Harmonic numbers operational matrix for solving fifth-order two point boundary value problems

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Abstract

The principal purpose of this paper is to present and implement two numerical algorithms for solving linear and nonlinear fifth-order two point boundary value problems. These algorithms are developed via establishing a new Galerkin operational matrix of derivatives. The nonzero elements of the derived operational matrix are expressed explicitly in terms of the well-known harmonic numbers. The key idea for the two proposed numerical algorithms is based on converting the linear or nonlinear fifth-order two BVPs into systems of linear or nonlinear algebraic equations by employing Petrov-Galerkin or collocation spectral methods. Numerical tests are presented aiming to ascertain the high efficiency and accuracy of the two proposed algorithms.

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1 Introduction

Many practical problems in various fields of applied science are described by linear or nonlinear boundary value problems (BVPs), therefore the studies concerning these types of equations have gained a great importance. Many authors are interested in the numerical treatment of the linear or nonlinear high-order BVPs. For example, in the series of papers [4,9,10], the authors handled $2n$th-order and $(2n+1)$th-order linear BVPs via utilizing the Galerkin and Petrov-Galerkin methods in various situations. In fact, the proposed algorithms in these papers are based on constructing suitable combinations of orthogonal polynomials, and after that applying a suitable spectral method. It is convenient to apply Galerkin method for even-order BVPs, while it is convenient to apply Petrov-Galerkin method for odd-order BVPs.

The investigation of odd-order BVPs is of interest. In particular, the third-order BVPs are of particular interest since they contain a type of operators which appears in some important types of partial differential equations like the Kortweg-de Vries equation. Moreover, seven-order BVPs are of recent interest. For some applications of seven-order BVPs, see, the recent paper of

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Siddiqi [24]. Also, fifth-order BVPs are of interest. The behavior of an induction motor with two rotor circuits is modeled by a fifth-order differential equation (see, [21, 25]). Moreover, fifth-order BVPs arise in the mathematical modelling of viscoelastic flows (see, [8], [13]). There are many numerical techniques used for obtaining solutions of fifth-order BVPs, among these techniques, the quartic B-spline collocation method [18], sextic spline method [16, 23], decomposition method [27], variational iteration method [29], non-polynomial sextic spline method [14], and quartic B-spline method [17].

Spectral methods have gained a popularity over the the past four decades. The appeal of these methods for applications in many fields such as computational fluid dynamics has expanded greatly. These methods are considered a class of techniques employed in applied mathematics in order to obtain numerical solutions for a large number of ordinary, partial, and fractional differential equations. The numerical solutions obtained by their applications are expressed in terms of certain basis functions, which may be expressed in terms of various orthogonal polynomials. Spectral methods have an advantage that they take on a global approach while finite element methods use a local approach. The fascinating merit of spectral methods is the high accuracy of them, the so-called convergence of infinite order (see, [7]). One can consult the book of Shizgal for some applications of spectral methods in chemistry and physics [22]. Other applications can be found in [15].

The approach of employing the operational matrices of derivatives is an effective tool for handling BVPs, and in particular, nonlinear BVPs. There are several articles in this direction. For example, the authors in [11] employed the tau operational matrix of derivatives of Chebyshev polynomials of the second kind for numerically solving the singular Lane-Emden equations. Recently, Abd-Elhameed in [1, 3] has developed two novel harmonic numbers operational matrices of derivatives. Moreover, he utilized these derived operational matrices for solving respectively, linear and nonlinear fourth- and sixth-order BVPs. The approach of utilizing operational matrices of derivatives is followed in several articles, see for example, [2, 19].

Our main idea through this paper is to establish a new operational matrix of derivatives expressed in terms of harmonic numbers to numerically solve the linear and nonlinear fifth-order BVPs via the application of the two spectral methods namely, Petrov-Galerkin or collocation methods.

The rest of the article is as follows. Section 2 is concerned with establishing a Galerkin operational matrix of derivatives of a certain combination of shifted Legendre polynomials. Section 3 is interested in implementing and presenting two numerical algorithms for solving linear and nonlinear fifth-order BVPs based on the application of the two numerical methods namely, harmonic Petrov-Galerkin method (HPGM) for linear problems and the harmonic collocation method (HCM) for nonlinear problems. In Section 4, we study the convergence of the suggested expansion and the error analysis. Some numerical experiments including some discussions and comparisons are given in Section 5 aiming to illustrate the applicability and efficiency of the two suggested algorithms. Finally, some conclusions are reported in Section 6.

2 Harmonic numbers operational matrix of derivatives

We select the following set of basis functions

$$\psi_k(t) = (t-a)^3 (b-t)^2 \, L_k^*(t), \quad k = 0, 1, 2, \ldots, \tag{2.1}$$

where $L_k^*(t)$ are defined on $[a, b]$ by

$$L_k^*(t) = L_k\left(\frac{2t-a-b}{b-a}\right), \quad k = 0, 1, \ldots,$$

$L_k(t)$ are the standard Legendre polynomials. It should be noted here that the polynomials $\{\psi_k(t) : k = 0, 1, 2, \ldots\}$ are linearly independent and orthogonal in the sense that

$$\int_a^b w(t)\psi_k(t)\ \psi_j(t)\, dx = \begin{cases} 0, & k \neq j, \\ \dfrac{b-a}{2\,k+1}, & k = j. \end{cases}$$

where $w(t) = \dfrac{1}{(t-a)^6\,(b-t)^4}$, Let us denote $H_w^r(I)(r = 0, 1, 2, \ldots)$, by the weighted Sobolev spaces, whose inner products and norms are, respectively, denoted by $(.,.)_{r,w}$, and $\|.\|_{r,w}$ (see, [7]).

Particularly, for homogeneous boundary conditions, we define the space

$$X^{(0)} = \{y \in H_w^3(I) : y(a) = y(b) = y'(a) = y'(b) = y''(a) = 0\}. \tag{2.2}$$

Let P_M denote the space of all polynomials of degree less than or equal to M. Setting $X_M = X^{(0)} \cap P_M$. It is clear that

$$X_M = \text{span}\{\psi_0(t), \psi_1(t), \ldots, \psi_M(t)\}.$$

Now, we expand any function $y(t) \in X^{(0)}$ in the following form

$$y(t) = \sum_{i=0}^{\infty} c_i\, \psi_i(t), \tag{2.3}$$

$$c_i = \frac{2i+1}{b-a} \int_a^b w(t)\, y(t)\, \psi_i(t)\ dt, \tag{2.4}$$

and, also, we approximate it by the first $(M + 1)$ terms

$$y(t) \simeq y_M(t) = \sum_{i=0}^{M} c_i\, \psi_i(t) = \boldsymbol{C}^T\, \boldsymbol{\Psi}(t), \tag{2.5}$$

where

$$\boldsymbol{C}^T = [c_0, c_1, \ldots, c_M], \quad \boldsymbol{\Psi}(t) = [\psi_0(t), \psi_1(t), \ldots, \psi_M(t)]^T. \tag{2.6}$$

Now, we intend to state and prove an important theorem concerning the first derivative of the basis functions $\psi_i(t)$. With the aid of this formula, the nonzero elements of the introduced operational matrix of derivatives can be explicitly given in terms of harmonic numbers . This operational matrix will be employed for establishing our results hereafter.

Theorem 1. If the polynomials $\psi_i(t)$ are selected as in (2.1), then the following relation holds for all $i \geq 1$,

$$\begin{aligned}\frac{d\psi_i(t)}{dt} =& \frac{2}{b-a} \sum_{\substack{j=0 \\ (i+j)\ even}}^{i-2} (2\,j+1)\,(H_j - H_i)\,\psi_j(t) + \\ & \frac{2}{b-a} \sum_{\substack{j=0 \\ (i+j)\ odd}}^{i-1} (2\,j+1)\ (1+5\,H_i - 5\,H_j)\ \psi_j(t) + \nu_i(t),\end{aligned} \tag{2.7}$$

where $\nu_i(t)$ are given by the formula

$$\nu_i(t) = (t-a)^2\,(b-t)\ \begin{cases} 2\,a+3\,b-5\,t, & i \text{ even}, \\ 2\,a-3\,b+t, & i \text{ odd}. \end{cases} \tag{2.8}$$

Proof. We will prove relation (2.7) on $[-1,1]$, hence the relation on $[a,b]$ can be easily deduced. We prove the following relation by induction

$$\frac{d\varphi_i(t)}{dt} = \sum_{\substack{j=0 \\ (i+j)\ even}}^{i-2} (2\,j+1)\,(H_j - H_i)\,\varphi_j(t) + \sum_{\substack{j=0 \\ (i+j)\ odd}}^{i-1} (2\,j+1)\ (1+5\,H_i - 5\,H_j)\ \varphi_j(t) + \theta_i(t), \tag{2.9}$$

where $\varphi_i(t) = (t+1)^3\,(1-t)^2\,L_i(t)$ and $\theta_i(t)$ are given by the formula

$$\theta_i(t) = (t+1)^2\,(1-t)\ \begin{cases} 1-5\,t, & i \text{ even}, \\ t-5, & i \text{ odd}. \end{cases} \tag{2.10}$$

For $i=1$, each of the two sides of (2.9) equals $(t-1)(t+1)^2(2t-1)(3t+1)$. Now, assume the validity of relation (2.9) for $(i-2)$ and $(i-1)$, and we must show its validity for i. It is clear that the polynomials $\varphi_i(t)$ have the same Legendre's recurrence relation, that is

$$\varphi_i(t) = \frac{2i-1}{i}\,t\,\varphi_{i-1}(t) - \frac{(i-1)}{i}\,\varphi_{i-2}(t), \quad i \geq 2, \tag{2.11}$$

and hence, by differentiation, we have

$$\frac{d\varphi_i(t)}{dt} = \frac{2i-1}{i}\,t\,\frac{d\varphi_{i-1}(t)}{dt} + \frac{2i-1}{i}\,\varphi_{i-1}(t) - \frac{(i-1)}{i}\,\frac{d\varphi_{i-2}(t)}{dt}. \tag{2.12}$$

If we apply the induction hypothesis to $\dfrac{d\varphi_{i-1}(t)}{dt}$ and $\dfrac{d\varphi_{i-2}(t)}{dt}$ in (2.12), then we get

$$\begin{aligned}\frac{d\varphi_i(t)}{dt} =& \frac{2i-1}{i}\,t\,\theta_{i-1}(t)+\frac{1-i}{i}\,\theta_{i-2}(t)+\frac{2i-1}{i}\,\varphi_{i-1}(t)+\frac{2i-1}{i}\sum_{\substack{j=0\\(i+j)\,even}}^{i-2} B_{i-1,j}\,t\,\varphi_j(t)\\ &+\frac{2i-1}{i}\sum_{\substack{j=0\\(i+j)\,odd}}^{i-3} A_{i-1,j}\,t\,\varphi_j(t)+\frac{1-i}{i}\sum_{\substack{j=0\\(i+j)\,even}}^{i-4} A_{i-2,j}\,\varphi_j(t)+\frac{1-i}{i}\sum_{\substack{j=0\\(i+j)\,odd}}^{i-3} B_{i-2,j}\,\varphi_j(t),\end{aligned}\tag{2.13}$$

where

$$A_{i,j} =(2j+1)(H_j-H_i),\quad B_{i,j}=(2j+1)\,\left(5\,(H_i-H_j)+1\right).$$

If we substitute the recurrence relation (2.11) written in the form

$$t\,\varphi_j(t)=\frac{j+1}{2\,j+1}\,\varphi_{j+1}(t)+\frac{j}{2\,j+1}\,\varphi_{j-1}(t),$$

into relation (2.13), and make use of the recurrence relation satisfied by the harmonic numbers H_i:

$$i\,H_i-(2i-1)\,H_{i-1}+(i-1)\,H_{i-2}=0,\tag{2.14}$$

then after proceeding some manipulations, we get

$$\begin{aligned}\frac{d\varphi_i(t)}{dt} =&\\ &\sum_{\substack{j=0\\(i+j)\,even}}^{i-2}(2\,j+1)\,(H_j-H_i)+\sum_{\substack{j=0\\(i+j)\,odd}}^{i-1}(2\,j+1)\,\left(1+5\,H_i-5\,H_j\right)\,\varphi_j(t)\\ &+\frac{2i-1}{i}\,t\,\theta_{i-1}(t)+\frac{1-i}{i}\,\theta_{i-2}(t)+\left[\left(-2+\tfrac{1}{i}\right)(-1+H_{i-1})+\left(1-\tfrac{1}{i}\right)\,H_{i-2}+H_i\right]\mu_{i+1}\,\varphi_0(t)\\ &+\left[\left(2-\tfrac{1}{i}\right)(-4+5\,H_{i-1})+\left(\tfrac{1}{i}-1\right)\,(1+5\,H_{i-2})-(1+5\,H_i)\right]\mu_i\,\varphi_0(t).\end{aligned}\tag{2.15}$$

where

$$\mu_i=\begin{cases}1, & i \text{ odd},\\ 0, & i \text{ even}.\end{cases}$$

If the recurrence relation (2.14) is applied again, then it can be easily shown that

$$\begin{aligned}&\left(-2+\tfrac{1}{i}\right)(-1+H_{i-1})+\left(1-\tfrac{1}{i}\right)\,H_{i-2}+H_i=2-\frac{1}{i},\\ &\left(2-\tfrac{1}{i}\right)(-4+5\,H_{i-1})+\left(\tfrac{1}{i}-1\right)\,(1+5\,H_{i-2})-(1+5\,H_i)=5\left(-2+\tfrac{1}{i}\right),\end{aligned}$$

and hence, relation (2.15) can be simplified to the following equivalent one

$$\begin{aligned}\frac{d\varphi_i(t)}{dt} =& \sum_{\substack{j=0\\(i+j)\,even}}^{i-2}(2\,j+1)\,(H_j-H_i)+\sum_{\substack{j=0\\(i+j)\,odd}}^{i-1}(2\,j+1)\,\left(1+5\,H_i-5\,H_j\right)\,\varphi_j(t)\\ &+\left(2-\tfrac{1}{i}\right)\,\mu_{i+1}\,\varphi_0(t)+5(-2+\tfrac{1}{i})\,\mu_i\,\varphi_0(t)+\frac{2i-1}{i}\,t\,\theta_{i-1}(t)+\frac{1-i}{i}\,\theta_{i-2}(t).\end{aligned}\tag{2.16}$$

Finally, if we note the relation

$$\left(2-\tfrac{1}{i}\right)\,\mu_{i+1}\,\varphi_0(t)+5(-2+\tfrac{1}{i})\,\mu_i\,\varphi_0(t)+\frac{2i-1}{i}\,t\,\theta_{i-1}(t)+\frac{1-i}{i}\,\theta_{i-2}(t)=\theta_i(t),$$

then relation (2.9) is proved. Q.E.D.

Now, if t in (2.9) is replaced by $\dfrac{2t-a-b}{b-a}$, then it is not difficult to show that the following relation is obtained:

$$\begin{aligned}\frac{d\psi_i(t)}{dt}=&\frac{2}{b-a}\sum_{\substack{j=0\\(i+j)\,even}}^{i-2}(2\,j+1)\,(H_j-H_i)\,\psi_j(t)+\\&\frac{2}{b-a}\sum_{\substack{j=0\\(i+j)\,odd}}^{i-1}(2\,j+1)\ (1+5\,H_i-5\,H_j)\ \psi_j(t)+\nu_i(t),\end{aligned}$$

and

$$\nu_i(t)=(t-a)^2\,(b-t)\ \begin{cases}2\,a+3\,b-5\,t, & i\ \text{even},\\[2ex] 2\,a-3\,b+t, & i\ \text{odd}.\end{cases}\tag{2.17}$$

This completes the proof of Theorem 1.
It should be noted here that the result of Theorem 1 enables one to express the first derivative of the vector $\mathbf{\Psi}(t)$ defined in (2.6) in the following matrix form:

$$\frac{d\mathbf{\Psi}(t)}{dt}=P\,\mathbf{\Psi}(t)+\boldsymbol{\nu}(t),\tag{2.18}$$

where $\boldsymbol{\nu}(t)=(\nu_0(t),\nu_1(t),\ldots,\nu_M(t))^T$, and $P=\left(p_{ij}\right)_{0\leqslant i,j\leqslant M}$, is the $(M+1)\times(M+1)$ operational matrix. From relation (2.7), the nonzero entries of P can be explicitly given by:

$$p_{ij}=\begin{cases}\dfrac{2}{b-a}\,(2j+1)\ (H_j-H_i)\,, & i>j,\ (i+j)\ \text{even},\\ \dfrac{2}{b-a}\,(2j+1)\ (5\,(H_i-H_j)+1)\,, & i>j,\ (i+j)\ \text{odd},\\ 0, & \textit{otherwise}.\end{cases}\tag{2.19}$$

As an example, if $M=5$, then one has

$$P=\frac{2}{b-a}\begin{pmatrix}0&0&0&0&0&0\\ 6&0&0&0&0&0\\ -\frac{3}{2}&\frac{21}{2}&0&0&0&0\\ \frac{61}{6}&-\frac{5}{2}&\frac{40}{3}&0&0&0\\ -\frac{25}{12}&\frac{77}{4}&-\frac{35}{12}&\frac{63}{4}&0&0\\ \frac{149}{12}&-\frac{77}{20}&\frac{295}{12}&-\frac{63}{20}&18&0\end{pmatrix}.$$

Corollary 1. The qth-derivative of the vector $\boldsymbol{\Psi}(t)$ is given by

$$\frac{d^q \boldsymbol{\Psi}(t)}{dt^q} = P^q \, \boldsymbol{\Psi}(t) + \sum_{m=0}^{q-1} P^{q-m-1} \, \frac{d^m}{dt^m} \, \boldsymbol{\nu}(t). \tag{2.20}$$

3 Spectral solutions of fifth-order two point BVPs

The main purpose of this section is twofold:

- Developing a Pertov-Galerkin algorithm for solving linear two point BVPs of fifth-order.
- Developing a collocation algorithm for solving the nonlinear two point BVPs of fifth-order.

3.1 Solution of linear two point BVPs of fifth-order

This section is interested in deriving spectral solutions for linear fifth-order BVPs by employing the operational matrix of derivatives that introduced in Section 2. Now, consider the following linear differential equation

$$y^{(5)}(t) + \sum_{r=0}^{4} A_r(t) \, y^{(r)}(t) = g(t), \quad t \in (a, b), \tag{3.1}$$

with the homogeneous boundary conditions

$$y(a) = y(b) = y'(a) = y'(b) = y''(a) = 0. \tag{3.2}$$

If $y(t)$ is approximated as in (2.5), then with the aid of Corollary 1, the derivative $y^{(r)}(t)$, $1 \le r \le 5$, can be approximated as:

$$y^{(r)}(t) \simeq \boldsymbol{C}^T \left(P^r \, \boldsymbol{\Psi}(t) + \sum_{m=0}^{r-1} P^{r-m-1} \, \frac{d^m}{dt^m} \, \boldsymbol{\nu}(t) \right). \tag{3.3}$$

Making use of relations (2.5) and (3.3), the residual $R(t)$ of Eq. (3.1) can be written as

$$\begin{aligned} R(t) =& \boldsymbol{C}^T \left(P^5 \, \boldsymbol{\Psi}(t) + \sum_{m=0}^{4} P^{4-m} \, \frac{d^m}{dt^m} \, \boldsymbol{\nu}(t) \right) \\ &+ \sum_{r=1}^{4} A_r(t) \, \boldsymbol{C}^T \left(P^r \, \boldsymbol{\Psi}(t) + \sum_{m=0}^{q-1} P^{r-m-1} \, \frac{d^m}{dt^m} \, \boldsymbol{\nu}(t) \right) \\ &+ A_0(t) \, \boldsymbol{C}^T \, \boldsymbol{\Psi}(t) - g(t). \end{aligned} \tag{3.4}$$

To obtain our proposed numerical solution for (3.1), we apply the Petrov-Galerkin method (see, [7]) to obtain the following $(M+1)$ linear equations in the unknowns coefficients, c_i, namely

$$\int_a^b R(t) \; L_i^*(t) \, dx = 0, \quad 0 \le i \le M. \tag{3.5}$$

From Eq. (3.5), a system consists of $(M+1)$ linear equations can be generated. This system of equations can be solved for the unknown components of the vector $\boldsymbol{C}$ via employing any suitable solver. The required numerical solution $y_M(t)$ given in (2.5) can be found.

Remark 1. It should be noted that problem (3.1), governed by the nonhomogeneous boundary conditions

$$y(a)=\xi_1,\, y(b)=\xi_2,\, y'(a)=\xi_3,\, y'(b)=\xi_4,\, y''(a)=\xi_5, \tag{3.6}$$

can be transformed into a problem similar to (3.1)-(3.2), (see, [12]).

3.2 Handling nonlinear fifth-order BVPs

Assume the following nonlinear problem

$$y^{(5)}(t)=F\left(t,y(t),y^{(1)}(t),y^{(2)}(t),y^{(3)}(t),y^{(4)}(t)\right), \tag{3.7}$$

with by the homogenous boundary conditions

$$y(a)=y(b)=y'(a)=y'(b)=y''(a)=0. \tag{3.8}$$

If $y(t)$ is approximated as in (2.5) and if the derivatives $y^{(r)}(t)$, $1\le r\le 5$, are approximated as in (3.3), then we get

$$\begin{aligned}&\boldsymbol{C}^T\,\left(P^5\boldsymbol{\Psi}(t)+\boldsymbol{\nu}_5(t)\right)\approx\\&F\Bigg(t,\boldsymbol{C}^T\,\boldsymbol{\Psi}(t),\boldsymbol{C}^T(P\,\boldsymbol{\Psi}(t)+\boldsymbol{\nu}(t)),\boldsymbol{C}^T(P^2\,\boldsymbol{\Psi}(t)+\boldsymbol{\nu}_2(t)),\\&\quad\boldsymbol{C}^T(P^3\,\boldsymbol{\Psi}(t)+\boldsymbol{\nu}_3(t)),\boldsymbol{C}^T(P^4\,\boldsymbol{\Psi}(t)+\boldsymbol{\nu}_4(t))\Bigg),\end{aligned} \tag{3.9}$$

where the vectors $\boldsymbol{\nu_r}(t)$ are given by

$$\boldsymbol{\nu_r}(t)=\sum_{m=0}^{r-1}P^{r-m-1}\,\frac{d^m}{dt^m}\,\boldsymbol{\nu}(t),\quad 2\le r\le 5,$$

and the components of the vector $\boldsymbol{\nu}(t)=(\nu_i)_{i\ge 0}$, and ν_i are given in (2.17).
The residual of equation (3.7) takes the form

$$\begin{aligned}R(t)=\boldsymbol{C}^T\,\left(P^5\boldsymbol{\Psi}(t)+\boldsymbol{\nu}_5(t)\right)-F\Bigg(&t,\boldsymbol{C}^T\,\boldsymbol{\Psi}(t),\boldsymbol{C}^T(P\,\boldsymbol{\Psi}(t)+\boldsymbol{\nu}(t)),\boldsymbol{C}^T(P^2\,\boldsymbol{\Psi}(t)+\boldsymbol{\nu}_2(t)),\\&\boldsymbol{C}^T(P^3\,\boldsymbol{\Psi}(t)+\boldsymbol{\nu}_3(t)),\boldsymbol{C}^T(P^4\,\boldsymbol{\Psi}(t)+\boldsymbol{\nu}_4(t))\Bigg).\end{aligned} \tag{3.10}$$

The application of the spectral collocation method requires that the residual must vanish at some nodes which called collocation points. These collocation points may be selected to be one of the following choices:

1. The $(M+1)$ roots of the shifted Legendre polynomial $L^*_{M+1}(t)$.
2. The $(M+1)$ roots of the Chebyshev polynomial of the first kind $T^*_{M+1}(t)$.
3. The $(M+1)$ roots of the Chebyshev polynomial of the second kind $U^*_{M+1}(t)$.

Thus for every choice of the collocation points, a nonlinear system of $(M+1)$ equations is generated. Hence, the approximate solution of (3.7)-(3.8) can be obtained by solving the resulting nonlinear system via a suitable numerical solver such as the well-known Newton's iterative method.

4 Investigation of convergence and error analysis

This section is concerned with investigating the convergence analysis of the suggested expansion. In this respect, two theorems are stated and proved. In the first, we show that the expansion in (2.3) of $y(t) = (t-a)^3\,(b-t)^2\,f(t) \in X^{(0)}$ converges uniformly to $y(t)$, provided that the fourth derivative of $f(t)$ is bounded, while in the second theorem, an estimation for the global error is derived.

Theorem 2. Any function $y(t) = (t-a)^3\,(b-t)^2\,f(t) \in X^{(0)}$, $w(t) = \dfrac{1}{(t-a)^6\,(b-t)^4}$ with $|f^{(4)}(t)| \leqslant L$, can be expressed as in (2.3). Moreover, the series is uniformly convergent to $y(t)$ and the unknowns coefficients, c_i in (2.3), satisfy the inequality

$$|c_i| < \frac{2\,L\,(b-a)^3\,(2i+1)}{(2i-1)(2i+3)(2i+7)}, \qquad \forall\, i > 3. \tag{4.1}$$

Proof. From Eq. (2.4), we have

$$c_i = \frac{2i+1}{b-a}\int_a^b f(t)\,L^*_i(t)\,dt.$$

Now by integration by parts four times and repeated use of the formula

$$\int_a^t L^*_i(t)\,dt := I_1(t) = \frac{b-a}{2(2i+1)}\left(L^*_{i+1}(t) - L^*_{i-1}(t)\right),$$

noting that

$$I_1(a) = I_1(b) = 0,$$

we get

$$c_i = \frac{(b-a)^3}{256\,(i-\frac{3}{2})_2\,(i+\frac{3}{2})_3}\int_a^b f^{(4)}(t)\,\tau_i(t)\,dt,$$

where

$$\tau_i(t) = \xi_{1,i}\,L^*_{i+4}(t) - \xi_{2,i}\,L^*_{i+2}(t) + \xi_{3,i}\,L^*_i(t) - \xi_{4,i}\,L^*_{i-2}(t) + \xi_{5,i}\,L^*_{i-4}(t),$$

and

$$\xi_{1,i} =(-5+2i)(-3+2i)(-1+2i), \qquad \xi_{2,i} = 4(-5+2i)(-3+2i)(5+2i),$$
$$\xi_{3,i} =6(-5+2i)(1+2i)(7+2i), \qquad \xi_{4,i} = 4(-3+2i)(5+2i)(7+2i),$$
$$\xi_{5,i} =(3+2i)(5+2i)(7+2i).$$

The assumption $\left|f^{(4)}(t)\right| \leq L$ and the boundedness of Legendre polynomials ($|L_i^*(t)| \leq 1$), implies that

$$|c_i| \leq \frac{2\,L\,(b-a)^3\,(2i+1)}{(2i-1)(2i+3)(2i+7)}. \tag{4.2}$$

Q.E.D.

Theorem 3. If $y(t)$ satisfies the hypotheses of Theorem 2 and if $y_M(t) = \sum_{i=0}^{M} c_i\,\psi_i(t)$, then the following error estimate holds

$$|y - y_M| < \frac{L\,(b-a)^8}{M}. \tag{4.3}$$

Proof. From the definition of y_M, one has

$$|y - y_M| = \left|\sum_{i=M+1}^{\infty} c_i\,\psi_i(t)\right|.$$

With the aid of Theorem 2, and noting that $|\psi_i(t)| < (b-a)^5$, we get

$$\begin{aligned} |y - y_M| &= 2\,L\,(b-a)^8 \sum_{i=M+1}^{\infty} \frac{(2i+1)}{(2i-1)(2i+3)(2i+7)} \\ &= \frac{L\,(b-a)^8\,(31+70M+44M^2+8M^3)}{(1+2M)(3+2M)(5+2M)(7+2M)}. \end{aligned}$$

Now, since

$$\frac{(31+70M+44M^2+8M^3)}{(1+2M)(3+2M)(5+2M)(7+2M)} < \frac{1}{M}.$$

Theorem 3 is now proved. Q.E.D.

5 Numerical results and discussions

In this section, we present four numerical examples accompanied with comparisons with some other techniques to numerically solve linear and nonlinear fifth-order BVPs.

Example 1. Let us consider the following linear fifth-order BVP (see [5, 25])

$$y^{(5)}(t) + y(t) = f(x), \quad 0 \leqslant t \leqslant 1,$$

with the boundary conditions

$$\begin{aligned} &y(0)=1, && y'(0)=0, && y''(0)=-1,\\ &y(1)=e(1-\sin(1)), && y'(1)=-e(\cos(1)+\sin(1)-1), && \end{aligned}$$

where $f(x)$ will be chosen to be consistent with the exact solution $y(t)=e^t(1-\sin t)$.

In Table 1, the maximum pointwise errors $E=|y-y_M|$ using $HPGM$ are tabulated for different

Table 1: Maximum pointwise errors of Example 1 for various values of M.

M	4	6	8	10	20
$HPGM$	1.18797×10^{-8}	2.10243×10^{-12}	1.88738×10^{-15}	1.33227×10^{-15}	1.77636×10^{-15}

Table 2: Comparison of the maximum absolute error for Example 1.

t	ICA [5]			$HPGM$
	AST	IPT	SQP	$M=10$
0.0	2.8×10^{-4}	5.2×10^{-4}	6.5×10^{-4}	0.0
0.1	3.2×10^{-4}	5.9×10^{-4}	7.3×10^{-4}	8.3×10^{-17}
0.2	3.2×10^{-4}	6.0×10^{-4}	7.5×10^{-4}	5.6×10^{-17}
0.3	2.9×10^{-4}	5.3×10^{-4}	6.7×10^{-4}	1.1×10^{-16}
0.4	2.1×10^{-4}	3.8×10^{-4}	4.9×10^{-4}	1.1×10^{-16}
0.5	1.1×10^{-4}	2.0×10^{-4}	2.6×10^{-4}	0.0
0.6	2.9×10^{-5}	5.2×10^{-5}	6.9×10^{-5}	2.2×10^{-16}
0.7	9.1×10^{-8}	3.4×10^{-7}	2.0×10^{-7}	2.2×10^{-16}
0.8	3.3×10^{-5}	6.3×10^{-5}	7.5×10^{-5}	2.2×10^{-16}
0.9	9.7×10^{-5}	1.8×10^{-4}	2.2×10^{-4}	2.2×10^{-16}
1.0	1.3×10^{-4}	2.4×10^{-4}	2.9×10^{-4}	4.4×10^{-16}

values of M. Ahmad et al. [5] have solved this problem using the recently intelligent computing approaches (ICA). The results obtained in [5] are based on artificial neural networks (ANN) models optimized with efficient local search methods like sequential quadratic programming (SQP), interior point technique (IPT) and active set technique (AST). We compare the absolute errors obtained by our method $HPGM$ in case of $M=10$ with those in [5]. This table ascertain that our proposed method give more accurate results comparable with those in [5].

Note 5.1. To be fair enough, the method produced by Ahmad et al. [5] will result a better accuracy for the case $h=0.001$, but this will need a higher computational cost comparable with HPGM.

Example 2. Consider the following linear fifth-order BVP (see, Wazwaz [27], Caglar et al. [6], Zhang [29] and Rashidinia et al. [20]).

$$y^{(5)}(t)=y(t)-15e^t-10\,t\,e^t,\quad 0\leqslant t\leqslant 1,$$

Table 3: Maximum pointwise errors of Example 2 for various values of M.

M	2	4	6	8	10
$HPGM$	2.5×10^{-5}	4.0×10^{-8}	8.6×10^{-12}	2.6×10^{-15}	5.3×10^{-16}

with the boundary conditions

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y(1) = 0, \quad y'(1) = -e.$$

The exact solution of this problem is $y(t) = t(1-t)e^t$.

In Table 3, the maximum pointwise errors $E = |y - y_M|$ using $HPGM$ are tabulated for different values of M, while in Table 4, we compare the best absolute errors obtained by our method $HPGM$ in case of $M = 12$ with others obtained by using B-spline methods (BSM) in [6, 20], the decomposition method (DM) in [27], and the variational iteration method (VIM) in [29]. In order to compare with the recently intelligent computing approaches (ICA) in [5], we, also, list the results obtained by Ahmad et al. [5] in Table 4. The obtained results show that our method is more accurate comparable with the methods reported in literature.

Table 4: Comparison of the maximum absolute error between different methods for Example 2.

t	BSM		DM	VIM	ICA [5]			$HPGM$
	[6]	[20]	[27]	[29]	AST	IPT	SQP	
0.1	7.0×10^{-4}	3.38×10^{-15}	1.0×10^{-9}	0.0	5.0×10^{-2}	4.9×10^{-2}	4.5×10^{-2}	6.4×10^{-18}
0.2	7.2×10^{-4}	4.47×10^{-15}	2.0×10^{-9}	1.0×10^{-5}	5.1×10^{-2}	4.9×10^{-2}	5.1×10^{-2}	6.6×10^{-18}
0.3	4.1×10^{-4}	3.44×10^{-15}	1.0×10^{-8}	1.0×10^{-5}	4.5×10^{-2}	4.3×10^{-2}	4.5×10^{-2}	2.9×10^{-18}
0.4	4.6×10^{-4}	6.88×10^{-16}	2.0×10^{-8}	1.0×10^{-4}	3.3×10^{-2}	3.1×10^{-2}	3.3×10^{-2}	2.9×10^{-18}
0.5	4.7×10^{-4}	3.17×10^{-15}	3.1×10^{-8}	3.2×10^{-4}	1.7×10^{-2}	1.6×10^{-2}	1.7×10^{-2}	9.2×10^{-18}
0.6	4.8×10^{-4}	7.28×10^{-15}	3.7×10^{-8}	3.6×10^{-4}	4.6×10^{-3}	3.8×10^{-3}	4.6×10^{-3}	2.6×10^{-18}
0.7	3.9×10^{-4}	1.05×10^{-14}	4.1×10^{-8}	1.4×10^{-4}	1.7×10^{-5}	8.7×10^{-5}	1.7×10^{-5}	1.8×10^{-18}
0.8	3.1×10^{-4}	1.15×10^{-14}	3.1×10^{-8}	3.1×10^{-4}	5.1×10^{-3}	5.9×10^{-3}	5.1×10^{-3}	5.5×10^{-17}
0.9	1.6×10^{-4}	8.68×10^{-15}	1.4×10^{-8}	5.8×10^{-4}	1.5×10^{-2}	1.6×10^{-2}	1.5×10^{-2}	2.5×10^{-16}

Example 3. Consider the following nonlinear fifth-order BVP (see, [18, 27–29]):

$$y^{(5)}(t) = e^{-t}\, y^2(t), \quad 0 \leqslant t \leqslant 1,$$

with the boundary conditions

$$y(0) = 0 = y'(0) = y''(0) = 1, \quad y(1) = y'(1) = e,$$

with the analytic solution $y(t) = e^t$.

Table 5 displays the maximum absolute errors $E = |y - y_M|$ when HCM is applied for various values of M for different choices of collocation points. In this table, we denote E_1, E_2 and E_3

by the maximum absolute errors if the selected collocation points are respectively, the zeros of the shifted Legendre polynomials $L^*_{M+1}(t)$, shifted Chebyshev polynomials of the first and second kinds $T^*_{M+1}(t)$, $U^*_{M+1}(t)$. In Table 6, we compare the absolute errors obtained HCM with the absolute errors of the following methods:

- An Enhanced Quartic B-spline Method (EQBSM) in [17]
- Quartic B-spline collocation method (QBSCM) in [18].
- The decomposition method (DM) in [27].
- collocation method using the Bessel polynomial (CBM) [28].
- The variational iteration method (VIM) in [29].

Table 6 shows that our algorithm is more accurate than the methods developed in the above mentioned methods.

Table 5: Maximum absolute error of $|y - y_M|$ for Example 3

M	E_1	E_2	E_3
2	3.76751×10^{-7}	5.80042×10^{-7}	2.41662×10^{-7}
4	4.20465×10^{-10}	4.20465×10^{-10}	9.42551×10^{-10}
6	6.55032×10^{-14}	2.56684×10^{-13}	2.89768×10^{-14}
8	1.30451×10^{-15}	1.32533×10^{-15}	1.36002×10^{-15}

Table 6: Comparison of different absolute errors $|y - y_M|$ in solving Example 3

t	0	0.2	0.4	0.6	0.8	1
EQBSM [17]	–	9.3×10^{-14}	4.5×10^{-13}	8.0×10^{-13}	4.9×10^{-13}	–
QBSCM [18]	0	2.7×10^{-11}	1.3×10^{-11}	3.0×10^{-11}	6.0×10^{-11}	0
DM [27]	0	2.0×10^{-9}	2.0×10^{-8}	3.7×10^{-8}	3.1×10^{-8}	0
CBM [28]$(M = 6)$	0	2.8×10^{-6}	1.5×10^{-5}	2.7×10^{-5}	1.9×10^{-5}	1.2×10^{-12}
CBM [28]$(M = 9)$	0	1.2×10^{-9}	6.9×10^{-9}	1.4×10^{-8}	1.2×10^{-8}	2.3×10^{-12}
CBM [28]$(M = 12)$	0	1.1×10^{-13}	6.2×10^{-13}	1.3×10^{-12}	1.3×10^{-12}	1.7×10^{-13}
VIM [29]	0	1.0×10^{-5}	1.0×10^{-4}	3.6×10^{-4}	3.1×10^{-4}	9.9×10^{-5}
$HCM(M = 6)$	0	1.2×10^{-14}	3.5×10^{-15}	5.8×10^{-15}	1.2×10^{-14}	0
$HCM(M = 8)$	0	0	0	0	4.4×10^{-16}	0

Example 4. Consider the following nonlinear fifth-order BVP (see, [6, 26]):

$$y^{(5)}(t) = -24\, e^{-5\, y(t)} + \frac{48}{(1+t)^5}, \quad 0 < t < 1,$$

with the boundary conditions

$$y(0) = 0,\ y'(0) = 1,\ y''(0) = -1,\ y(1) = \ln(2),\ y'(1) = \frac{1}{2},$$

with the analytic solution $y(t) = \ln(1 + t)$.
In Table 7, we display a comparison between the absolute errors obtained by the application of our methods (HCM) for the two cases correspond to $M = 12$, and $M = 16$, with the errors resulted from the application of using B-spline methods (BSM) in [6, 26]. The results of this table ascertain that our method is more accurate if compared with these two methods.

Table 7: Comparison between different methods for Example 4

t	BSM		HCM	
	[6]	[26]	$M = 12$	$M = 16$
0	0	0	0	0
0.1	0	2×10^{-8}	8.34353×10^{-15}	5.49826×10^{-17}
0.2	0.015	1.2×10^{-7}	1.88638×10^{-14}	4.85723×10^{-17}
0.3	0.029	2.8×10^{-7}	5.73994×10^{-14}	6.19079×10^{-17}
0.4	0.028	4.5×10^{-7}	1.06721×10^{-13}	1.1254×10^{-16}
0.5	0.026	5.6×10^{-7}	1.14803×10^{-13}	9.36751×10^{-17}
0.6	0.024	5.8×10^{-7}	6.96845×10^{-14}	9.36751×10^{-16}
0.7	0.026	4.8×10^{-7}	1.97955×10^{-14}	3.77302×10^{-17}
0.8	0.033	3.0×10^{-7}	6.63965×10^{-16}	4.59702×10^{-17}
0.9	0.046	1.0×10^{-7}	1.55778×10^{-15}	1.11022×10^{-16}
1	0	0	0	2.08167×10^{-17}

6 Conclusions

In this article we have presented two numerical algorithms for solving linear or nonlinear fifth-order BVPs. The proposed algorithms are built on establishing a new operational matrix of derivatives of a certain combination of shifted Legendre polynomials. This operational matrix is utilized along with the application of the Petrov-Galerkin and collocation spectral methods for obtaining the desired spectral solutions. The new suggested algorithms produce very accurate results even when employing a small number of terms of the suggested expansion.

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