x), \tag{23}$$

where,

$$\delta_m = \sum_{j,i,h,\ell=0}^{\infty} \frac{(-1)^{j+\ell+m} \theta^{h+1} \lambda^j (\lambda-1)^i}{h!(m+1)\alpha^h} \left[2\lambda\,(2j+i+2)^h - (\lambda-1)\,(2j+i+1)^h\right]$$
$$\times \begin{pmatrix} k+w-1 \\ j \end{pmatrix} \begin{pmatrix} k+w-j-1 \\ i \end{pmatrix} \begin{pmatrix} h \\ \ell \end{pmatrix} \begin{pmatrix} -\alpha(\ell+1)-1 \\ m \end{pmatrix}.$$

From (23) in (22), we have

$$f_{X_{k:n}}(x) = \sum_{w=0}^{n-k} \sum_{m=0}^{\infty} \delta^*_m\, \pi_{m+1}(x), \tag{24}$$

where, $\delta^*_m = \frac{(-1)^w \begin{pmatrix} n-k \\ w \end{pmatrix}}{\beta(k,n-k+1)} \delta_m$.

Moreover, the rth moment of kth order statistic for TGO-G family is given by

$$E\left(x^r_{k:n}\right) = \sum_{w=0}^{n-k} \sum_{m=0}^{\infty} \delta^{**}_m\, \psi_{r,m}, \tag{25}$$

where, $\delta^{**}_m = (m+1)\delta^*_m$.

4.6 Entropy measures

The concept of entropy has been used in different fields such as queuing theory, statistics and reliability estimation. The Rényi entropy is defined as

$$I_R(X) = (1-\beta)^{-1} \log \int_{-\infty}^{\infty} f(x)^{\beta}\, dx,\ \ \beta > 0,\ \ \beta \neq 0.$$

From (6), we have

$$f(x)^{\beta} = \sum_{w=0}^{\infty} \xi_w\, g(x)^{\beta}\, G(x)^{w},$$

where, $\xi_w = \sum_{j,h,\ell=0}^{\infty} \frac{(-1)^{j+\ell+w}(2\lambda)^j \theta^{\beta+h}(\beta+j)^h(1-\lambda)^{\beta-h}}{\alpha^h\, h!} \binom{\beta}{j}\binom{h}{\ell}\binom{-\alpha(\beta+\ell)-\beta}{w}$.
Consequently, the Rényi entropy for the TGO-G family is given by

$$I_R(X) = (1-\beta)^{-1} \log \left(\sum_{w=0}^{\infty} \xi_w \int_{-\infty}^{\infty} g(x)^{\beta}\, G(x)^{w} dx \right). \tag{26}$$

The Shanon entropy is given by

$$\eta_x = -E\left\{\log f(x)\right\}.$$

From (6), we have

$$\log f(x) = \log(\theta) + \log\,(g(x)) - (\alpha+1)\log\left(\bar{G}(x;\varphi)\right) + \frac{\theta}{\alpha}\,\left(1-\bar{G}(x;\varphi)^{-\alpha}\right)$$
$$+\log\left\{1-\lambda\left(1-2e^{\frac{\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]}\right)\right\}.$$

Using $\log(1-z) = -\sum_{i=1}^{\infty}\frac{z^i}{i}$ in the above equation and after some manipulations, we arrive at

$$\eta_x = -\log(\theta) - E\left(\log g(x)\right) + (\alpha+1)\sum_{i=1}^{\infty}\left(\frac{1}{i}\right) E\left(G(x)^i\right)$$
$$-\frac{\theta}{\alpha}\left\{1-\sum_{j=0}^{\infty}(-1)^j\binom{-\alpha}{j} E\left(G(x)^j\right)\right\}$$
$$+\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{r,h,\ell=0}^{\infty}\frac{(-1)^{j+h+\ell}2^j\lambda^i(\theta j)^r}{\alpha^r\, r!}E\left(G(x)^{\ell}\right). \tag{27}$$

4.7 Stress strength model

The stress strength model is a common criterion used in different applications in physics and engineering such as strength failure and system collapse. Let X_1 and X_2 be two independent random variables with $\mathrm{TGO-G}\,(\theta_1,\alpha_1,\lambda_1,\varphi)$ and $\mathrm{TGO-G}\,(\theta_2,\alpha_2,\lambda_2,\varphi)$distributions. Then, the stress strength model is given by

$$R = \Pr(X_2 < X_1) = \int_0^{\infty} f_1\left(\theta_1,\alpha_1,\lambda_1,\varphi\right) F_2\left(\theta_2,\alpha_2,\lambda_2,\varphi\right)\, dx.$$

Using (5) and (6), we have

$$f_1\left(\theta_1,\alpha_1,\lambda_1,\varphi\right)F_2\left(\theta_2,\alpha_2,\lambda_2,\varphi\right)=\sum_{i=0}^{\infty}\nu_i g(x)G(x)^i,$$

where,

$$\nu_i=\sum_{r,j=0}^{\infty}(-1)^{j+i}\left(\alpha_1^r r!\right)^{-1}\theta_1^{r+1}\binom{r}{j}\left\{\left[1-\lambda_1\left(1-2^{r+1}\right)\right]\binom{-\alpha_1(j+1)}{i}+\sum_{w,\ell=0}^{\infty}\rho_{w,\ell}\right\},$$

and

$$\rho_{w,\ell}=(-1)^{\ell}\theta_2^{w}\left(\alpha_2^{w}w!\right)^{-1}\left(\lambda_2-2^{w}-1\right)\left[1-\lambda_1\left(1-2^{r+1}\right)\right]\binom{w}{\ell}\binom{-\alpha_1(j+1)-\alpha_2\ell-1}{i}.$$

Consequently, the stress strength model is given below

$$R=\sum_{i=0}^{\infty}\nu_i^{*}, \tag{28}$$

where, $\nu_i^{*}=(i+1)^{-1}\nu_i$.

4.8 Moment of residual and reversed residual life

The moment of residual and reversed residual life uniquely determine $F(x)$. The $n$thmoment of the residual life, say $m_n(t)$, of a random variable X with TGO-G distribution is

$$m_n(t)=\sum_{i,h=0}^{\infty}\sum_{j=i}^{\infty}\sum_{w=0}^{n}\frac{q_w a_{i,j,h}}{\bar{F}(t)}\int_t^{\infty}x^w G(x)^h\,dx, \tag{29}$$

where, $q_w=\frac{(-1)^{n-w}t^{n-w}\Gamma(n+1)}{w!\,\Gamma(n-w+1)}$.

The nth moment of the reversed residual life, say $M_n(t)$, of a random variable X with TGO-G distribution is

$$M_n(t)=\sum_{i,h=0}^{\infty}\sum_{j=i}^{\infty}\sum_{w=0}^{n}\frac{d_w\,a_{i,j,h}}{F(t)}\int_0^{t}x^w G(x)^h\,dx, \tag{30}$$

where, $d_w=\frac{(-1)^{w}t^{n-w}\Gamma(n+1)}{w!\,\Gamma(n-w+1)}$.

5 Characterizations results

This section is devoted to the characterizations of the TGO-G distribution in different directions: *(i)* based on the ratio of two truncated moments; *(ii)* in terms of the hazard function; *(iii)* based on the conditional expectation of certain function of the random variable. Note that *(i)* can be employed also when the cdf does not have a closed form. We would also like to mention that due to the nature of TGO-G distribution, our characterizations may be the only possible ones. We present our characterizations (i)-(iii) in three subsections.

5.1 Characterizations based on two truncated moments

This subsection deals with the characterizations of TGO-G distribution based on the ratio of two truncated moments. Our first characterization employs a theorem due to Glanzel (1987), see Theorem 1 of Appendix A. The result, however, holds also holds also when the interval H is not closed, since the condition of the Theorem is on the interior of H.

Proposition 5.1. Let $X : \Omega \to \mathrm{R}$ be a continuous random variable and let,

$$q_1 = \left\{1 - \lambda + 2\lambda\, e^{\frac{\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]}\right\}^{-1} \quad \text{and} \quad q_2(x) = q_1(x)\; e^{\frac{\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]}$$

for $x \in \mathrm{R}$. The random variable X has pdf (6) if and only if the function ξ defined in Theorem 1 is of the form

$$\xi(x) = \frac{1}{2} e^{\frac{\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]}, \;\; x \in \mathrm{R}.$$

Proof. Suppose the random variable X has pdf (6), then

$$(1 - F(x))\; E\left[q_1(X)\right| X \geq x] = e^{\frac{\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]}, \quad x \in \mathrm{R},$$

and

$$(1 - F(x))\; E\left[q_2(X)\right| X \geq x] = \frac{1}{2} e^{\frac{2\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]}, \quad x \in \mathrm{R}.$$

Further,

$$\xi(x)\, q_1(x) - q_2(x) = -\frac{q_1(x)}{2}\; e^{\frac{\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]} < 0 \;\; \text{for} \;\; x \in \mathrm{R}.$$

Conversely, if ξ is of the above form, then

$$s'(x) = \frac{\xi'(x)\, q_1(x)}{\xi(x)\; q_1(x) - q_2(x)} = \theta\, g(x;\varphi)\; \bar{G}(x;\varphi)^{-\alpha-1}, \;\; x \in \mathrm{R},$$

and consequently

$$s(x) = \frac{\theta}{\alpha}\; \bar{G}(x;\varphi)^{-\alpha}, \;\; x \in \mathrm{R}.$$

Now, according to Theorem 1, X has density (6).

Corollary 5.1. Let $X : \Omega \to \mathrm{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 5.1. The random variable X has pdf (6) if and only if there exist functions q_2 and ξ defined in Theorem 1 satisfying the differential equation

$$\frac{\xi'(x)\, q_1(x)}{\xi(x)\; q_1(x) - q_2(x)} = \theta\, g(x;\varphi)\; \bar{G}(x;\varphi)^{-\alpha-1}, \;\; x \in \mathrm{R}.$$

Corollary 5.2. The general solution of the differential equation in Corollary 5.1 is

$$\xi(x) = e^{-\frac{\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]} \left[-\int \theta\, g(x;\varphi)\; \bar{G}(x;\varphi)^{-\alpha-1} e^{\frac{\theta}{\alpha}\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]}\, (q_1(x))^{-1}\, q_2(x)\, dx + D\right],$$

where D is a constant. We like to point out that a set of functions satisfying the above differential equation is given in Proposition 5.1 with $D = 0$. Clearly, there are other triplets $(q_1,\; q_2,\; \xi)$ which satisfy conditions of Theorem 1.

5.2 Characterization in terms of hazard function

The hazard function, h_F, of a twice differentiable distribution function, F, satisfies the following first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h'_F(x)}{h_F(x)} - h_F(x).$$

It should be mentioned that for many univariate continuous distributions, the above equation is the only differential equation available in terms of the hazard function. In this subsection we present non-trivial characterizations of TGO-G distribution for two cases:
$\lambda = 0$ and $\lambda = 1$ in terms of the hazard function.

Proposition 5.2. Let $X : \Omega \to \mathrm{R}$ be a continuous random variable. The random variable X has pdf (6) (for $\lambda = 0$) if and only if its hazard function $h_F(x)$ satisfies the following differential equation

$$h'_F(x) - \frac{g'(x;\varphi)}{g(x;\varphi)} h_F(x) = \theta(\alpha+1)\, g(x;\varphi)^2\ \bar{G}(x;\varphi)^{-\alpha-2}, \quad x \in \mathrm{R}.$$

Proof. If X has pdf (6), for $\lambda = 0$, then clearly the above differential equation holds. If the differential equation holds, then

$$\frac{d}{dx}\left\{(g(x;\varphi))^{-1}\, h_F(x)\right\} = \theta \frac{d}{dx}\left\{\bar{G}(x;\varphi)^{-\alpha-1}\right\},$$

from which we arrive at the hazard function corresponding to pdf (6) when $\lambda = 0$.

Remark 5.1. For $\lambda = 1$,we have the following differential equation

$$h'_F(x) - \frac{g'(x;\varphi)}{g(x;\varphi)} h_F(x) = 2\theta(\alpha+1)\, g(x;\varphi)\ \bar{G}(x;\varphi)^{-\alpha-2}, \quad x \in \mathrm{R}.$$

5.3 Characterizations based on the conditional expectation of certain function of the random variable

In this subsection we employ a single function ψ of X and characterize the distribution of Xin terms of the truncated moment of $\psi(X)$. The following proposition has already appeared in Hamadan's previous work (2013), so we will just state it here which can be used to characterize the TGO-G distribution.

Proposition 5.3. Let $X : \Omega \to (e, f)$ be a continuous random variable with cdf F. Let $\psi(x)$ be a differentiable function on (e, f) with $\lim_{x\to e^+} \psi(x) = 1$. Then for $\delta \neq 1$,

$$E\left[\psi(X)|\, X \geq x\right] = \delta\psi(x), \quad x \in (e, f),$$

if and only if

$$\psi(x) = (1 - F(x))^{\frac{1}{\delta}-1}, \qquad x \in (e, f).$$

Remark 5.2. For$(e, f) = \mathrm{R}$, $\lambda = 1$, $\psi(x) = e^{\left[1-\bar{G}(x;\varphi)^{-\alpha}\right]}$ and $\delta = \frac{2\theta}{2\theta+\alpha}$, Proposition 5.3. provides a characterization of the TGO-G distribution for $\lambda = 1$.

6 Maximum likelihood estimation

This section discusses the maximum likelihood estimates (MLEs) of the model parameters of the TGO-G family. Let $x_1, x_2, ..., x_n$ be a random sample from TGO-G family with parameter vector $\Theta = (\theta, \alpha, \lambda, \varphi)^T$, then the corresponding log-likelihood function is given by

$$\ell = n\log(\theta) + \sum_{i=1}^{n} \log\left(g(x_i, \varphi)\right) - (\alpha+1)\sum_{i=1}^{n} \log\left(\bar{G}(x_i, \varphi)\right) + \frac{\theta}{\alpha}\sum_{i=1}^{n}\left(1 - \bar{G}(x;\varphi)^{-\alpha}\right)$$
$$+ \sum_{i=1}^{n} \log\left\{1 - \lambda + 2\lambda e^{\frac{\theta}{\alpha}\left(1-\bar{G}(x;\varphi)^{-\alpha}\right)}\right\}. \quad (31)$$

The components of the score vector $\nabla\ell = \left(\frac{\partial\ell}{\partial\theta}, \frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\lambda} \frac{\partial\ell}{\partial\varphi_r}\right)$are the following:

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} + \frac{1}{\alpha}\sum_{i=1}^{n}\left(1 - \bar{G}(x;\varphi)^{-\alpha}\right) + \frac{2\lambda}{\alpha}\sum_{i=1}^{n}\left\{\frac{\left(1 - \bar{G}(x;\varphi)^{-\alpha}\right) e^{\frac{\theta}{\alpha}\left(1-\bar{G}(x;\varphi)^{-\alpha}\right)}}{1 - \lambda + 2\lambda e^{\frac{\theta}{\alpha}\left(1-\bar{G}(x;\varphi)^{-\alpha}\right)}}\right\}, \quad (32)$$

$$\frac{\partial\ell}{\partial\lambda} = \sum_{i=1}^{n}\left\{\frac{-1 + 2e^{\frac{\theta}{\alpha}\left(1-\bar{G}(x;\varphi)^{-\alpha}\right)}}{1 - \lambda + 2\lambda e^{\frac{\theta}{\alpha}\left(1-\bar{G}(x;\varphi)^{-\alpha}\right)}}\right\}, \quad (33)$$

$$\frac{\partial\ell}{\partial\alpha} = -\sum_{i=1}^{n}\log\left(\bar{G}(x;\varphi)\right) + \frac{\theta}{\alpha}\sum_{i=1}^{n}\left\{\bar{G}(x;\varphi)^{-\alpha}\left[\log\left(\bar{G}(x;\varphi)\right) + \frac{1}{\alpha}\right] - \frac{1}{\alpha}\right\}$$
$$+\frac{2\lambda\theta}{\alpha}\sum_{i=1}^{n}\left\{\frac{\left(\bar{G}(x;\varphi)^{-\alpha}\left[\log\left(\bar{G}(x;\varphi)\right) + \frac{1}{\alpha}\right] - \frac{1}{\alpha}\right) e^{\frac{\theta}{\alpha}\left(1-\bar{G}(x;\varphi)^{-\alpha}\right)}}{1 - \lambda + 2\lambda e^{\frac{\theta}{\alpha}\left(1-\bar{G}(x;\varphi)^{-\alpha}\right)}}\right\}, \quad (34)$$

and (for for $(r = 1, ..., q)$

$$\frac{\partial\ell}{\partial\varphi_r} = \sum_{i=1}^{n}\left(\frac{g_r'(x_i,\varphi)}{g_r(x_i,\varphi)}\right) + (\alpha+1)\sum_{i=1}^{n}\left(\frac{G_r'(x_i,\varphi)}{\bar{G}_r(x_i,\varphi)}\right) - \theta\sum_{i=1}^{n}\left(\frac{G_r'(x_i,\varphi)}{\bar{G}_r(x_i,\varphi)^{\alpha+1}}\right)$$
$$-2\lambda\theta\sum_{i=1}^{n}\left(\frac{G_r'(x_i,\varphi)\bar{G}_r(x_i,\varphi)^{-(\alpha+1)} e^{\frac{\theta}{\alpha}\left(1-\bar{G}_r(x_i,\varphi)^{-\alpha}\right)}}{1 - \lambda + 2\lambda e^{\frac{\theta}{\alpha}\left(1-\bar{G}_r(x_i,\varphi)^{-\alpha}\right)}}\right), \quad (35)$$

where,$g_r'(x_i,\varphi) = \partial g_r(x_i,\varphi)/\partial\varphi_r$, $G_r'(x_i,\varphi) = \partial G_r(x_i,\varphi)/\partial\varphi_r$.

The MLEs, say $\hat{\Theta} = (\hat{\theta}, \hat{\alpha}, \hat{\lambda}, \hat{\varphi})$ of $\Theta = (\theta, \alpha, \lambda, \varphi)^T$can be obtained by equating the system of nonlinear equations (32) through (35) with zero and solving them simultaneously. Mention that if analytical solutions are not possible we use certain software Package.

7 Simulation study

In this section, a brief simulation study is conducted to examine the performance of the MLEs of TGOLx parameters. Quantile function is used to generate random observations from the TGOLx distribution. We generate 1000 samples of size, $n = 20, 120, 220$ and $n = 500$ from the TGOLx distribution. The evaluation of estimates was based on the bias of the MLEs of the model parameters, the mean squared error (MSE) of the MLEs. The empirical study was conducted with software R and the results are given in Table 1. It is observed, from Table 1, that the biases and MSEs decrease as n increases. The simulation study shows that the maximum likelihood method is appropriate for estimating the parameters of the TGOLx distribution.

Table 1: Biases and MSEs for the MLEs of the parameters of the TGOLx distribution.

n	parameter	Initial	Bais	MSE	Initial	Bais	MSE
20	α	1	0.6801959	0.058183456	0.5	0.5989014	0.05786768
	λ	2.5	1.0592599	0.049993870	1.5	1.5179616	0.04519296
	θ	1.5	1.9286206	0.069414132	0.5	1.8112338	0.08081967
	a	1.5	1.1510636	0.061151668	2.5	0.5610504	0.05962089
	b	2.5	1.5105637	0.07914550	2	0.5666119	0.05907682
120	α	1	0.6404050	0.03215263	0.5	0.5631270	0.02824470
	λ	2.5	0.9097647	0.03191560	1.5	1.4893997	0.03136887
	θ	1.5	1.9723259	0.04703798	0.5	1.6857998	0.04041504
	a	1.5	1.0838723	0.03377313	2.5	0.5187906	0.03292594
	b	2.5	1.3020527	0.03735997	2	0.5173752	0.03461471
220	α	1	0.6021129	0.01629026	0.5	0.5130746	0.01030776
	λ	2.5	0.8998808	0.02226033	1.5	1.5027060	0.01498938
	θ	1.5	1.8688384	0.03265637	0.5	1.5833461	0.02426799
	a	1.5	1.0093226	0.01888700	2.5	0.5238198	0.02110809
	b	2.5	1.2652241	0.02294630	2	0.5421806	0.01827662
500	α	1	0.5638233	0.008183456	0.5	0.4900580	0.004445937
	λ	2.5	0.9146213	0.009993870	1.5	1.4699461	0.008834193
	θ	1.5	1.8565628	0.019414132	0.5	1.5863443	0.010317182
	a	1.5	0.9341439	0.011151668	2.5	0.3823062	0.006861445
	b	2.5	1.2733167	0.007914550	2	0.5399559	0.007359099

8 Applications

In this section, we introduce two applications to real data to demonstrate the applicability of the TGO-G family. We concerned with on the TGOLLx distribution introduced in Subsection 3.2. The second data set consists of 63 observations of the strengths of 1.5 cm glass fibers which obtained by workers at the UK National Physical Laboratory. The data are: 0.55, 0.74, 0.77, 0.81, 0.84,0.93, 1.04, 1.11,1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49,1.49, 1.50, 1.50, 1.51, 1.52, 1.53,1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61,1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68,1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76,1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24. The data has previously been used by Smith and Naylor (1987), Bourguinon et al. (2014), Merovci et al. (2016) and Reyad and Othman (2017).
The second data set for breaking stress of carbon Öbers of 50 mm length (GPa) was reported by Nicholas and Padgett (2006). The data are: 0.39,0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67,

2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90. This data was also used by Cordeiro and Lemonte (2011) and Yousof et al. (2017a).
First, we obtain the MLEs for the unknown parameters of all competitive models and then comparing the results via goodness-of-fit statistics; Anderson-Darling (A^*), Cramér–von Mises (W^*), AIC (Akaike information criterion), and BIC (Bayesian information criterion). The better model corresponds to smaller of these criteria. The values for the Kolmogorov Smirnov (KS) statistic and its p-value are also provided.
We compare the TGO-Lx distribution with those of the Gompertz Lomax (GzLx) Oguntunde et al., 2017), Weibull Lomax (WLx)(Tahir et al., 2015) beta Lomax (BLx) (Lemonte and Cordeiro, 2013), Kumaraswamy Lomax (KwLx) (Lemonte and Cordeiro, 2013), exponentiated Lomax (ELx) (El-Bassiouny et al., 2015) and Topp-Leone Lomax (TLLx) (Al-Shomrani et al., 2016). The MLEs and some statistics of the models for the first data set and second data set are presented in Tables (2), (3), (4) and (5) respectively

Table 2: The MLEs for the first data set.

Distribution	Estimates with standard error in parenthesis						
	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\gamma}$
TGOLx	0.6974 (0.3121)	0.1227 (0.0874)	−0.6660 (0.3209)		0.2476 (0.2306)	42.8375 (23.9727)	—
GzLx	—	—	—	8.1791 (1.9036)	0.5069 (3.3751)	0.0046 (0.0047)	1.5158 (8.9789)
WLx	1.0629 (0.6492)	0.0649 (1.8323)	—	0.1069 (65.7718)	—	6.0947 (27.9028)	—
BLx	10.8769 (3.24968)	0.0329 (1.5668)	—	26.7645 (16.3236)	—	18.1737 (2.4957)	—
KwLx	15.1182 (2.2553)	0.0483 (1.8471)	—	45.3107 (19.69574)	—	9.8352 (33.5077)	—
ELx	34.97716 (38.793)	33.07455 (10.333)	—	—	94.59875 (101.676)	—	—
TLLx	69.84608 (91.2567)	32.63070 (10.0717)	—	—	52.17284 (69.5289)	—	—

Table 3: Some statistics for the models fitted to the first data set.

Distribution	Estimates with standard error in parenthesis						
	-LL	A*	W*	KS	P-value	AIC	BIC
TGOLx	13.79668	0.7460	0.1294	0.12029,	0.3217	36.5537	44.2693
GzLx	14.5027	0.9462	0.1473	0.1542	0.0998	37.0055	45.5780
WLx	15.3399	1.3315	0.2057	0.1517	0.1100	38.6798	47.2524
BLx	24.4034	3.1986	0.5726	0.2182	0.0049	56.8068	65.3793
KwLx	18.1027	1.9915	0.3230	0.1854	0.0263	44.2055	52.7779
ELx	31.8183	4.3567	0.7993	0.2298	0.0025	69.6366	76.0660
TLLx	31.6781	4.3368	0.7956	0.2299	0.0025	69.3563	75.7857

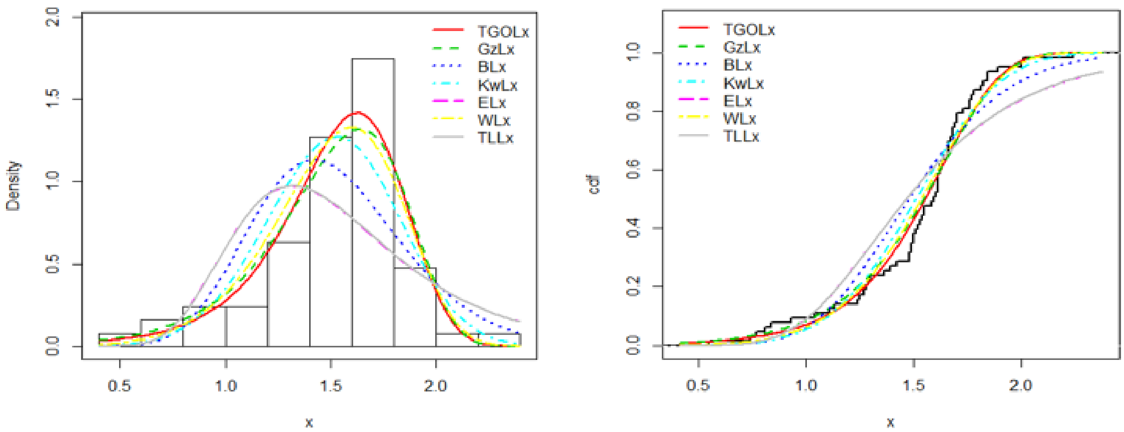

Figure 6: Estimated pdfs and cdfs plots of the TGOLx distribution for data set 1.

Table 4: The MLEs for the second data set.

Distribution	Estimates with standard error in parenthesis						
	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\gamma}$
TGOLx	0.0660 (0.1101)	1.2842 (2.5356)	0.6978 (0.4437)		0.2454 (1.1347)	54.4920 (24.0972)	—
GzLx	—	—	—	0.5921 (0.1531)	8.7421 (2.7485)	0.0031 (0.0007)	1.7668 (0.7151)
WLx	2.8127 (0.7718)	0.1241 (2.1393)	—	11.6216 (30.8346)	—	2.9447 (6.9306)	—
BLx	7.5729 (1.3036)	68.4371 (303.1534)	—	7.0222 (32.3429)	—	181.8549 (37.7779)	—
KwLx	3.95530 (1.6199)	568.3499 (221.1471)	—	2.86126 (10.5489)	—	37.3109 (195.0610)	—
ELx	54.6166 (52.2379)	9.7254 (2.3069)	—	—	57.6515 (53.5301)	—	—
TLLx	54.6327 (69.8100)	9.3738 (2.2139)	—	—	106.3314 (138.4358)	—	—

Table 5: Some statistics for the models fitted to the second data set.

Distribution	Estimates with standard error in parenthesis						
	-LL	A*	W*	KS	P-value	AIC	BIC
TGOLx	85.2528	0.3715	0.0604	0.0746	0.8556	180.3057	189.0937
GzLx	86.1675	0.4409	0.0694	0.0953	0.5865	180.5351	191.454
WLx	86.9479	0.4941	0.0825	0.0818	0.7690	182.6959	194.2649
BLx	91.2880	1.3397	0.2474	0.1335	0.1895	190.5762	199.3348
KwLx	86.4514	0.5838	0.1065	0.0868	0.7021	180.9030	189.6616
ELx	95.9593	2.0000	0.3632	0.1580	0.0741	197.9186	204.4876
TLLx	95.6768	1.9505	0.3546	0.1559	0.0805	197.3538	203.9227

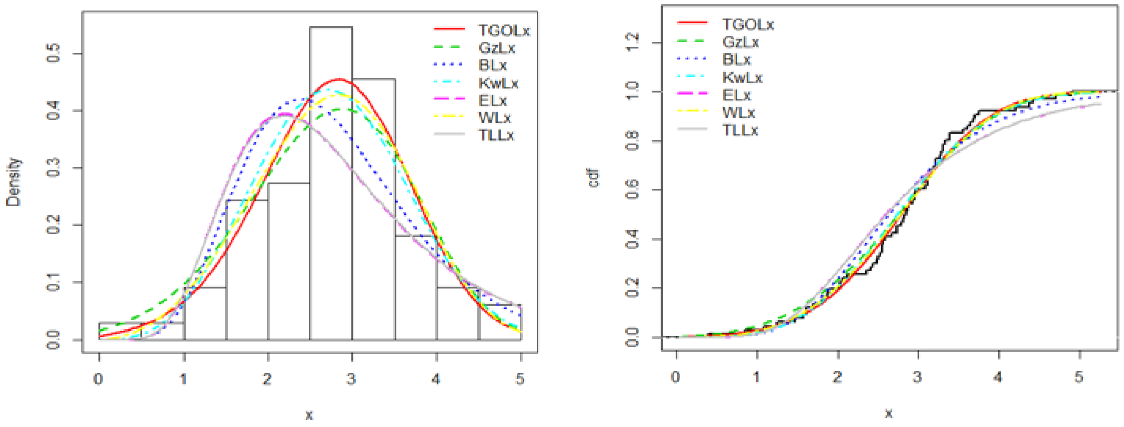

Figure 7: Estimated pdfs and cdfs plots of the TGOLx distribution for data set 2.

The values in Tables 3 and 4 showed that the TGOLx distribution has the lowest values for A*, W*, AIC, BIC, KS and largest P-values among all fitted models (for the two real data sets). So, the TGOLx model could be chosen as the best model.
The estimated pdfs and cdfs plots are displayed in Figures 6 and 7. It is clear from Figures 6 and 7, that the new TGOLx distribution provides the best fits for the two data sets.

9 Conclusions

We propose a new class of distributions, called the transmuted Gompertz-G (TGO-G) family that generalizes the Gompertz-G class introduced by Alizadeh et al. (2016a). We have studied most mathematical properties of the new family such as quantile function, probability weighted moment, ordinary and incomplete moments, generating function, Rényi and shanonof entropies, Lorenz and Bonferroni curves, stress strength model, moment of residual and reversed residual life, characterizations and order statistics. The method of maximum likelihood is used to estimate the model parameters. Two real data sets are used to demonstrate the usefullness of the new family.

References

[1] Alizadeh, M., Cordeiro, G. M., Pinho, L. G. B. and Ghosh, I. (2016a). The Gompertz-G family of distributions. Journal of Statistical Theory and Practice, 11(1): 179-207.

[2] Alizadeh, M., Altun, E., Cordeiro, G. M. and Rasekhi, M. (2018). The odd power Cauchy family of distributions: properties, regression models and applications. Journal of Statistical Computation and Simulation, 88(4):785-805.

[3] Alizadeh, M., Cordeiro, G. M., Nascimento, A. D. C., Lima, M. D. S. and Ortega, E. M. M. (2016b). Odd-Burr generalized family of distributions with some applications. Journal of Statistical Computation and Simulation, 83, 326-339.

[4] Al-Shomrani, A., Arif, O., Shawky, K., Hanif, S. and Shahbaz, M. Q. (2016). Topp-Leone family of distributions: some properties and application. Pak.j.stat.oper.res. Vol.XII(3): 443-451.

[5] Bourguignon, M,. Silva, R. B. and Cordeiro, G. M. (2014). The Weibull-G family of probability distributions. Journal of Data Science, 12: 53–68.

[6] Cordeiro, G. M. and de Castro, M. (2011). A new family of generalized distributions. Journal of Statistical Computation and Simulation, 81, 883-898.

[7] Cordeiro, G. M. and Lemonte, A. J. (2013). The β- Birnbaum-Saunders distribution: An improved distribution for fatigue life modeling. Comput. Stat. Data Anal, 55: 1445-1461

[8] Cordeiro, G. M., Ortega, E. M. M., Popovic, B. V. and Pescim, R. R. (2014). The Lomax generator of distributions: properties, minification process and regression model. Appl. Math. Comput., 247: 465-486.

[9] El-Bassiouny, A. H., Abdo, N. F., Shahen, H. S. (2015). Exponential lomax distribution. International Journal of Computer applications, 121(13): 24-29.

[10] Eugene, N., Lee, C., Famoye, F. (2002). Beta-normal distribution and its applications. Commun. Stat. Theory Methods, 31: 497-512

[11] Glanzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families. Mathematical Statistics and Probability Theory (Bad Tatzmannsdorf, 1986)7, Reidel, Dordrecht: 75-84.

[12] Glanzel, W. (1990). Some consequences of a characterization theorem based on truncated moments. Statistics, A Journal of Theoretical and Applied Statistics, 21(4): 613-618.

[13] Gupta, R. C., Gupta, P. L. and Gupta, R. D. (1998). Modeling failure time data by Lehmann alternatives. Commun. Stat. Theory Methods, 27: 887-904.

[14] Hamedani, G. G. On certain generalized gamma convolution distribution II, Technical Report No. 484, 2013, MSCS, Marquette University.

[15] Hamedani, G. G., Yousof, H. M., Rasekhi, M., Alizadeh, M., Najibi, S. M. (2018). Type I general exponential class of distributions. Pak.j.stat.oper.res. Forthcoming.

[16] Lemonte, A., Cordeiro, G. (2013). An extended lomax distribution. Statistics, 47: 800-816.

[17] Merovci, F., Khaleel, M. A., Ibrahim, N. A. and Shitan, M. (2016). The beta type X distribution:properties with application, SpringerPlus, 5:697.

[18] Nasir, M. A., Jamal, F., Silva, G. O. and Tahir, M. H. (2018). Odd Burr-G Poisson family of distributions. Journal of Statistics Applications & Probability, 7(1): 1-19.

[19] Nichols, M. D. and Padgett, W. J. (2006). A bootstrap control chart for Weibull percentiles. Quality and Reliability Engineering International, 22: 141-151.

[20] Oguntunde, P. E., Khaleel, M. A., Ahmed, M. T., Adejumo, A. O. and Odetunmibi, O. A. (2017). A New generalization of the Lomax distribution with increasing, decreasing and constant failure rate. Modelling and Simulation in Engineering, Volume 2017, Article ID 6043169, 6 pages.

[21] Reyad, H. M. and Othman, S, A. (2017). The Topp-Leone Burr-XII Distribution: Properties and Applications. British Journal of Mathematics & Computer Science, 21(5): 1-15.

[22] Shaw, W. T. and Buckley, I. R. C. (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map. Research report.

[23] Smith, R. L. and Naylor, J. C. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. Applied Statistics, 36: 258-369.

[24] Tahir, M. H., Cordeiro, G. M., Mansoorz, M. and Zubair, M. (2015). The Weibull-Lomax distribution: properties and applications. Hacettepe Journal of Mathematics and Statistics, 44(2): 461–480

[25] Torabi, H. and Montazari, N. H. (2014). The logistic-uniform distribution and its application. Commun. Stat.Simui. Comput, 43: 2551-2569.

[26] Yousof, H. M., Alizadeh, M., Jahanshahic, S. M., Ramires, T. G., Ghosh, I. and Hamedani, G. G. (2017a). The transmuted Topp-Leone G family of distributions: theory, characterizations and applications. Journal of Data Science, 15: 723-740

[27] Yousof, H. M., Rasekhi, M., Afify, A. Z., Ghosh, I., Alizadeh, M. and Hamedani, G. G. (2017b). The beta Weibull-G family of distribution: Theory, characterizations and applications. Pakistan Journal of Statistics, 32(2), 95:116.

Appendix A

Theorem 1. Let $(\Omega\,,\ F,\ \mathrm{P})$be a given probability space and let $H=[a,b]$ be an interval for some $d<b$ ($a=-\infty,\ b=\infty$ might as well be allowed). Let $X:\Omega\rightarrow H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$E\left[q_2(X)|\, X\geq x\right]=E\left[q_1(X)|\, X\geq x\right]\xi(x),\quad x\in H,$$

is defined with some real function η. Assume that $q_1,\ q_2\in C^{-1}(H),\ \xi\in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi\, q_1=q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions $q_1,\ q_2$ and ξ, particularly

$$F(x)=\int_a^x C\left|\frac{\xi'(u)}{\xi(u)\,q_1(u)-q_2(u)}\right|\exp\left(-s(u)\right)\ du\,,$$

where the function s is a solution of the differential equation $s'=\frac{\xi'\,q_1}{\xi\ q_1-q_2}$ and C is the normalization constant, such that $\int_H dF=1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence (see Glanzel (1990)), in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution function $\{F_n\}$ such that the functions q_{1n}, q_{2n} and ξ_n $(n \in N)$ satisfy the conditions of Theorem 1 and let $q_{1n} \to q_1$, $q_{2n} \to q_2$ for some continuously differentiable real functions q_1 and q_2. Let, finally, X be a random variable with distribution F. Under the condition that $q_{1n}(X)$ and $q_{2n}(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if ξ_n converges to ξ, where

$$\xi(x) = \frac{E\left[q_2(X)|\, X \geq x\right]}{E\left[q_1(X)|\, X \geq x\right]}$$

This stability theorem makes sure that the convergence of distribution function is reflected by corresponding convergence of the function q_1, q_2 and ξ_n, respectively. It guarantees, for instance, the convergence of characterization on the Wald distribution to that of the Levy-Smirrnov distribution if $\alpha \to \infty$.

A further consequence of the stability property of Theorem 1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions q_1, q_2 and, specially, ξ should be as simple as possible. Since the function triplet is not uniquely determined it is often possible to choose ξ as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.