Existence of a pair of new recurrence relations for the Meixner-Pollaczek polynomials

E.I. Jafarov[1], A.M. Jafarova[2], S.M. Nagiyev[1]

[1] Institute of Physics, Azerbaijan National Academy of Sciences
Javid av. 33, AZ-1143, Baku, Azerbaijan

[2] Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences
Vahabzade str. 9, AZ-1141, Baku, Azerbaijan

E-mail: {ejafarov,smnagiyev}@physics.science.az, aynura.jafarova@imm.az

Abstract

We report on existence of pair of new recurrence relations (or difference equations) for the Meixner-Pollaczek polynomials. Proof of the correctness of these difference equations is also presented. Next, we found that subtraction of the forward shift operator for the Meixner-Pollaczek polynomials from one of these recurrence relations leads to the difference equation for the Meixner-Pollaczek polynomials generated via cosh difference differentiation operator. Then, we show that, under the limit $\varphi \to 0$, new recurrence relations for the Meixner-Pollaczek polynomials recover pair of the known recurrence relations for the generalized Laguerre polynomials. At the end, we introduced differentiation formula, which expresses Meixner-Pollaczek polynomials with parameters $\lambda > 0$ and $0 < \varphi < \pi$ via generalized Laguerre polynomials.

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1 Introduction

Orthogonal polynomials play important role in the explicit solution of huge number of problems, mainly coming from physics and mathematics. Their advantage is that by employing some of known properties of them (three-term recurrence relations, differential/difference equations, forward/backward shift operators, generating functions etc), one can construct number of exactly-solvable dynamical systems with various behaviour of the eigenvalues and eigenfunctions. Best example that can be provided here is the Hermite polynomials, which are solutions of the second order differential equation. In quantum mechanics, upon explicit solution of the Schrödinger equation of the nonrelativistic 1D harmonic oscillator within the condition that non-commuting momentum and position operators satisfy the relation $[\hat{x}, \hat{p}] = i$, these polynomials appear in the wavefunctions of the stationary oscillator states, which have the discrete eigenvalues $E_n = n + 1/2, \ n = 1, 2, ...$ [1]. Then, another two important properties of these orthogonal polynomials - their forward and backward shift operators and three-term recurrence relations play the role of starting point for construction of the dynamical symmetry algebra of the non-relativistic quantum harmonic oscillator, which is the Heisenberg-Weil algebra [2]. Hermite polynomials appear in the Askey table of the orthogonal polynomials in the lower level, which means that they can be considered as simplest realization within the orthogonal polynomials [3]. They are expressed through ${}_2F_0$ hypergeometric functions and have no free parameter. As a next step, one can drop the so-called canonical commutation relation, i.e. $[\hat{x}, \hat{p}] \neq 0$. However, instead of it, one can require here satisfaction of the

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Heisenberg-Lie equations. Under such a condition, the Schrödinger equation of the nonrelativistic 1D harmonic oscillator with non-commuting momentum and position operators again can be solved explicitly. Its wavefunctions of the stationary states are expressed through the generalized Laguerre polynomials and dynamical symmetry is $osp\,(1|2)$ Lie superalgebra [4]. From mathematical viewpoint, two well-known recurrence relations for the generalized Laguerre polynomials are starting bricks for construction of this model [5]. Here, one need to note that various generalizations of both Hermite and Laguerre polynomials already exists and some examples are Hermite based Appell matrix polynomials [6] and special polynomials as mixing of Laguerre-Gould Hopper polynomials with Sheffer sequences [7]. Also, 'algebraic zero' conditions with respect to weight and contour of such ortogonal polynlomials are already studied in detail [8]. Besides such generalizations, one observes that the Meixner-Pollaczek polynomials appearing in the next level of the Askey table of the orthogonal polynomials can be best candidates to generalize both Hermite and Laguerre polynomials and quantum harmonic oscillator models with stationary states wavefunctions expressed through them. They have two parameters, one is $\lambda > 0$ and another one is an angle $0 < \varphi < \pi$. Already, their special case $\varphi = \pi/2$ has been used to construct two interesting models of the quantum harmonic oscillator. Dynamical symmetry of both models is $su(1,1)$ Lie algebra [9, 10]. There exists general limit between Meixner-Pollaczek polynomials of parameter $\lambda > 0$ under the case $\lambda \to \infty$ [3] as well as one can apply same limit for both quantum harmonic oscillator models with a special case $\varphi = \pi/2$ and show that constructed models and their $su(1,1)$ dynamical symmetry algebra correctly reduce to the Hermite oscillator model with Heisenberg-Weil dynamical symmetry algebra. It is necessary to note that, Meixner-Pollaczek polynomials are related with Krawtchouk and Meixner polynomials via simple transformation [11], which are already used to construct different kind of the quantum harmonic oscillator models in the discrete configurational space [12, 13, 14]. The open question here is possibility of the generalization of pair of recurrence relations for the generalized Laguerre polynomials for the case of Meixner-Pollaczek polynomials.

Aim of this paper is to show that the Meixner-Pollaczek polynomials with parameter $\lambda > 0$ and angle $0 < \varphi < \pi$ satisfy pair of new recurrence relations/difference equations differing from those appearing in [3]. Our paper is structured as follows. In Section 2, we give basic information about the Meixner-Pollaczek polynomials, including their definition in terms of the ${}_2F_1$ hypergeometric functions, orthogonality relation and forward and backward shift operators. Then, we introduce pair of new recurrence relations/difference equations for the Meixner-Pollaczek polynomials, prove their correctness by using direct computations as well as show that subtraction of the forward shift operator for the Meixner-Pollaczek polynomials from one of these recurrence relations leads to the difference equation for the Meixner-Pollaczek polynomials generated via cosh difference differentiation operator and these recurrence relations reduce to the pair of known recurrence relations for the generalized Laguerre polynomials, in Section 3. Also, in this section, we introduce differentiation formula, which expresses Meixner-Pollaczek polynomials with parameters $\lambda > 0$ and $0 < \varphi < \pi$ via generalized Laguerre polynomials. Conclusions and further discussions of possible application of the derived equations are discussed in Section 4.

2 Definition and known properties of the Meixner-Pollaczek polynomials

Known properties of the Meixner-Pollaczek polynomials are used extensively for solution of wide range applied mathematics and theoretical physics problems since their introduction in [15]. Main properties of these polynomials, such as, their definition through the hypergeometric functions, three-term recurrence relations, difference equation, explicit solution of which leads to these polynomials, forward and backward shift operators and generating functions of them as well as various limit relations and special cases between them and other orthogonal polynomials belonging to the Askey scheme of the orthogonal polynomials are listed in [3]. However, information provided there does not mean that these properties are unique and another generalizations or various finite-difference equations, solutions of which can lead to the Meixner-Pollaczek polynomials does not exist. As a support of this statement, one can provide the following brief information about recently published works achieving attractive results through involvement of the Meixner-Pollaczek polynomials: multivariable biorthogonal generalizations of the Meixner-Polaczek polynomials are presented in [16], an integral representation of the Meixner-Pollaczek polynomials in terms of a multidimensional generalization of the Barnes type integral are presented in [17], the asymptotics of zeros of the Meixner-Pollaczek polynomials are studied in [18, 19], new asymptotic expansions of the Hahn-type ${}_3F_2$ polynomials in terms of the Meixner-Pollaczek polynomials are derived and new limit relation between the continuous dual Hahn and Meixner-Pollaczek polynomials is derived in [20], new generalization of the Meixner-Pollaczek polynomials with an additional ψ parameter is introduced in [21], expansions of Dirichlet beta function and Riemann zeta function in the series of Meixner-Pollaczek polynomials are derived in [22, 23] and by using Meixner-Pollaczek polynomials with specific parameters, a class of new generalized coherent states is constructed in [24]. Also, it is necessary to highlight very recent paper [25], where, it is proven that the Meixner-Pollaczek polynomials are solutions of a second-order divided-difference equation of hypergeometric-type and then, the inversion, connection, multiplication and linearization problems have been solved for these polynomials.

Meixner-Pollaczek polynomials are defined through the ${}_2F_1$ hypergeometric functions by the following expression [3]:

$$P_n^{(\lambda)}(x;\varphi) = \frac{(2\lambda)_n}{n!} e^{in\varphi}\, {}_2F_1\left(\begin{array}{c} -n, \lambda + ix \\ 2\lambda \end{array} ; \quad 1 - e^{-2i\varphi} \right). \tag{2.1}$$

They satisfy the following orthogonality relation in the continuous configurational space:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(2\varphi - \pi)x} \left|\Gamma(\lambda + ix)\right|^2 P_m^{(\lambda)}(x;\varphi)\, P_n^{(\lambda)}(x;\varphi)\, dx = \frac{\Gamma(n + 2\lambda)}{(2\sin\varphi)^{2\lambda}\, n!} \delta_{mn}, \tag{2.2}$$

$$\lambda > 0, \quad 0 < \varphi < \pi.$$

These polynomials are explicit solutions of the following difference equation:

$$\begin{aligned} &2i\left[x\cos\varphi - (n + \lambda)\sin\varphi\right] P_n^{(\lambda)}(x;\varphi) \\ &= e^{-i\varphi}(\lambda + ix)\, P_n^{(\lambda)}(x - i;\varphi) - e^{i\varphi}(\lambda - ix)\, P_n^{(\lambda)}(x + i;\varphi). \end{aligned} \tag{2.3}$$

At same time, the Meixner-Pollaczek polynomials satisfy the following three-term recurrence relations

$$\begin{aligned} &2\left[x\sin\varphi+(n+\lambda)\cos\varphi\right]P_n^{(\lambda)}\left(x;\varphi\right) \\ &=(n+1)\,P_{n+1}^{(\lambda)}\left(x;\varphi\right)+(n+2\lambda-1)\,P_{n-1}^{(\lambda)}\left(x;\varphi\right), \end{aligned} \tag{2.4}$$

and the following forward and backward shift operators (or recurrence relations):

$$\left(e^{i\varphi}-e^{-i\varphi}\right)P_{n-1}^{\left(\lambda+\frac{1}{2}\right)}\left(x;\varphi\right)=P_n^{(\lambda)}\left(x+\tfrac{i}{2};\varphi\right)-P_n^{(\lambda)}\left(x-\tfrac{i}{2};\varphi\right), \tag{2.5}$$

$$\begin{aligned} &(n+1)P_{n+1}^{\left(\lambda-\frac{1}{2}\right)}\left(x;\varphi\right) \\ &=e^{i\varphi}\left(\lambda-\tfrac{1}{2}-ix\right)P_n^{(\lambda)}\left(x+\tfrac{i}{2};\varphi\right)+e^{-i\varphi}\left(\lambda-\tfrac{1}{2}+ix\right)P_n^{(\lambda)}\left(x-\tfrac{i}{2};\varphi\right). \end{aligned} \tag{2.6}$$

Taking into account definition of the formal derivatives $\delta f\left(x\right)=f\left(x+\frac{i}{2}\right)-f\left(x-\frac{i}{2}\right)$ and $\delta x=\left(x+\frac{i}{2}\right)-\left(x-\frac{i}{2}\right)=i$, one can reformulate (2.5) and (2.6) as follows:

$$\frac{\delta P_n^{(\lambda)}\left(x;\varphi\right)}{\delta x}=2\sin\varphi P_{n-1}^{\left(\lambda+\frac{1}{2}\right)}\left(x;\varphi\right), \tag{2.7}$$

$$\frac{\delta\left[\omega\left(x;\lambda,\varphi\right)P_n^{(\lambda)}\left(x;\varphi\right)\right]}{\delta x}=-(n+1)\omega\left(x;\lambda-\tfrac{1}{2},\varphi\right)P_{n+1}^{\left(\lambda-\frac{1}{2}\right)}\left(x;\varphi\right), \tag{2.8}$$

where

$$\omega\left(x;\lambda,\varphi\right)=\Gamma\left(\lambda+ix\right)\Gamma\left(\lambda-ix\right)e^{(2\pi-\varphi)x}.$$

One of the important properties of (2.5)-(2.8) is that under the following limit from Meixner-Pollaczek to the generalized Laguerre polynomials

$$\lim_{\varphi\to 0}P_n^{\left(\frac{1}{2}\alpha+\frac{1}{2}\right)}\left(-\frac{x}{2\varphi};\varphi\right)=L_n^{(\alpha)}\left(x\right), \tag{2.9}$$

they recover both forward and backward operators for the generalized Laguerre polynomials:

$$\frac{d}{dx}L_n^{(\alpha)}\left(x\right)=-L_{n-1}^{(\alpha+1)}\left(x\right), \tag{2.10}$$

$$\frac{d}{dx}\left[e^{-x}x^{\alpha}L_n^{(\alpha)}\left(x\right)\right]=(n+1)\,e^{-x}x^{\alpha-1}L_{n+1}^{(\alpha-1)}\left(x\right), \tag{2.11}$$

where, the generalized Laguerre polynomials are defined through the ${}_1F_1$ hypergeometric functions by the following way:

$$L_n^{(\alpha)}\left(x\right)=\frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{array}{c}-n\\ \alpha+1\end{array};x\right).$$

3 Pair of new recurrence relations for the Meixner-Pollaczek polynomials

Below, we are going to define pair of new recurrence relations/difference equations for the Meixner-Pollaczek polynomials, which differ from (2.4)-(2.6).

Proposition 3.1. There exists a pair of recurrence relations for the Meixner-Pollaczek polynomials of the following kind:

$$\begin{aligned} &2\left(x-i\lambda\right)\sin\varphi\cdot P_n^{\left(\lambda+\frac{1}{2}\right)}\left(x-\tfrac{i}{2};\varphi\right) \\ &= \left(n+1\right)P_{n+1}^{(\lambda)}\left(x;\varphi\right)-e^{i\varphi}\left(n+2\lambda\right)P_n^{(\lambda)}\left(x;\varphi\right), \end{aligned} \tag{3.1}$$

$$P_n^{(\lambda)}\left(x;\varphi\right)=P_n^{\left(\lambda+\frac{1}{2}\right)}\left(x-\tfrac{i}{2};\varphi\right)-e^{-i\varphi}P_{n-1}^{\left(\lambda+\frac{1}{2}\right)}\left(x-\tfrac{i}{2};\varphi\right). \tag{3.2}$$

Proof. We are going to prove the correctness of both recurrence relations (3.1) and (3.2) by performing direct straightforward computations and using known properties of the shifted factorials and ${}_2F_1$ hypergeometric functions.

First, we remind that, if one of the numerator parameters equals $-n$, where n is nonnegative integer, then the hypergeometric function ${}_2F_1$ is defined as finite sum of the following form:

$${}_2F_1\left(\begin{array}{c}-n,a\\ b\end{array};x\right)=\sum_{k=0}^{n}\frac{(-n)_k\,(a)_k}{(b)_k}\frac{x^k}{k!}, \tag{3.3}$$

where, $(a)_k$ is the Pochhammer symbol (or shifted factorial) defined as $(a)_0=1$ and $(a)_k=a(a+1)(a+2)\cdots(a+k-1)$, $k=1,2,3,\ldots$. Then, by using (3.3), one can rewrite (3.1) as follows:

$$\begin{aligned} &2\left(x-i\lambda\right)\left(2\lambda+1\right)_{n-1}e^{-i\varphi}\sin\varphi\sum_{k=0}^{n-1}\frac{\left(-n+1\right)_k\left(\lambda+ix+1\right)_k}{\left(2\lambda+1\right)_k\,k!}\left(1-e^{-2i\varphi}\right)^k \\ &= \left(2\lambda\right)_n\sum_{k=0}^{n}\frac{\left(-n\right)_k\left(\lambda+ix\right)_k}{\left(2\lambda\right)_k\,k!}\left(1-e^{-2i\varphi}\right)^k \\ &-\left(n+2\lambda-1\right)\left(2\lambda\right)_{n-1}\sum_{k=0}^{n-1}\frac{\left(-n+1\right)_k\left(\lambda+ix\right)_k}{\left(2\lambda\right)_k\,k!}\left(1-e^{-2i\varphi}\right)^k. \end{aligned} \tag{3.4}$$

Taking into account that

$$\sum_{k=0}^{n-1}\left(-n+1\right)_k A_k=\sum_{k=0}^{n}\left(-n+1\right)_k A_k, \tag{3.5}$$

then, (3.4) can be reformulated as follows:

$$
\begin{aligned}
2\left(x-i\lambda\right)\left(2\lambda+1\right)_{n-1}e^{-i\varphi}\sin\varphi\sum_{k=0}^{n}\frac{\left(-n+1\right)_k\left(\lambda+ix+1\right)_k}{\left(2\lambda+1\right)_k k!}\left(1-e^{-2i\varphi}\right)^k \\
=\left(2\lambda\right)_n\sum_{k=0}^{n}\frac{\left(-n\right)_k\left(\lambda+ix\right)_k}{\left(2\lambda\right)_k k!}\left(1-e^{-2i\varphi}\right)^k \\
-\left(n+2\lambda-1\right)\left(2\lambda\right)_{n-1}\sum_{k=0}^{n}\frac{\left(-n+1\right)_k\left(\lambda+ix\right)_k}{\left(2\lambda\right)_k k!}\left(1-e^{-2i\varphi}\right)^k.
\end{aligned}
\tag{3.6}
$$

Using the following transformation formulae for the Pochhammer symbols

$$
\begin{aligned}
\left(2\lambda+1\right)_{n-1}&=\frac{\left(2\lambda\right)_n}{2\lambda},\\
\left(2\lambda\right)_{n-1}&=\frac{\left(2\lambda\right)_n}{2\lambda+n-1},\\
\left(2\lambda+1\right)_k&=\frac{2\lambda+k}{2\lambda}\left(2\lambda\right)_k,\\
\left(-n+1\right)_k&=\frac{n-k}{n}\left(-n\right)_k,\\
\left(\lambda+ix+1\right)_k&=\frac{\lambda+ix+k}{\lambda+ix}\left(\lambda+ix\right)_k,
\end{aligned}
$$

and substituting them in (3.6), after some simplifications one finds that

$$
\sum_{k=0}^{n}\frac{\left(-n\right)_k\left(\lambda+ix\right)_k}{\left(2\lambda\right)_k k!}\left(1-e^{-2i\varphi}\right)^k\left[k+\frac{\left(n-k\right)\left(\lambda+ix\right)\left(1-e^{-2i\varphi}\right)}{2\lambda+k}\right]=0. \tag{3.7}
$$

Due to existence of k in one and $(n-k)$ in second summation term, one can terminate both summations as follows:

$$
\sum_{k=1}^{n}\frac{\left(-n\right)_k\left(\lambda+ix\right)_k}{\left(2\lambda\right)_k\left(k-1\right)!}\left(1-e^{-2i\varphi}\right)^k=\sum_{k=0}^{n-1}\frac{\left(-n\right)_{k+1}\left(\lambda+ix\right)_{k+1}}{\left(2\lambda\right)_{k+1}k!}\left(1-e^{-2i\varphi}\right)^{k+1}, \tag{3.8}
$$

correctness of which can be easily observed. Therefore, correctness of (3.8) also proves correctness of recurrence relation (3.1) for the Meixner-Pollaczek polynomials.

We prove (3.2) by using similar approach. Taking into account (3.3), one can rewrite (3.2) as follows:

$$
\begin{aligned}
2\lambda\sum_{k=0}^{n}\frac{\left(-n\right)_k\left(\lambda+ix\right)_k}{\left(2\lambda\right)_k k!}\left(1-e^{-2i\varphi}\right)^k=\left(2\lambda+n\right)\sum_{k=0}^{n}\frac{\left(-n\right)_k\left(\lambda+ix+1\right)_k}{\left(2\lambda+1\right)_k k!}\left(1-e^{-2i\varphi}\right)^k \\
-ne^{-2i\varphi}\sum_{k=0}^{n-1}\frac{\left(-n+1\right)_k\left(\lambda+ix+1\right)_k}{\left(2\lambda+1\right)_k k!}\left(1-e^{-2i\varphi}\right)^k.
\end{aligned}
\tag{3.9}
$$

Now, one applies (3.5) and extend second summation of right hand-side from $(n-1)$ to n as follows:

$$2\lambda\sum_{k=0}^{n}\frac{(-n)_k\,(\lambda+ix)_k}{(2\lambda)_k\,k!}\left(1-e^{-2i\varphi}\right)^k=(2\lambda+n)\sum_{k=0}^{n}\frac{(-n)_k\,(\lambda+ix+1)_k}{(2\lambda+1)_k\,k!}\left(1-e^{-2i\varphi}\right)^k \\ -ne^{-2i\varphi}\sum_{k=0}^{n}\frac{(-n+1)_k\,(\lambda+ix+1)_k}{(2\lambda+1)_k\,k!}\left(1-e^{-2i\varphi}\right)^k. \tag{3.10}$$

Further straightforward computations lead to the following summation:

$$\sum_{k=0}^{n}\frac{(-n)_k\,(\lambda+ix)_k}{(2\lambda)_k\,k!}\left(1-e^{-2i\varphi}\right)^k\left[\frac{k}{\lambda+ix}+\frac{(n-k)(\lambda+ix+k)\left(1-e^{-2i\varphi}\right)}{(\lambda+ix)(2\lambda+k)}\right]=0. \tag{3.11}$$

Again, due to existence of k in first and $(n-k)$ in second summation term, (3.11) can be reformulated as follows:

$$\sum_{k=1}^{n}\frac{(-n)_k\,(\lambda+ix+1)_{k-1}}{(2\lambda)_k\,(k-1)!}\left(1-e^{-2i\varphi}\right)^k=\sum_{k=0}^{n-1}(k-n)\frac{(-n)_k\,(\lambda+ix+1)_k}{(2\lambda)_{k+1}\,k!}\left(1-e^{-2i\varphi}\right)^{k+1}. \tag{3.12}$$

Taking into account that

$$(k-n)\,(-n)_k=(-n)_{k+1}\,,$$

as well as changing summation in right hand-side from 0 to $n-1$ as from 1 to n, one easily observes correctness of (3.12), which also proves (3.2). Q.E.D.

Now, taking into account existence of (3.1)-(3.2) pairs to known forward and backward operators (2.5) and (2.6) of the Meixner-Pollaczek polynomials, one can discover new 'hidden' properties of the Meixner-Pollaczek polynomials. One example is introduced in the proposition below.

Proposition 3.2. Subtraction of (2.5) from (3.2) leads to the following difference equation for the Meixner-Pollaczek polynomials:

$$\cosh\left(\tfrac{i}{2}\partial_x\right)P_n^{(\lambda)}(x;\varphi)=P_n^{\left(\lambda+\frac{1}{2}\right)}(x;\varphi)-\cos\varphi P_{n-1}^{\left(\lambda+\frac{1}{2}\right)}(x;\varphi)\,. \tag{3.13}$$

Proof. Correctness of (3.13) can be checked easily by trivial substraction of (2.5) from (3.2). One needs to highlight that, existence of (3.13) is already known and it is introduced in [26]. There, its correctness is proven by using known generating function for the Meixner-Pollaczek polynomials [3]. Q.E.D.

Main property of recurrence relations (3.1)-(3.2) is that, unlike forward and backward operators (2.5) and (2.6), limit from Meixner-Pollaczek polynomials to the generalized Laguerre polynomials does not reduce them to forward and backward operators for the generalized Laguerre polynomials (2.10)-(2.11), but, to the recurrence relations for the generalized Laguerre polynomials.

Proposition 3.3. Under the limit (2.9), recurrence relations (3.1) and (3.2) reduce to the following recurrence relations for the generalized Laguerre polynomials:

$$xL_n^{(\alpha+1)}(x) = (n+\alpha+1)\,L_n^{(\alpha)}(x) - (n+1)\,L_{n+1}^{(\alpha)}(x)\,, \tag{3.14}$$

$$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) - L_{n-1}^{(\alpha+1)}(x)\,. \tag{3.15}$$

Proof. As we noted above, proof of both recurrence relations is based on limit relation (2.9) between Meixner-Pollaczek and generalized Laguerre polynomials. Limit from (3.2) to (3.15) is trivial and for the limit from (3.1) to (3.14), one need to take into account that

$$\lim_{\varphi\to 0}(i\alpha + i - x/\varphi)\sin\varphi = (i\alpha + i - x/\varphi)\sum_{k=0}^{\infty}\frac{(-1)^k\varphi^{2k+1}}{(2k+1)!} = -x.$$

Q.E.D.

However, this is not only known way to connect these two orthogonal polynomials. In [27], new differentiation formula in terms of Meixner-Pollaczek polynomials is introduced to express generalized Laguerre polynomials in terms of Meixner-Pollaczek polynomials. Below, we introduce similar differentiation formula, which expresses Meixner-Pollaczek polynomials in terms of generalized Laguerre polynomials.

Proposition 3.4. For $\lambda > 0$ and $0 < \varphi < \pi$, Meixner-Pollaczek polynomials can be expressed via generalized Laguerre polynomials by the following differentiation formula:

$$(2\sin\varphi)^{ix}\,e^{x\left(\varphi-\frac{\pi}{2}\right)}\Gamma(\lambda+ix)\,P_n^{(\lambda)}(x;\varphi) = L_n^{(2\lambda-1)}\left(e^{i\partial_x}\right)(2\sin\varphi)^{ix}\,e^{x\left(\varphi-\frac{\pi}{2}\right)}\Gamma(\lambda+ix)\,. \tag{3.16}$$

Proof. We drop proof of this differentiation formula, but, one need note that its correctness can be proven easily by using Mellin transform between Meixner-Pollaczek and generalized Laguerre polynomials similar to the proof in [27].

Q.E.D.

Pair of recurrence relations (3.14)-(3.15) play very important role in the non-canonical approach to the quantum mechanics, which will be discussed briefly in final section.

4 Discussions and conclusion

Both Meixner-Pollaczek and generalized Laguerre polynomials play important role on explicit solution of number of problems of applied mathematics and quantum physics. The Meixner-Pollaczek polynomials, being expressed through ${}_2F_1$ hypergeometric functions and having two quasi-free parameters, can be considered as some generalization of the generalized Laguerre polynomials expressed by ${}_1F_1$ hypergeometric function and having one quasi-free parameter. One of the main relations between these two polynomials is the limit relation (2.9), that reduces Mexiner-Pollaczek polynomials to the generalized Laguerre polynomials, when, $\varphi \to 0$. Another connection is an application of the Mellin transform to them, under which, both polynomials are mapped onto each other [27]. In addition to the forward and backward operators for the generalized Laguerre polynomials (2.10)-(2.11), these polynomials are also explicit solutions of the pair of known recurrence relations (3.14)-(3.15). In his seminal paper [28], Wigner solved operator equations for the non-relativistic quantum harmonic oscillator $\left[\hat{H},\hat{p}\right]=i\hat{x}$ and $\left[\hat{H},\hat{x}\right]=-i\hat{p}$ under general commutation relation $[\hat{x},\hat{p}]\neq 0$. Then, by using (2.10)-(2.11), it was found that dynamical symmetry algebra of such an oscillator model is $osp\,(1|2)$ Lie superalgebra. Both Wigner quantum oscillator model and its dynamical symmetry algebra $osp\,(1|2)$ recover known Hermite quantum oscillator model and Heisenberg-Weil algebra, when, quasi-free parameter a of the generalized Laguerre polynomials becomes equal to 1/2. One can show that, when, $a=1/2$, then both recurrence relations for the generalized Laguerre polynomials (3.14)-(3.15) reduce the known normalized recurrence relation for the Hermite polynomials [3]. In this paper, we aimed to look for pair of the recurrence relations for the Meixner-Polaczek polynomials with behavior similar to (3.14)-(3.15). We proposed that these recurrence relations have the form (3.1)-(3.2). Next, we found that subtraction of (2.5) from (3.2) leads to the difference equation for the Meixner-Pollaczek polynomials (3.13). Then, it is shown that, under the limit relation (2.9), both recurrence relations recover (3.14)-(3.15). However, we highlighted that this is not only the approach to connect Meixner-Pollaczek and Laguerre polynomials. Therefore, we expressed Meixner-Pollaczek polynomials with parameters $\lambda>0$ and $0<\varphi<\pi$ via generalized Laguerre polynomials by differentiation formula (3.16). Further, these recurrence relations can be used to find some generalization of $osp\,(1|2)$ Lie superalgebra and its realization as quantum oscillator model in terms of the Meixner-Pollaczek polynomials.

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References

[1] L.D. Landau and E.M. Lifshitz, Quantum Mechanics: Non-Relativistic Theory, 689 pages. Butterworth-Heinemann, Oxford (1981)

[2] M. Moshinsky and Y.F. Smirnov, The Harmonic Oscillator in Modern Physics, 414 pages. Harwood Academic Publishers, Amsterdam (1996)

[3] R. Koekoek, P.A. Lesky and R.F. Swarttouw, Hypergeometric orthogonal polynomials and their q-analogues, 578 pages. Springer-Verlag, Berlin (2010)

[4] Y. Ohnuki and S. Kamefuchi, Quantum Field Theory and Parastatistics, 490 pages. Springer-Verlag, Berlin-Heidelberg (1982)

[5] A. Erdelyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, Higher Transcendental Functions, Vol. 2, McGraw-Hill (1953)

[6] S. Khan, A.A. Al-Gonah and G. Yasmin, Some properties of Hermite based Appell matrix polynomials, *Tbilisi Math. J.* **10** (2017), 121–131

[7] N. Raza, S. Khan and M. Ali, Properties of certain new special polynomials associated with Sheffer sequences, *Tbilisi Math. J.* **9** (2016), 245–270

[8] F.G. Abdullayev and G.A. Abdullayeva, The "algebraic zero" condition for orthogonal polynomials over a contour in the weighted Lebesgue spaces, *Proc. IMM ANAS* **42** (2016), 154–173

[9] N.M. Atakishiev, R.M. Mir-Kasimov and Sh.M. Nagiev, Quasipotential models of a relativistic oscillator, *Theor. Math. Phys.* **44** (1980), 592–603

[10] A.U. Klimyk, The $su(1,1)$-models of quantum oscillator, *Ukr. J. Phys.* **51** (2006), 1019–1027

[11] N.M. Atakishiyev and S.K. Suslov, The Hahn and Meixner polynomials of an imaginary argument and some of their applications, *J. Phys. A: Math. Gen.* **18** (1985), 1583–1596

[12] N.M. Atakishiyev, E.I. Jafarov, S.M. Nagiyev and K.B. Wolf, Meixner oscillators, *Rev. Mex. Fis.* **44** (1998), 235–244

[13] E.I. Jafarov, J. Van der Jeugt, Discrete series representations for $sl(2|1)$, Meixner polynomials and oscillator models, *J. Phys. A: Math. Theor.* **45** (2012), 485201

[14] N.M. Atakishiyev, A.M. Jafarova and E.I. Jafarov, Meixner Polynomials and Representations of the 3D Lorentz group $SO(2,1)$, *Comm. Math. Anal.* **17** (2014), 14–23

[15] J. Meixner, Orthogonale Polynomsysteme Mit Einer Besonderen Gestalt Der Erzeugenden Funktion, *J. London Math. Soc.* **s19** (1934), 6-13

[16] M.V. Tratnik, Multivariable Meixner, Krawtchouk, and Meixner-Pollaczek polynomials, *J. Math. Phys.* **30** (1989), 2740–2749

[17] K. Mimachi, Barnes-Type Integral and the Meixner-Pollaczek Polynomials, *Lett. Math. Phys.* **48** (1999), 365–373

[18] Y. Chen and M.E.H. Ismail, Asymptotics of extreme zeros of the Meixner-Pollaczek polynomials, *J. Comp. App. Math.* **82** (1997), 59–78

[19] X. Li and R. Wong, On the asymptotics of the Meixner-Pollaczek polynomials and their zeros, *Constr. Approx.* **17** (2001), 59–90

[20] C. Ferreira, J.L. Lpez and E.P. Sinusa, Asymptotic relations between the Hahn-type polynomials and Meixner-Pollaczek, Jacobi, Meixner and Krawtchouk polynomials, *J. Comp. App. Math.* **217** (2008), 88–109

[21] S. Kanas and A. Tatarczak, Generalized Meixner-Pollaczek polynomials, *Adv. Differ. Equ.* **2013** (2013), 131

[22] A. Kuznetsov, Integral representations for the Dirichlet L-functions and their expansions in Meixner-Pollaczek polynomials and rising factorials, *Integral Transform. Spec. Funct.* **18** (2007), 827–835

[23] A. Kuznetsov, Expansion of the Riemann Ξ Function in Meixner-Pollaczek Polynomials, *Canad. Math. Bull.* **51** (2008), 561–569

[24] Z. Mouayn, A new class of coherent states with Meixner-Pollaczek polynomials for the Goldman-Krivchenkov Hamiltonian, *J. Phys. A: Math. Theor.* **43** (2010), 295201

[25] D.D. Tcheutia, P. Njionou Sadjang, W. Koepf and M. Foupouagnigni, Divided-difference equation, inversion, connection, multiplication and linearization formulae of the continuous Hahn and the Meixner-Pollaczek polynomials, *Ramanujan J.* **44** (2017), 1–24

[26] Sh.M. Nagiev, Dynamical symmetry group of the relativistic Coulomb problem in the quasipotential approach, *Theor. Math. Phys.* **80** (1989), 697–702

[27] T. Koornwinder, Meixner-Pollaczek polynomials and the Heisenberg algebra, *J. Math. Phys.* **30** (1989), 767–769

[28] E.P. Wigner, Do the equations of motion determine the quantum mechanical commutation relations?, *Phys. Rev.* **77** (1950), 711–712