General Solution and Stability of Quattuorvigintic Functional Equation in Matrix Paranormed Spaces

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Abstract

In this current work, we acquire the general solution and Hyers-Ulam stability for a new form of Quattuorvigintic functional equation in Matrix Paranormed Space by using the direct and fixed point methods.

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1 Introduction

In 1940, an interesting talk presented by S. M. Ulam [12] triggered the study of stability problems for various functional equations. He raised a question concerning the stability of homomorphism. In the following year, 1941, D. H. Hyers [4] was able to give a partial solution to Ulam's question. The result of Hyers was generalized by Aoki [1] for additive mappings. In 1978, Th. M. Rassias [10] succeeded in extending the result of Hyers theorem by weakening the condition for the Cauchy difference. The stability phenomenon that was presented by Th. M. Rassias is called the generalized Hyers-Ulam stability. This concept actually means that if one is studying a Hyers-Ulam stable system, one need not have to reach the exact solution, which usually is quite difficult or time consuming. This is quite useful in many applications for example optimization, numerical analysis, biology, life sciences, economics etc., where finding the exact solution is quite difficult.

In 1994, a generalization of the Rassias theorem was obtained by Gavruta [3] by replacing the unbounded Cauchy difference by a general control function. A further generalization of the Hyers-Ulam stability for a large class of mapping was obtained by Isac and Th. M. Rassias [5]. They also presented some applications in non-linear analysis, especially in fixed point theory. This terminology may also be applied to the cases of other functional equations [2, 9]. Also, the generalized Hyers-Ulam stability of functional equations and inequalities in matrix paranormed spaces has been studied by number of authors [6, 8, 11].

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In this paper, we introduce the following new functional equation

$$\begin{aligned}
&\mathcal{D}\varphi(v,w) = \varphi(v+12w) - 24\varphi(v+11w) + 276\varphi(v+10w) - 2024\varphi(v+9w) \\
&+ 10626\varphi(v+8w) - 42504\varphi(v+7w) + 134596\varphi(v+6w) - 346104\varphi(v+5w) \\
&+ 735471\varphi(v+4w) - 1307504\varphi(v+3w) + 1961256\varphi(v+2w) - 2496144\varphi(v+w) \\
&+ 2704156\varphi(v) - 2496144\varphi(v-w) + 1961256\varphi(v-2w) - 1307504\varphi(v-3w) \\
&+ 735471\varphi(v-4w) + 134596\varphi(v-6w) - 42504\varphi(v-7w) + 10626\varphi(v-8w) \\
&- 2024\varphi(v-9w) + 276\varphi(v-10w) - 346104\varphi(v-5w) - 24\varphi(v-11w) \\
&+ \varphi(v-12w) - 1.124000728 \times 10^{21}\varphi(w). \qquad (1.1)
\end{aligned}$$

is said to be Quattuorvigintic functional equation since the function $f(x) = cx^{24}$ is its solution. In this paper, we determine the general solution of the functional equation (1.1) and we also prove the Ulam-Hyers stability of the functional equation (1.1) in matrix paranormed spaces by using fixed point approach and direct method.

2 General Solution of Quattuorvigintic Functional Equation in (1.1)

Theorem 2.1. Let $\mathcal{F}$ and $\mathcal{G}$ be the vector spaces. If $\varphi : \mathcal{F} \to \mathcal{G}$ be a function (1.1) for all $v, w \in \mathcal{F}$, then φ is Quattuorvigintic.

Proof. Letting $v = 0$ and $w = 0$ in (1.1), we obtain that $\varphi(0) = 0$. Substituting (v, w) with (v, v) and $(v, -v)$ in (1.1), respectively, and subtracting two resulting equations, we can arrive at $\varphi(-v) = \varphi(v)$, that is to say, φ is an even function.
Replacing (v, w) by $(12v, v)$ and $(0, 2v)$ respectively, and subtracting the two resulting equations, we arrive at

$$\begin{aligned}
&24\varphi(23v) - 300\varphi(22v) + 2024\varphi(21v) - 10350\varphi(20v) + 42504\varphi(19v) \\
&- 136620\varphi(18v) + 346104\varphi(17v) - 724845\varphi(16v) + 1307504\varphi(15v) \\
&- 2003760\varphi(14v) + 2496144\varphi(13v) - 2569560\varphi(12v) + 2496144\varphi(11v) \\
&- 2307360\varphi(10v) + 1307504\varphi(9v) + 346104\varphi(7v) - 1442100\varphi(6v) \\
&+ 42504\varphi(5v) + 1950630\varphi(4v) + 2024\varphi(3v) - \frac{24!}{2}\varphi(2v) + 24!\varphi(v) = 0 \qquad (2.1)
\end{aligned}$$

$\forall v \in \mathcal{F}$. Replacing $v = 11v$ and $w = v$ in (1.1) and multiplying the resulting equation by 24, then (2.1) gives

$$\begin{aligned}
&276\varphi(22v) - 4600\varphi(21v) + 38226\varphi(20v) - 212520\varphi(19v) + 883476\varphi(18v) \\
&- 2884200\varphi(17v) + 7581651\varphi(16v) - 16343800\varphi(15v) + 29376336\varphi(14v) \\
&- 44574000\varphi(13v) + 57337896\varphi(12v) - 62403600\varphi(11v) + 57600096\varphi(10v) \\
&- 45762640\varphi(9v) - 31380096\varphi(8v) - 17305200\varphi(7v) + 6864396\varphi(6v) \\
&- 3187800\varphi(5v) + 2970726\varphi(4v) - 253000\varphi(3v) + \frac{24!}{2}\varphi(2v) + 24!(25)\varphi(v) = 0 \qquad (2.2)
\end{aligned}$$

$\forall v \in \mathcal{F}$. Replacing $v = 10v$ and $w = v$ in (1.1) and multiplying the resulting equation by 276, then (2.2) gives

$$\begin{aligned}&2024\varphi(21v) - 37950\varphi(20v) + 346104\varphi(19v) - 2049300\varphi(18v) + 8846904\varphi(17v)\\&- 29566845\varphi(16v) + 79180904\varphi(15v) - 173613660\varphi(14v) + 316297104\varphi(13v)\\&- 483968760\varphi(12v) + 626532144\varphi(11v) - 688746960\varphi(10v) + 643173104\varphi(9v)\\&- 509926560\varphi(8v) + 343565904\varphi(7v) - 196125600\varphi(6v) + 92336904\varphi(5v)\\&- 34177770\varphi(4v) + 11478104\varphi(3v) - \frac{24!}{2}\varphi(2v) - 24!(301)\varphi(v) = 0\end{aligned} \tag{2.3}$$

$\forall v \in \mathcal{F}$. Replacing $v = 9v$ and $w = v$ in (1.1) and multiplying the resulting equation by 2024, then (2.3) gives

$$\begin{aligned}&10626\varphi(20v) - 212520\varphi(19v) + 2047276\varphi(18v) - 12660120\varphi(17v) + 56461251\varphi(16v)\\&- 193241400\varphi(15v) + 526900836\varphi(14v) - 1172296200\varphi(13v) + 2162419336\varphi(12v)\\&- 3343050000\varphi(11v) + 4363448496\varphi(10v) - 4830038640\varphi(9v) + 4542268896\varphi(8v)\\&- 3626016240\varphi(7v) + 2450262496\varphi(6v) - 1396256400\varphi(5v) + 666336726\varphi(4v)\\&- 260946224\varphi(3v) + \frac{24!}{2}\varphi(2v) - 24!(2325)\varphi(v) = 0\end{aligned} \tag{2.4}$$

$\forall v \in \mathcal{F}$. Replacing $v = 8v$ and $w = v$ in (1.1) and multiplying the resulting equation by 10626, then (2.4) gives

$$\begin{aligned}&42504\varphi(19v) - 885500\varphi(18v) + 8846904\varphi(17v) - 56450625\varphi(16v)\\&+ 258406104\varphi(15v) - 903316260\varphi(14v) + 2505404904\varphi(13v) - 5652695510\varphi(12v)\\&+ 10550487500\varphi(11v) - 16476857760\varphi(10v) + 21693987500\varphi(9v) - 24192092760\varphi(8v)\\&+ 22898009900\varphi(7v) - 18390043760\varphi(6v) + 12497281100\varphi(5v) - 7148788746\varphi(4v)\\&+ 3417009904\varphi(3v) - \frac{24!}{2}\varphi(2v) - 24!(12951)\varphi(v) = 0\end{aligned} \tag{2.5}$$

$\forall v \in \mathcal{F}$. Replacing $v = 7v$ and $w = v$ in (1.1) and multiplying the resulting equation by 42504, then (2.5) gives

$$\begin{aligned}&134596\varphi(18v) - 2884200\varphi(17v) + 29577471\varphi(16v) - 193241400\varphi(15v)\\&+ 903273756\varphi(14v) - 3215463480\varphi(13v) + 9058108906\varphi(12v) - 20709971880\varphi(11v)\\&+ 39097292260\varphi(10v) - 61667237520\varphi(9v) + 81904011820\varphi(8v) - 92039436720\varphi(7v)\\&+ 87706060820\varphi(6v) - 70863986420\varphi(5v) + 48426381370\varphi(4v)\\&- 27855180580\varphi(3v) - \frac{24!}{2}\varphi(2v) - 24!(55455)\varphi(v) = 0\end{aligned} \tag{2.6}$$

$\forall v \in \mathcal{F}$. Replacing $v = 6v$ and $w = v$ in (1.1) and multiplying the resulting equation by 134596, then (2.6) gives

$$\begin{aligned}
&346104\varphi(17v) - 7571025\varphi(16v) + 79180904\varphi(15v) - 526943340\varphi(14v)\\
&+ 2505404904\varphi(13v) - 9057974310\varphi(12v) + 25874242100\varphi(11v) - 59894162460\varphi(10v)\\
&+ 114317570900\varphi(9v) - 182073200800\varphi(8v) + 243931561100\varphi(7v)\\
&- 276262654800\varphi(6v) + 265110241700\varphi(5v) - 215587979700\varphi(4v)\\
&+ 148402050100\varphi(3v) - \frac{24!}{2}\varphi(2v) + 24!(190051)\varphi(v) = 0
\end{aligned} \tag{2.7}$$

$\forall v \in \mathcal{F}$. Replacing $v = 5v$ and $w = v$ in (1.1) and multiplying the resulting equation by 346104, then (2.7) gives

$$\begin{aligned}
&735471\varphi(16v) - 16343800\varphi(15v) + 173571156\varphi(14v) - 1172296200\varphi(13v)\\
&+ 5652830106\varphi(12v) - 20709971880\varphi(11v) + 59893816360\varphi(10v) - 140231884100\varphi(9v)\\
&+ 270459163700\varphi(8v) - 434867331600\varphi(7v) + 587671074700\varphi(6v) - 670904491200\varphi(5v)\\
&+ 649037957800\varphi(4v) - 534074197600\varphi(3v) - \frac{24!}{2}\varphi(2v) + 24!(536155)\varphi(v) = 0
\end{aligned} \tag{2.8}$$

$\forall v \in \mathcal{F}$. Replacing $v = 4v$ and $w = v$ in (1.1) and multiplying the resulting equation by 735471, then (2.8) gives

$$\begin{aligned}
&1307504\varphi(15v) - 29418840\varphi(14v) + 316297104\varphi(13v) - 2162284740\varphi(12v)\\
&+ 10550487500\varphi(11v) - 39097638360\varphi(10v) - 114317570900\varphi(9v) + 270459163700\varphi(8v)\\
&+ 526781594100\varphi(7v) - 854978826900\varphi(6v) + 1166425626000\varphi(5v) - 1347605475000\varphi(4v)\\
&+ 1333027786000\varphi(3v) - \frac{24!}{2}\varphi(2v) + 24!(1271626)\varphi(v) = 0
\end{aligned} \tag{2.9}$$

$\forall v \in \mathcal{F}$. Replacing $v = 3v$ and $w = v$ in (1.1) and multiplying the resulting equation by 1307504, then (2.9) gives

$$\begin{aligned}
&1961256\varphi(14v) - 44574000\varphi(13v) + 484103356\varphi(12v) - 3343050000\varphi(11v)\\
&+ 16476511660\varphi(10v) - 61668545020\varphi(9v) + 182104580900\varphi(8v)\\
&- 435210551400\varphi(7v) + 857234271300\varphi(6v) - 1411817977000\varphi(5v) + 1971686940000\varphi(4v)\\
&- 2378651809000\varphi(3v) + \frac{24!}{2}\varphi(2v) - 24!(2579130)\varphi(v) = 0
\end{aligned} \tag{2.10}$$

$\forall v \in \mathcal{F}$. Replacing $v = 2v$ and $w = v$ in (1.1) and multiplying the resulting equation by 1961256, then (2.10) gives

$$\begin{aligned}
&2496144\varphi(13v) - 57203300\varphi(12v) + 626532144\varphi(11v) - 4365755856\varphi(10v)\\
&+ 21739750140\varphi(9v) - 82413938380\varphi(8v) + 247557577300\varphi(7v) - 606052946600\varphi(6v)\\
&+ 1235893313000\varphi(5v) - 2138815370000\varphi(4v) + 3195724134000\varphi(3v)\\
&- \frac{24!}{2}\varphi(2v) + 24!(4540386)\varphi(v) = 0
\end{aligned} \tag{2.11}$$

$\forall v \in \mathcal{F}$. Replacing $v = v$ and $w = v$ in (1.1) and multiplying the resulting equation by 2496144, then (2.11) gives

$$\begin{aligned} &2704156\varphi(12v) - 64899744\varphi(11v) + 746347056\varphi(10v) - 5473211744\varphi(9v) \\ &+ 28734361660\varphi(8v) - 114937446600\varphi(7v) + 363968581000\varphi(6v) \\ &- 935919208200\varphi(5v) + 1988828317000\varphi(4v) - 3535694787000\varphi(3v) \\ &+ \frac{24!}{2}\varphi(2v) - 24!(7036530)\varphi(v) = 0 \end{aligned} \tag{2.12}$$

$\forall v \in \mathcal{F}$. Replacing $v = 0$ and $w = v$ in (1.1), one gets

$$\begin{aligned} &\varphi(12v) - 24\varphi(11v) + 276\varphi(10v) - 2024\varphi(9v) + 10626\varphi(8v) \\ &- 42504\varphi(7v) + 134596\varphi(6v) - 346104\varphi(5v) + 735471\varphi(4v) \\ &- 1307504\varphi(3v) + 1961256\varphi(2v) - 3.102242009 \times 10^{23}\varphi(v) = 0 \end{aligned} \tag{2.13}$$

$\forall v \in \mathcal{F}$. On simplification we arrive at

$$\varphi(2) = (2^{24})\varphi(u) \tag{2.14}$$

Q.E.D.

3 Hyers-Ulam stability of (1.1) in Matrix paranormed Spaces: A Fixed Point Approach

Throughout this section, let $(\mathcal{F}, \{\|.\|_n\})$ be a Matrix Banach Spaces and $(\mathcal{G}, \{P_n(.)\})$ be a Matrix Frechet Spaces. Note that $P(2v) \leq 2P(v) \qquad \forall v \in \mathcal{G}$.
For a mapping $\varphi : \mathcal{F} \to \mathcal{G}$, define
$\mathcal{D}\varphi : \mathcal{F}^2 \to \mathcal{G}$ and $\mathcal{D}\varphi_n : M_n(\mathcal{F}^2) \to M_n(\mathcal{G})$ by (1.1) and

$$\begin{aligned} \mathcal{D}\varphi_n([x_{ij}], [y_{ij}]) = {}& \varphi([x_{ij}] + 12[y_{ij}]) - 24\varphi([x_{ij}] + 11[y_{ij}]) \\ &+ 276\varphi_n([x_{ij}] + 10[y_{ij}]) - 2024\varphi_n([x_{ij}] + 9[y_{ij}]) + 10626\varphi_n([x_{ij}] + 8[y_{ij}]) \\ &- 42504\varphi_n([x_{ij}] + 7[y_{ij}]) + 134596\varphi_n([x_{ij}] + 6[y_{ij}]) - 346104\varphi_n([x_{ij}] \\ &+ 5[y_{ij}]) + 735471\varphi_n([x_{ij}] + 4[y_{ij}]) - 1307504\varphi_n([x_{ij}] + 3[y_{ij}]) \\ &+ 1961256\varphi_n([x_{ij}] + 2[y_{ij}]) - 2496144\varphi_n([x_{ij}] + [y_{ij}]) + 2704156\varphi_n([x_{ij}]) \\ &- 2496144\varphi_n([x_{ij}] - [y_{ij}]) + 1961256\varphi_n([x_{ij}] - 2[y_{ij}]) - 1307504\varphi_n([x_{ij}] \\ &- 3[y_{ij}]) + 735471\varphi_n([x_{ij}] - 4[y_{ij}]) + 134596\varphi_n([x_{ij}] - 6[y_{ij}]) \\ &- 42504\varphi_n([x_{ij}] - 7[y_{ij}]) + 10626\varphi_n([x_{ij}] - 8[y_{ij}]) - 2024\varphi_n([x_{ij}] \\ &- 9[y_{ij}]) + 276\varphi_n([x_{ij}] - 10[y_{ij}]) - 346104\varphi_n([x_{ij}] - 5[y_{ij}]) \\ &- 24\varphi_n([x_{ij}] - 11[y_{ij}]) + \varphi_n([x_{ij}] - 12[y_{ij}]) - 1.124000728 \times 10^{21}\varphi_n([y_{ij}]) \end{aligned}$$

for all $v, w \in \mathcal{F}$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(\mathcal{F})$.

Theorem 3.1. Let $\xi : \mathcal{F}^2 \to [0, \infty)$ be a mapping such that there exists an $\rho < 1$ with

$$\xi(v, w) \leq 24\rho\xi\left(\frac{v}{2}, \frac{w}{2}\right) \tag{3.1}$$

Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an mapping satisfying the inequality

$$\|\mathcal{D}\varphi_n\left([x_{ij}],[y_{ij}]\right)\|_n \leq \sum_{i,j=1}^{n} \xi(x_{ij},y_{ij}) \tag{3.2}$$

for all $x=[x_{ij}], y=[y_{ij}] \in M_n(F)$. Then there exists a unique Quattuorvigintic mapping $\mathcal{Q}_{24} : \mathcal{F} \to \mathcal{G}$ such that

$$\|\varphi_n([x_{ij}]) - \mathcal{Q}_{24n}([x_{ij}])\|_n \leq \sum_{i,j=1}^{n} \frac{1}{2^{24}-\rho}\Psi(x_{ij}), \tag{3.3}$$

where

$$\begin{aligned}\Psi(x_{ij}) = \frac{2}{24!}\Bigg[&\frac{1}{2}\xi(0,2x_{ij}) + \xi(12x_{ij},x_{ij}) + 24\xi(11x_{ij},x_{ij}) + 276\xi(10x_{ij},x_{ij})\\ &+2024\xi(9x_{ij},x_{ij}) + 10626\xi(8x_{ij},x_{ij}) + 42504\xi(7x_{ij},x_{ij}) + 134596\xi(6x_{ij},x_{ij})\\ &+346104\xi(5x_{ij},x_{ij}) + 735471\xi(4x_{ij},x_{ij}) + 1307504\xi(3x_{ij},x_{ij})\\ &+1961256\xi(2x_{ij},x_{ij}) + 2496144\xi(x_{ij},x_{ij}) + 1352078\xi(0,x_{ij})\Big]\end{aligned}$$

for all $x=[x_{ij}] \in M_n(\mathcal{F})$.

Proof. Letting $n=1$ in (3.2). Then (3.2) is equivalent to
$\|\varphi(v+12w) - 24\varphi(v+11w) + 276\varphi(v+10w) - 2024\varphi(v+9w) + 10626\varphi(v+8w)$
$-42504\varphi(v+7w) + 134596\varphi(v+6w) - 346104\varphi(v+5w) + 735471\varphi(v+4w)$
$-1307504\varphi(v+3w) + 1961256\varphi(v+2w) - 2496144\varphi(v+w) + 2704156\varphi(v) - 2496144\varphi(v-w)$
$+1961256\varphi(v-2w) - 1307504\varphi(v-3w) + 735471\varphi(v-4w) + 134596\varphi(v-6w)$
$-2024\varphi(v-9w) + 276\varphi(v-10w) - 346104\varphi(v-5w) - 24\varphi(v-11w)$

$$-42504\varphi(v-7w) + 10626\varphi(v-8w) + \varphi(v-12w) - 1.124000728\times 10^{21}\varphi(w)\big\| \leq \xi(v,w) \tag{3.4}$$

Applying the same procedure of Theorem 2.1 and using (2.14), we get

$$\left\|\varphi(v) - \frac{1}{2^{24}}\varphi(2v)\right\| \leq \frac{1}{2^{24}}\Psi(v) \tag{3.5}$$

where $\Psi(v) = \frac{2}{24!}\left[\frac{1}{2}\xi(0,2v) + \xi(12v,v) + 24\xi(11v,v) + 276\xi(10v,v) + 2024\xi(9v,v)\right.$
$+10626\xi(8v,v) + 42504\xi(7v,v) + 134596\xi(6v,v) + 346104\xi(5v,v) + 735471\xi(4v,v)$
$\left.+1307504\xi(3v,v) + 1961256\xi(2v,v) + 2496144\xi(v,v) + 1352078\xi(0,v)\right]$.

We Consider the set $\mathcal{M} = \{\varphi : \mathcal{F} \to \mathcal{G}\}$ and introduce the generalized metric d on $\mathcal{M}$ as follows:

$$d(l,m) = \inf\{\mu \in [0,\infty] \mid \|l(v) - m(v)\| \leq \mu\Psi(v) \quad \forall \quad v \in \mathcal{F}\}$$

It is easy to check that $(\mathcal{M}, d)$ is a complete generalized Metric([7], Lemma 2.1).
Define the mapping $\mathcal{P} : \mathcal{M} \to \mathcal{M}$ by $\mathcal{P}\varphi(v) = \frac{1}{2^{24}}\varphi(2v) \qquad \forall\varphi \in \mathcal{M}$, and $\forall v \in \mathcal{F}$.
Let $l, m \in \mathcal{M}$ and ν be an arbitary constant with $d(l,m) = \nu$.

Then $\|l(v) - m(v)\| \le \nu\Psi(v)$ for all $v \in \mathcal{F}$.
Utilizing (3.1), we find that

$$\|\mathcal{P}l(v) - \mathcal{P}m(v)\| = \left\| \frac{1}{2^{24}} l(2v) - \frac{1}{2^{24}} m(2v) \right\| \le \rho\nu\Psi(v) \qquad \forall v \in \mathcal{F}.$$

It follows from (3.5) that $d(\varphi, \mathcal{P}\varphi) \le \frac{1}{2^{24}}$.
Applying the Theorem 2.2 in [2], there existence a mapping $\mathcal{Q}_{24} : \mathcal{F} \to \mathcal{G}$ such that

$$\mathcal{Q}_{24}(2v) = 2^{24}\mathcal{Q}_{24}(v) \qquad \forall v \in \mathcal{F}.$$

Moreover, we have $d\left(\mathcal{P}^n\varphi, \mathcal{Q}_{24}\right) \to 0$, which implies $\mathcal{Q}_{24}(v) = \lim_{n\to\infty} \frac{\varphi(2^n v)}{2^{24n}}$.
Also, $d(\varphi, \mathcal{Q}_{24}) \le \frac{1}{1-\rho} d(\mathcal{P}\varphi, \varphi)$ implies the inequality

$$\|\varphi(v) - \mathcal{Q}_{24}(v)\| \le \frac{1}{2^{24} - \rho}\Psi(v) \tag{3.6}$$

It follows from (3.1) and (3.2) that
$\|\mathcal{D}\mathcal{Q}_{24}(v, w)\| = \lim_{n\to\infty} \frac{1}{2^{24n}} \|\mathcal{D}\varphi\left(2^n v, 2^n w\right)\| \le \lim_{n\to\infty} \frac{1}{2^{24n}} = 0$
Therefore the mapping $\mathcal{Q}_{24} : \mathcal{F} \to \mathcal{G}$ is Quattuorvigintic Mapping. By Using Lemma 2.2 in [6] and (3.6), we reach at (3.3). Thus $\mathcal{Q}_{24n} : \mathcal{F} \to \mathcal{G}$ is unique Quattuorvigintic mapping. Q.E.D.

Corollary 3.2. Let κ and θ be a non-negative real numbers with $\kappa < 24$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be an mapping satisfying

$$\|D\varphi_n\left([x_{ij}], [y_{ij}]\right)\|_n \le \sum_{i,j=1}^{n} \theta\left(P(x_{ij})^\kappa + P(y_{ij})^\kappa\right) \tag{3.7}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then there exists unique Quattuorvigintic mapping $\mathcal{Q}_{24n} : \mathcal{F} \to \mathcal{G}$ such that

$$\|\varphi_n[x_{ij}] - \mathcal{Q}_{24n}[x_{ij}]\|_n \le \sum_{i,j=1}^{n} \frac{\gamma_\kappa}{(2^{24} - 2^\kappa)} P(x_{ij})^\kappa \tag{3.8}$$

where $\gamma_\kappa = 2\frac{\theta}{24!}\left(12^\kappa + 24(11)^\kappa + 276(10)^\kappa + 2024(9)^\kappa + 10626(8)^\kappa + 42504(7)^\kappa\right.$
$\left. +134596(6)^\kappa + 346104(5)^\kappa + 735471(4)^\kappa + 1307504(3)^\kappa + \frac{3922513}{2}(2)^\kappa + 10884752\right)$

Proof. The proof follows from Theorem 3.1 by taking $\xi(v, w) = \theta\left(P(v)^\kappa + P(w)^\kappa\right)$ for all $v, w \in \mathcal{F}$. Then we can choose $\rho = 2^{\kappa-24}$ and we get the desired result. Q.E.D.

Theorem 3.3. Let $\xi : \mathcal{G}^2 \to [0, \infty)$ be a function such that there exists $\rho < 1$ with

$$\xi(v, w) \le \frac{\rho}{24}\xi\left(2v, 2w\right) \tag{3.9}$$

Suppose that $\varphi : \mathcal{G} \to \mathcal{F}$ be an mapping satisfying the inequality

$$P\left(\mathcal{D}\varphi_n\left([x_{ij}],[y_{ij}]\right)\right) \le \sum_{i,j=1}^{n} \xi(x_{ij},y_{ij}) \tag{3.10}$$

for all $x=[x_{ij}], y=[y_{ij}] \in M_n(\mathcal{G})$. Then there exists a unique Quattuorvigintic mapping $\mathcal{Q}_{24} : \mathcal{G} \to \mathcal{F}$ such that

$$P_n\left(\varphi_n([x_{ij}]) - \mathcal{Q}_{24n}([x_{ij}])\right) \le \sum_{i,j=1}^{n} \frac{\rho}{\rho - 2^{24}} \Psi(x_{ij}). \tag{3.11}$$

for all $x=[x_{ij}] \in M_n(\mathcal{G})$.

Proof. Taking $n=1$ in (3.10). Then (3.10) is equivalent to
$P\left(\varphi(v+12w) - 24\varphi(v+11w) + 276\varphi(v+10w) - 2024\varphi(v+9w)10626\varphi(v+8w) - 42504\varphi(v+7w)\right.$
$+134596\varphi(v+6w) - 346104\varphi(v+5w) + 735471\varphi(v+4w) - 1307504\varphi(v+3w)$
$+1961256\varphi(v+2w) - 2496144\varphi(v+w) + 2704156\varphi(v) - 2496144\varphi(v-w) + 134596\varphi(v-6w)$
$+1961256\varphi(v-2w) - 1307504\varphi(v-3w) + 735471\varphi(v-4w) - 42504\varphi(v-7w) + 10626\varphi(v-8w)$
$-2024\varphi(v-9w) + 276\varphi(v-10w) - 346104\varphi(v-5w) - 24\varphi(v-11w)$

$$\left.+\varphi(v-12w) - 1.124000728\times 10^{21}\varphi(w)\right) \le \xi(v,w) \tag{3.12}$$

Applying the same procedure of Theorem 2.1 and using (2.14), we get

$$P\left(\varphi(v) - 2^{24}\varphi(\frac{v}{2})\right) \le \frac{\rho}{2^{24}}\Psi(v) \tag{3.13}$$

where $\Psi(v) = \dfrac{2}{24!}\left[\dfrac{1}{2}\xi(0,2v) + \xi(12v,v) + 24\xi(11v,v) + 276\xi(10v,v) + 2024\xi(9v,v)\right.$
$+10626\xi(8v,v) + 42504\xi(7v,v) + 134596\xi(6v,v) + 346104\xi(5v,v) + 735471\xi(4v,v)$
$+1307504\xi(3v,v) + 1961256\xi(2v,v) + 2496144\xi(v,v) + 1352078\xi(0,v)]$
Also, $d(\varphi,\mathcal{Q}_{24}) \le \dfrac{1}{1-\rho} d(\mathcal{P}\varphi,\varphi)$ implies the inequality

$$P\left(\varphi(v) - \mathcal{Q}_{24}(v)\right) \le \frac{\rho}{\rho - 2^{24}}\Psi(v) \tag{3.14}$$

By using Lemma 2.1 in [6] and (3.14), we get the result (3.11). Q.E.D.

Corollary 3.4. Let κ, θ be a non-negative real number with $\kappa > 24$. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be an mapping satisfying

$$P_n\left(D\varphi_n\left([x_{ij}],[y_{ij}]\right)\right) \le \sum_{i,j=1}^{n} \theta\left(\|x_{ij}\|^{\kappa} + \|y_{ij}\|^{\kappa}\right) \tag{3.15}$$

for all $x=[x_{ij}], y=[y_{ij}] \in M_n(G)$. Then there exists unique Quattuorvigintic mapping $\mathcal{Q}_{24n} : \mathcal{F} \to \mathcal{G}$ such that

$$P_n\left(\varphi_n[x_{ij}] - \mathcal{Q}_{24n}[x_{ij}]\right) \le \sum_{i,j=1}^{n} \frac{\gamma_\kappa}{(2^k - 2^{24})}\|x_{ij}\|^{\kappa} \tag{3.16}$$

where $\gamma_\kappa = 2\dfrac{\theta}{24!}\left(12^\kappa + 24(11)^\kappa + 276(10)^\kappa + 2024(9)^\kappa + 10626(8)^\kappa + 42504(7)^\kappa\right.$
$\left. +134596(6)^\kappa + 346104(5)^\kappa + 735471(4)^\kappa + 1307504(3)^\kappa + \dfrac{3922513}{2}(2)^\kappa + 10884752\right)$

Proof. The proof follows from Theorem 3.3 by taking $\xi(v,w) = \theta\left(\|v\|^\kappa + \|w\|^\kappa\right)$ for all $v, w \in \mathcal{G}$. Then we can choose $\rho = 2^{24-\kappa}$ and we get the desired result. Q.E.D.

4 Hyers-Ulam stability of (1.1) in Matrix paranormed Spaces: Direct Method

Theorem 4.1. Let κ, θ be non-negative real numbers with $\kappa > 24$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a mapping such that

$$P_n\left(\mathcal{D}\varphi_n\left([x_{ij}], [y_{ij}]\right)\right) \le \sum_{i,j=1}^{n} \theta\left(\|x_{ij}\|^\kappa + \|y_{ij}\|^\kappa\right) \tag{4.1}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(\mathcal{F})$. Then there exists a unique Quattuorvigintic mapping $\mathcal{Q}_{24} : \mathcal{F} \to \mathcal{G}$ such that

$$P_n\left(\varphi_n([x_{ij}]) - \mathcal{Q}_{24n}([x_{ij}])\right) \le \sum_{i,j=1}^{n} \frac{2\theta}{2^\kappa - 2^{24}} \|x_{ij}\|^\kappa \delta_{24} \tag{4.2}$$

where $\delta_{24} = \dfrac{2}{24!}\left[12^\kappa + 24(11)^\kappa + 276(10)^\kappa + 2024(9)^\kappa + 10626(8)^\kappa + 42504(7)^\kappa\right.$
$\left. +134596(6)^\kappa + 346104(5)^\kappa + 735471(4)^\kappa + 1307504(3)^\kappa + \dfrac{3922513}{2}(2)^\kappa + 10884752\right]$
for all $x = [x_{ij}] \in M_n(\mathcal{F})$.

Proof. Let $n = 1$ in (4.1). Then (4.1) is equivalent to

$$P\left(\mathcal{D}(v,w)\right) \le \theta\left(\|v\|^\kappa + \|w\|^\kappa\right) \tag{4.3}$$

Applying the same procedure of Theorem 2.1 and using (2.14), we get

$$P\left(\varphi(v) - 2^{24}\varphi(\frac{v}{2})\right) \le \frac{\theta}{2^\kappa} \|v\|^\kappa \delta_{24}. \tag{4.4}$$

where $\delta_{24} = \dfrac{2}{24!}\left[12^\kappa + 24(11)^\kappa + 276(10)^\kappa + 2024(9)^\kappa + 10626(8)^\kappa + 42504(7)^\kappa\right.$
$\left. +134596(6)^\kappa + 346104(5)^\kappa + 735471(4)^\kappa + 1307504(3)^\kappa + \dfrac{3922513}{2}(2)^\kappa + 10884752\right].$
One can easily prove that

$$P\left(2^{24p}\varphi(\frac{v}{2^p}) - 2^{24q}\varphi(\frac{v}{2^q})\right) \le \frac{1}{2^\kappa} \sum_{l=p}^{q-1} \frac{2^{24l}}{2^{\kappa l}} \theta \|v\|^\kappa \delta_{24} \tag{4.5}$$

for all $v, w \in \mathcal{F}$ and non-negative integers p, q with $p < q$. It follows from (4.5) that the sequence $\left\{2^{24l}\varphi(\frac{v}{2^l})\right\}$ is Cauchy for all $v \in \mathcal{F}$. Since $\mathcal{G}$ is Complete, the sequence $\left\{2^{24l}\varphi(\frac{v}{2^l})\right\}$ converges. So, one can define the mapping $\mathcal{Q}_{24} : \mathcal{F} \to \mathcal{G}$ by

$$\mathcal{Q}_{24} = \lim_{n\to\infty} \left\{2^{24l}\varphi(\frac{v}{2^l})\right\}$$

$\forall v \in \mathcal{F}$. Moreover, letting $p=0$ and passing the limit $q \to \infty$ in (4.5), we reach that

$$P\left(\varphi(v) - \mathcal{Q}_{24}\right) \leq \frac{2\theta}{2^{\kappa} - 2^{24}} \|v\|^{\kappa} \delta_{24} \tag{4.6}$$

for all $v \in \mathcal{F}$. It follows from (4.3) that

$$\begin{aligned} P\left(\mathcal{G}\mathcal{Q}_{24}(v,w)\right) &\leq \lim_{n\to\infty} P\left(\mathcal{G}\varphi(2^{24l}v, 2^{24l}w)\right) \\ &\leq \frac{2^{24l}}{2^{l\kappa}}\theta\left(\|v\|^{\kappa} + \|w\|^{\kappa}\right) = 0 \end{aligned}$$

Now let $\mathcal{T} : \mathcal{F} \to \mathcal{G}$ be another Quattuorvigintic mapping satisfying (4.2). Then we have

$$\begin{aligned} P\left(\mathcal{Q}_{24}(v) - \mathcal{T}(v)\right) &= P\left(24^{l}\left(\mathcal{Q}_{24}(\frac{v}{2^{l}}) - \mathcal{T}(\frac{v}{2^{l}})\right)\right) \\ &\leq \frac{24^{l}}{2^{l\kappa}}\theta\left(\|v\|^{\kappa} + \|w\|^{\kappa}\right) \end{aligned}$$

which tends to zero as $l \to \infty$ for all $v \in \mathcal{F}$. So we can conclude that $\mathcal{Q}_{24}(v) = \mathcal{T}(v)$ for all $v \in \mathcal{F}$. This proves the uniqueness of $\mathcal{Q}_{24}$. By Lemma 2.1 in [6] and (4.6), we reach at (4.2). Q.E.D.

Theorem 4.2. Let κ, θ be non-negative real numbers with $\kappa < 24$. Let $\varphi : \mathcal{G} \to \mathcal{F}$ be a mapping such that

$$\|\mathcal{D}\varphi_n\left([x_{ij}], [y_{ij}]\right)\|_n \leq \sum_{i,j=1}^{n} \theta\left(P(x_{ij})^{\kappa} + P(y_{ij})^{\kappa}\right) \tag{4.7}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(\mathcal{G})$. Then there exists a unique Quattuorvigintic mapping $\mathcal{Q}_{24} : \mathcal{G} \to \mathcal{F}$ such that

$$\|\varphi_n([x_{ij}]) - \mathcal{Q}_{24n}([x_{ij}])\| \leq \sum_{i,j=1}^{\infty} \frac{2\theta}{2^{24} - 2^{\kappa}} P(x_{ij})^{\kappa} \delta_{24} \tag{4.8}$$

where $\delta_{24} = \dfrac{2}{24!(2^{24})}\Big[12^{\kappa} + 24(11)^{\kappa} + 276(10)^{\kappa} + 2024(9)^{\kappa} + 10626(8)^{\kappa} + 42504(7)^{\kappa} + 134596(6)^{\kappa}$

$+346104(5)^{\kappa} + 735471(4)^{\kappa} + 1307504(3)^{\kappa} + \dfrac{3922513}{2}(2)^{\kappa} + 10884752\Big]$ for all $x = [x_{ij}] \in M_n\left(\mathcal{G}\right)$.

Proof. Let $n = 1$ in (4.7). Then (4.7) is equivalent to

$$\|\mathcal{D}\varphi(v,w)\| \leq \theta\left(P(v)^{\kappa} + P(w)^{\kappa}\right) \tag{4.9}$$

for all $v, w \in \mathcal{G}$.
Applying the same procedure of Theorem 2.1 and using (2.14), we get

$$\left\|\frac{1}{16777216}\varphi(2v) - \varphi(v)\right\| \leq \theta P(v)^{\kappa}\delta_{24} \tag{4.10}$$

where $\delta_{24} = \dfrac{2}{24!(2^{24})}\left[12^{\kappa} + 24(11)^{\kappa} + 276(10)^{\kappa} + 2024(9)^{\kappa} + 10626(8)^{\kappa} + 42504(7)^{\kappa} + 134596(6)^{\kappa} + 346104(5)^{\kappa} + 735471(4)^{\kappa} + 1307504(3)^{\kappa} + \dfrac{3922513}{2}(2)^{\kappa} + 10884752\right]$.

One can easily show that

$$\left\|\frac{1}{2^{24}}\varphi(2^{p}v) - \frac{1}{2^{24}}\varphi(2^{q}v)\right\| \leq \sum_{l=p}^{q-1}\frac{2^{l\kappa}}{2^{24l}}\theta P(x_{ij})^{\kappa}\delta_{24} \tag{4.11}$$

The rest of the proof is similar to the proof of Theorem 4.1. Q.E.D.

5 Counter-examples

The following examples illustrates the fact that functional equation (1.1) is not stable for $\kappa = 24$ in corollaries 3.2 and 3.4.

Example 5.1. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$\beta(v) = \begin{cases} \varepsilon v^{24}, & |v| < 1 \\ \varepsilon, & otherwise \end{cases}$$

where $\varepsilon > 0$ is a constant, and define a function $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(v) = \sum_{n=0}^{\infty}\frac{\beta(2^{n})v}{2^{24n}}$$

for all $v \in \mathbb{R}$. Then φ satisfies the inequality

$$\|\mathcal{D}\varphi(v,w)\| \leq \frac{6.204484017 \times 10^{23}}{16777215}(16777216)^{2}\varepsilon\left(|v|^{24} + |w|^{24}\right) \tag{5.1}$$

for all $v, w \in \mathbb{R}$. Then there does not exists a Quattuorvigintic mapping $\mathcal{Q}_{24} : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ such that

$$|\varphi(v) - \mathcal{Q}_{24}(v)| \leq \lambda\,|v|^{24} \quad \forall v \in \mathbb{R}. \tag{5.2}$$

Proof. It is easy to see that φ is bounded by $\dfrac{16777216}{16777215}\varepsilon$ on $\mathbb{R}$.

If $|v|^{24} + |w|^{24} = 0$, then (5.1) is trivial. If $|v|^{24} + |w|^{24} \geq \dfrac{1}{2^{24}}$, then there exists a non-negative integer k such that

$$\frac{1}{2^{24(k+1)}} \leq |v|^{24} + |w|^{24} \leq \frac{1}{2^{24k}} \tag{5.3}$$

Hence From definition of φ and (5.3), we arrive that

$$|\mathcal{D}\varphi(v,w)| \leq \frac{(6.204484017 \times 10^{23})}{16777215}(16777216)^{2}\varepsilon\left(|v|^{24} + |w|^{24}\right).$$

Therefore, φ satisfies (5.1) for all $v, w \in \mathbb{R}$. Now, we claim that functional equation (1.1) is not stable for $\kappa = 24$ in Corollary (3.2) and (3.4).
Suppose on the contrary that there exists a Quattuorvigintic mapping $\mathcal{Q}_{24} : \mathbb{R} \to \mathbb{R}$ and a constant $\lambda > 0$ satisfying (5.2).
Then there exists a constant $c \in \mathbb{R}$ such that $\mathcal{Q}_{24}(v) = cv^{24}$ for any $v \in \mathbb{R}$. Thus we obtain the following inequality

$$|\varphi(v)| \leq (\lambda + |c|) \, |v|^{24} \tag{5.4}$$

Let $m \in \mathbb{N}$ with $m\varepsilon > \lambda + |c|$. If $v \in (0, \dfrac{1}{2^{m-1}})$, then $2^n v \in (0,1)$ for all $n = 0, 1, 2...m-1$ and for this case we get

$$\varphi(v) = \sum_{n=0}^{\infty} \frac{\beta(2^n)v}{2^{24n}} \geq \sum_{n=0}^{m-1} \frac{\varepsilon(2^n v)^{24}}{2^{24n}} = m\varepsilon v^{24} > (\lambda + |c|) \, |v|^{24}$$

which is a contradiction to (5.4). Therefore the Quattuorvigintic functional equation (1.1) is not stabel for $\kappa = 24$. Q.E.D.

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