Almost periodicity of stochastic operators on $\ell^1(\mathbb{N})$

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Abstract

We characterize from a functional analytic point of view almost periodicity of operators on ℓ^1 given by infinite column-stochastic matrices. Some of the equivalent properties occur, under the name of Foster's condition, in the theory of stochastic processes. The results are applied to flows in infinite networks.

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1 Introduction

Stochastic operators on ℓ^1, i.e., positive operators which preserve the norm of positive elements, arise, e.g., as transition matrices of Markov processes with countably infinitely many states (e.g., [4, 6, 13, 15, 24]) or as adjacency matrices of (infinite) weighted directed graphs (e.g., [3, 5, 13]). They can be represented as infinite column-stochastic matrices

$$\mathbb{A} = (p_{ji})_{i,j\in\mathbb{N}}$$

with $0 \leq p_{ji}$, $i, j \in \mathbb{N}$, and $\sum_{i\in\mathbb{N}} p_{ji} = 1$, $j \in \mathbb{N}$.

A central theme in the theory of these operators is to describe the (asymptotic) behavior of the powers $\mathbb{A}^n$, $n \in \mathbb{N}$. A simple functional analytic property is the basis for such an analysis (cf., e.g., [18, § 2.4] or [2, 21]).

Definition 1.1. A bounded linear operator $\mathbb{A}$ on a Banach space X is *almost periodic* if the set $\{\mathbb{A}^n \mid n \in \mathbb{N}_0\}$ is relatively compact for the strong operator topology, i.e., if for all $x \in X$ the orbit $\{\mathbb{A}^n x \mid n \in \mathbb{N}_0\}$ is relatively compact in X.

We shall frequently use that, if $\mathbb{A}$ is power-bounded, then it suffices that $\{\mathbb{A}^n x \mid n \in \mathbb{N}_0\}$ is relatively compact for all x in some dense subset of X (cf. [10, Corollary A.5]).

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In this paper we analyze this property in the case of stochastic operators on $X = \ell^1$ and give a list of equivalent properties (cf. Theorems 2.11 and Theorem 3.7). These properties occur (and are known) in stochastic processes or graph theory, but our functional analytic approach yields a unifying view and new perspectives.

As a first step we show that, in our situation, weak almost periodicity (defined in analogy to Definition 1.1) coincides with almost periodicity.

Lemma 1.2. For an operator $\mathbb{A} \in \mathcal{L}(\ell^1)$ the set $\{\mathbb{A}^n \mid n \in \mathbb{N}_0\}$ is relatively compact in the *weak* operator topology if and only if it is relatively compact in the *strong* operator topology.

Proof. If $\{\mathbb{A}^n \mid n \in \mathbb{N}_0\}$ is relatively compact in the weak operator topology, then for all $x \in \ell^1$ the weak closure of the orbit $\{\mathbb{A}^n x \mid n \in \mathbb{N}_0\}$ is weakly compact. By Eberlein's theorem (e.g., [23, Corollary 2 to Theorem 11.1]) it is weakly sequentially compact, hence every sequence $(\mathbb{A}^{n_k} x)_{k\in\mathbb{N}}$ has a weakly convergent subsequence. Since in ℓ^1 weakly convergent sequences converge in norm ([1, p. 137] or [16, p. 283/284]), we obtain that the orbit $\{\mathbb{A}^n x \mid n \in \mathbb{N}_0\}$ is relatively norm-compact. This means relative compactness of $\{\mathbb{A}^n \mid n \in \mathbb{N}_0\}$ in the *strong* operator topology.
The converse implication is trivially true. Q.E.D.

Almost periodicity allows the application of the Jacobs-Glicksberg-deLeeuw decomposition theorem in which case the "convergence to a group" occurs in the strong operator topology.

Theorem 1.3. Suppose that the column-(sub-)stochastic operator $\mathbb{A} \in \mathcal{L}(\ell^1)$ is almost periodic. Then there is a positive, contractive projection $Q \in \mathcal{L}(\ell^1)$ such that the decomposition

$$\ell^1 = Q\ell^1 \oplus \ker Q$$

is $\mathbb{A}$-invariant and

$$\begin{aligned} Q\ell^1 = Y_r &:= \operatorname{lin}\{y \in \ell^1 \mid \mathbb{A}y = \lambda y,\ |\lambda| = 1\}, \\ \ker Q = Y_s &:= \{y \in \ell^1 \mid \|\mathbb{A}^n y\| \to 0\}. \end{aligned}$$

Moreover, Y_r is a closed sublattice of ℓ^1 and the restriction $R := \mathbb{A}_{|Y_r}$ is invertible with positive contractive inverse.

This result is well-known and can be found e.g. in [8, Lemma 4] (a general reference is [18, Chapter 2, Theorems 4.4 & 4.5]).

In Section 2 we begin with the irreducible case, and our main result characterizing almost periodicity is Theorem 2.11.

In Section 3 we treat the general case and show (Theorem 3.7) that almost periodicity of $\mathbb{A}$ means that $\mathbb{A}$ can be written as

$$\mathbb{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix},$$

where A is a direct sum of countably many irreducible stochastic matrices A_j with A_j almost periodic, while $D^n \to 0$ strongly. This can be thought of as a generalization of the normal form of positive finite matrices given in [22, Proposition II.8.8].

We conclude with an application of the results to the asymptotics of the C_0-semigroup belonging to a difference equation on $L^1([0,1], \ell^1)$, generalizing results on flows in infinite networks proved in of [8].

2 The irreducible case

In this section, we discuss almost periodicity of an irreducible, infinite, column-stochastic matrix $\mathbb{A} \in \mathcal{L}(\ell^1)$. Here we use the notion of (ideal) irreducibility as in [22, Definition III.8.1] which in our situation means that for all $i, j \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $(\mathbb{A}^n)_{i,j} = \langle \mathbb{A}^n e_j, e_i \rangle > 0$ (use [22, Proposition III.8.3]), where e_j denotes the canonical basis vector with jth entry 1 and 0 else.

2.1 Spectral characterization

Based on the Jacobs-Glicksberg-deLeeuw-decomposition stated in Theorem 1.3, we find the following first characterization of almost periodicity.

Proposition 2.1. If $\mathbb{A} \in \mathcal{L}(\ell^1)$ is irreducible and column-stochastic, then the following assertions are equivalent.

(i) $\mathrm{P}\sigma(\mathbb{A}) \cap \Gamma \neq \varnothing$ (Γ denotes the unit circle).

(i') 1 is an eigenvalue of $\mathbb{A}$.

(i") There exists a strictly positive fixed vector $0 \ll x \in \ell^1$ of $\mathbb{A}$.

(ii) $\{\mathbb{A}^n \mid n \in \mathbb{N}_0\}$ is relatively strongly/weakly compact, i.e., $\mathbb{A}$ is almost periodic.

Without the assumption of irreducibility, the implications (i") $\Rightarrow$ (ii) $\Rightarrow$ (i) $\iff$ (i') still hold.

Proof. Suppose (i) with $\mathbb{A}y = \lambda y$ for some $0 \neq y \in \ell^1$, $|\lambda| = 1$. By positivity, $|y| \leq \mathbb{A}|y|$ and $\|\mathbb{A}|y|\| \leq \|y\|$ since $\mathbb{A}$ is contractive. Hence $|y|$ is a positive fixed vector, and we have (i').

Arguing as before, existence of some non-zero fixed vector y as supposed in (i') implies existence of a positive fixed vector $x := |y|$ of $\mathbb{A}$, which, by irreducibility, must be a sequence of strictly positive numbers (compare [22, Theorem V.5.2]), which is (i”).

Starting with (i”), strict positivity of x yields that the order interval $[-x,x]$ spans a dense subspace of ℓ^1. For all $-x \le y \le x$ the orbit $\{\mathbb{A}^n y : n \in \mathbb{N}_0\}$ is contained in $[-x,x]$, which is compact. Therefore, for all $y \in \operatorname{lin}[-x,x]$ the orbit is relatively compact. Since $\mathbb{A}$ is power-bounded, we obtain (ii).

For (ii) $\Rightarrow$ (i), note that $\mathrm{P}\sigma(\mathbb{A}) \cap \Gamma = \varnothing$ implies, by Theorem 1.3, that the powers of $\mathbb{A}$ converge strongly to 0. This is impossible since $\|\mathbb{A}^n e_i\| = 1$ for all $n, i \in \mathbb{N}$. Q.E.D.

For the action of $\mathbb{A}$ on Y_r, we note that there is a Perron-Frobenius result saying that an irreducible stochastic matrix has at most finitely many unimodular eigenvalues forming a subgroup of the circle group (cf., e.g., [8, Corollary 6] or [25], [28, Theorem 2.2] in a stochastic context).

Corollary 2.2. If $\mathbb{A}$ is irreducible, column-stochastic and almost periodic, then there is $m \in \mathbb{N}$ such that (in the notation of Theorem 1.3)

$$\mathrm{P}\sigma(\mathbb{A}) \cap \Gamma = \mathrm{P}\sigma(R) = \left\{1, e^{\frac{2\pi i}{m}}, \dots, e^{\frac{2\pi i(m-1)}{m}}\right\},$$

and

$$\|\mathbb{A}^{nm} y - Qy\| L^1([0,\,1], \ell^1) \xrightarrow{n\to\infty} 0 \quad \text{for all } y \in \ell^1.$$

It can be shown, using graph theoretical terms, that the number m depends only on the zero/non-zero-pattern of the entries of $\mathbb{A}$ (compare [8, Lemma 8]).

2.2 A summability condition

We now split the matrix $\mathbb{A}$ into a block matrix

$$\mathbb{A} = \begin{pmatrix} \widetilde{A}_1 & \widetilde{A}_2 \\ \widetilde{A}_3 & \widetilde{A}_4 \end{pmatrix}, \tag{2.1}$$

where $\widetilde{A}_1 \in M_{|F|\times|F|}(\mathbb{R})$, $\widetilde{A}_2 \in M_{|F|\times\infty}(\mathbb{R})$, and $\widetilde{A}_3 \in M_{\infty\times|F|}(\mathbb{R})$, according to a given finite index set F (and after a relabelling of the indices if necessary). Considering the blocks $\widetilde{A}_i$ as operators A_i on ℓ^1, $\mathbb{A}$ is of the form

$$\mathbb{A} = K + A_4$$

for positive contractions A_4 and $K := A_1 + A_2 + A_3$.

Since K is of finite rank (note that the range of A_1 and A_2 is a subset of $\langle e_1, \dots, e_{|F|}\rangle$, and A_3 has only finitely many columns), $\mathbb{A}$ can be considered as a perturbation of A_4 by the compact operator K. We ask for a condition on the “cut” matrix A_4 (relative to A_3) needed to ensure almost periodicity of $\mathbb{A}$.

Proposition 2.3. For an irreducible, infinite stochastic matrix $\mathbb{A}$ the following assertions are equivalent.

(i) $\mathbb{A}$ is almost periodic.

(ii) For one/all finite sets $F \subset \mathbb{N}$ and $\mathbb{A}$ decomposed as in (2.1) the series $\sum_{k=0}^{\infty} A_4^k A_3 e_j$ converges absolutely in ℓ^1 for all basis vectors e_j, $j \in F$.

(ii') For one/all finite sets $F \subset \mathbb{N}$ and $\mathbb{A}$ decomposed as in (2.1) the series $\sum_{k=0}^{\infty} A_4^k A_3 e_j$ and $\sum_{k=0}^{\infty} A_4^k e_j$ converge absolutely in ℓ^1 for all basis vectors e_j, $j \in \mathbb{N}$.

Without irreducibility, (ii') (for some fixed F) implies (i).

The key to the proof is the following expansion for the powers of $\mathbb{A}$.

Lemma 2.4. For all $n \geq 2$ we have

$$\mathbb{A}^n = A_4^{n-1}\mathbb{A} + \sum_{k=0}^{n-2} A_4^k (K\mathbb{A}^{n-1-k}). \tag{2.2}$$

Proof. For $n = 2$ we have $\mathbb{A}^2 = (A_4 + K)\mathbb{A} = A_4^{2-1}\mathbb{A} + \sum_{k=0}^{0} A_4^k (K\mathbb{A}^{2-1-k})$. For the induction it suffices to apply the formula for n to the second term of the expansion $\mathbb{A}^{n+1} = (K + A_4)\mathbb{A}^n = K\mathbb{A}^n + A_4\mathbb{A}^n$. Q.E.D.

Proof of Proposition 2.3. By Proposition 2.1, (i) implies the existence of a strictly positive fixed vector x. We consider (2.2) applied to x, i.e.,

$$x = \mathbb{A}^n x = A_4^{n-1} x + \sum_{k=0}^{n-2} A_4^k K x, \quad n \geq 2. \tag{2.3}$$

From $0 \leq A_4^n y \leq \mathbb{A}^n x = x$ for $y \in [0, x]$ we see that the orbit $\{A_4^n y : n \in \mathbb{N}\}$ is contained in the compact order interval $[0, x]$, hence is relatively compact. By strict positivity of x and as in the proof of Proposition 2.1, $\{A_4^n \mid n \in \mathbb{N}\}$ is relatively compact in the strong operator topology.

If $\mathrm{P}\sigma(A_4) \cap \Gamma \neq \varnothing$, there is again a fixed vector $0 < (0, y)^\top$ of A_4 (y is supported in F^c). Since $\mathbb{A}^n = (A_4 + K)^n = A_4^n + S_n$ for some $S_n \geq 0$,

$$\mathbb{A}^n \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix} + S_n \begin{pmatrix} 0 \\ y \end{pmatrix} \quad \text{for all } n \in \mathbb{N}.$$

Since $\mathbb{A}$ is irreducible, there has to be n with $S_n(0, y)^\top > 0$ and thus $\mathbb{A}^n(0, y)^\top > (0, y)^\top$, which contradicts $\|\mathbb{A}\| \leq 1$. Hence $\mathrm{P}\sigma(A_4) \cap \Gamma = \varnothing$, and Theorem 1.3 implies that the first term in (2.3) converges strongly to 0.

Consequently, the second term in (2.3) must converge to x as $n \to \infty$. This implies convergence of $\sum_{k=1}^{\infty} A_4^k K e_j = \sum_{k=1}^{\infty} A_4^k A_3 e_j$ for all $j \in F$. Clearly, the additivity of the 1-norm on positive elements implies absolute convergence of the series in question.

For (ii) $\Rightarrow$ (ii') we note that convergence of $\sum_{k=0}^{\infty} A_4^k A_3 \mathbb{1}_F$ implies that $\sum_{k=0}^{\infty} A_4^k e_j$ converges for all those $j \in F^c$ such that $(A_4^{k_0} A_3 \mathbb{1}_F)_j > 0$ for some $k_0 \in \mathbb{N}$. Since $\mathbb{A}$ is irreducible, this is actually true for every $j \in F^c$. In fact, by (2.2) and irreducibility,

$$0 < (\mathbb{A}^n \mathbb{1}_F)_j = \left(A_4^{n-1} A_3 \mathbb{1}_F\right)_j + \sum_{k=1}^{n-2} \left(A_4^k A_3 \mathbb{1}_F\right)_j$$

for some $n \geq 2$. Hence there is $k_0 \in \mathbb{N}$ such that $(A_4^{k_0} A_3 \mathbb{1}_F)_j > 0$, and $\sum_{k=0}^{\infty} A_4^k e_j$ converges.

In order to show (ii') $\Rightarrow$ (i), we use that all $(A_1 + A_2)\mathbb{A}^m$ are positive contractions with range living on the first $|F|$ entries, hence the estimate

$$(A_1 + A_2)\mathbb{A}^m e_j \leq \mathbb{1}_F \quad \text{and} \quad A_3 \mathbb{A}^m e_j = A_3(A_1 + A_2)\mathbb{A}^{m-1} e_j \leq A_3 \mathbb{1}_F$$

holds for all $m \in \mathbb{N}$. Property (ii) implies convergence of convergence of $\sum_{k=1}^{\infty} A_4^k A_3 \mathbb{1}_F$, and for $z := \sum_{k=1}^{\infty} A_4^k A_3 \mathbb{1}_F$ we can estimate

$$\begin{aligned} 0 \leq \sum_{k=0}^{n-2} A_4^k (K \mathbb{A}^{n-1-k}) e_j &= (A_1 + A_2)\mathbb{A}^{n-1} e_j + \sum_{k=1}^{n-2} A_4^k (A_3 \mathbb{A}^{n-1-k}) e_j \\ &\leq \mathbb{1}_F + \sum_{k=1}^{n-2} A_4^k A_3 \mathbb{1}_F \leq \mathbb{1}_F + z. \end{aligned}$$

Hence $\left\{\sum_{k=0}^{n-2} A_4^k (K \mathbb{A}^{n-1-k}) e_j \mid 2 \leq n \in \mathbb{N}\right\}$ is uniformly order bounded. Order intervals in ℓ^1 are compact, so we obtain relative strong compactness of $\left\{\sum_{k=0}^{n-2} A_4^k (K \mathbb{A}^{n-1-k}) e_j \mid n \geq 2\right\}$.

The convergence of $\sum_{k=0}^{\infty} A_4^k e_j$ for all j implies strong stability of A_4^k, i.e., $\|A_4^k y\| \to 0$ for all $y \in \ell^1$ as $k \to \infty$. Hence the first term of (2.2) applied to e_j, $j \notin F$, converges to 0 as $n \to \infty$, and (i) holds.

Since (i) is independent of the choice of F, the equivalences hold for every choice of F. Q.E.D.

2.3 Positive recurrence

We now relate the summability condition on the blocks of the above matrix decomposition to some stochastic concepts.

An infinite stochastic matrix $\mathbb{A}$ can be regarded as the (transposed) *transition matrix* of a discrete-time homogeneous Markov chain with countably many states (cf., e.g., [24, Chapter 5]). Here, the entries $(\mathbb{A})_{i,j} = p_{ji}$, $i, j \in \mathbb{N}$, are the one-step transition probabilities.

In this context, a non-zero positive fixed vector is called a *finite invariant measure* (or *stationary distribution*) or, if normalized, *invariant probability measure.* In stochastic processes, the existence of such an invariant probability measure is related to *positive recurrence.*

Definition 2.5. An index $j \in \mathbb{N}$ is called

(a) *recurrent* if the probability of ever returning to j is equal to 1, i.e.,

$$\sum_{n=1}^{\infty} F_j^n = 1,$$

where F_j^n denotes the probability of returning to j for the first time in the nth step;

(b) *positive recurrent* if j is recurrent and the expected "first return time" $\sum_{n=1}^{\infty} n\, F_j^n$ is finite.

An irreducible (Markov chain defined by) $\mathbb{A}$ is called *(positive) recurrent* if one/all states are (positive) recurrent (compare our Proposition 2.6 and [24, Theorem 5.2]).

As in [26, Lemma 4.1] we obtain the following characterization.

Proposition 2.6. An irreducible, infinite stochastic matrix $\mathbb{A}$ is almost periodic if and only if j is positive recurrent for one/all $j \in \mathbb{N}$.

Proof. By Proposition 2.3, almost periodicity of $\mathbb{A}$ and $\sum_{k=0}^{\infty} \|A_4^k A_3 e_j\| < \infty$ are equivalent, where $\mathbb{A}$ is splitted as in (2.1) according to $F = \{j\}$, for one/all $j \in \mathbb{N}$.

Almost periodicity of $\mathbb{A}$ implies that for fixed $F = \{j\}$ and $\mathbb{A}$ splitted accordingly the series $\sum_{k=0}^{\infty} \|A_4^k A_3 e_j\|$ converges.

In particular, the sequence $\|A_4^{k-1} A_3 e_j\|$ converges to 0. Since the quantity $\|A_4^{k-1} A_3 e_j\|$ expresses the probability of not having re-entered j during the first k steps given that the random walk started in j, the probability of re-entering j during the first k steps equals

$$1 - \|A_4^{k-1} A_3 e_j\| = \sum_{n=1}^{k} F^n.$$

Therefore, the probability of eventually re-entering j is given by

$$\lim_{k\to\infty} \left(1 - \|A_4^{k-1} A_3 e_j\|\right) \leq 1$$

which by $\|A_4^{k-1} A_3 e_j\| \to 0$ must be equal to 1. By definition, this means that j is recurrent.

It remains to show that $\sum_{n=1}^{\infty} nF^n$ is finite. For every $k \geq 2$ we have $0 \leq \|A_4^{k-2}A_3e_j\| = 1 - \sum_{n=1}^{k-1} F^n$ and, since $\|A_4^{k-2}A_3e_j\| \to 0$ as $k \to \infty$, $\sum_{n=1}^{\infty} F^n = 1$. Thus

$$\sum_{n=k}^{\infty} F^n = 1 - \sum_{n=1}^{k-1} F^n = \|A_4^{k-2}A_3e_j\| \quad \text{for all } k \geq 2. \tag{2.4}$$

Therefore,

$$\sum_{n=1}^{\infty} nF^n = \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} F^n \overset{(2.4)}{=} 1 + \sum_{k=2}^{\infty} \|A_4^{k-2}A_3e_j\| \tag{2.5}$$

is finite and j is positive recurrent.

Conversely, recurrence of j implies, by the above arguments, first that $1 - \|A_4^{k-1}A_3e_j\|$ converges to 1, hence $\|A_4^{k-1}A_3e_j\| \to 0$. Then we have (2.4) and read (2.5) backwards in order to see that *positive* recurrence of j yields that $\sum_{k=2}^{\infty} \|A_4^{k-2}A_3e_1\|$ exists. Q.E.D.

The following consequences are well-known in the stochastic context.

Remark 2.7. In the case that $\mathbb{A}$ is irreducible,

(1) positive recurrence holds for one index if and only if it holds for all indices ([24, Theorem 5.2]),

(2) positive recurrence is equivalent to the existence of a finite invariant measure ([24, Theorem 5.5 & Corollary]) which by our Proposition 2.1 means almost periodicity of $\mathbb{A}$.

2.4 A Lyapunov-type inequality

The summability condition obtained in Proposition 2.3 results in a matrix-vector inequality.

Proposition 2.8. For a column-stochastic matrix $\mathbb{A} \in \mathcal{L}(\ell^1)$, decomposed as in (2.1), the following assertions are equivalent.

(i) There exist $\varepsilon > 0$ and a sequence $0 \leq h = (h_i)_{i\in\mathbb{N}}$ with $h_{|F} = 0$, satisfying

$$(1 - A_4^{\top})h \geq \varepsilon \mathbb{1}_{F^c}, \tag{2.6a}$$

$$(A_3^{\top} h)_j < \infty \quad \forall j \in \mathbb{N}. \tag{2.6b}$$

(i') There exists a sequence $0 \leq h = (h_i)_{i\in\mathbb{N}}$ with $h_{|F} = 0$, satisfying

$$(1 - A_4^{\top})h = \mathbb{1}_{F^c}, \tag{2.7a}$$

$$(A_3^{\top} h)_j < \infty \quad \forall j \in \mathbb{N}. \tag{2.7b}$$

(ii) The series $\sum_{k=0}^{\infty} A_4^k e_j$ and $\sum_{k=0}^{\infty} A_4^k A_3 e_j$ converge (absolutely) in ℓ^1 for all basis vectors e_j, $j \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii): We assume that (2.6a) and (2.6b) are satisfied, hence

$$\begin{aligned}\sum_{k=0}^{n} {A_4^\top}^k \mathbb{1}_{F^c} &= \frac{1}{\varepsilon}\sum_{k=0}^{n} {A_4^\top}^k \varepsilon \mathbb{1}_{F^c} \\ &\overset{(2.6a)}{\leq} \frac{1}{\varepsilon}\sum_{k=0}^{n} {A_4^\top}^k (1 - A_4^\top) h \\ &= \frac{1}{\varepsilon}\left(h - {A_4^\top}^{n+1} h\right) \leq \frac{1}{\varepsilon} h \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{2.8}$$

This means that the left hand side is bounded coordinatewise while it increases monotonically with n, thus $\sum_{k=0}^{\infty} {A_4^\top}^k \mathbb{1}_{F^c}$ converges coordinatewise. Similarly,

$$\sum_{k=0}^{n} A_3^\top {A_4^\top}^k \mathbb{1}_{F^c} \leq \frac{1}{\varepsilon} A_3^\top h\,,$$

where the right hand side is a vector with finite entries by assumption (2.6b), hence we obtain the coordinatewise convergence of the series.

Since for positive elements $0 \leq y \in \ell^1$ we have $\|y\| = \langle y, \mathbb{1}\rangle$, this coordinatewise convergence expresses just that

$$\sum_{k=0}^{\infty} \langle e_j,\, {A_4^\top}^n \mathbb{1}_{F^c}\rangle = \sum_{k=0}^{\infty} \langle A_4^n e_j,\, \mathbb{1}_{F^c}\rangle = \sum_{k=0}^{\infty} \|A_4^n e_j\|_1 \tag{2.9}$$

exists for all $j \in F^c$, and analogously for the series with A_3.

(ii) $\Rightarrow$ (i'): If coordinatewise convergence holds, we define

$$h := \sum_{k=0}^{\infty} {A_4^\top}^k \mathbb{1}_{F^c}, \tag{2.10}$$

and obtain, by the same calculation as above,

$$(1 - A_4^\top) \sum_{k=0}^{\infty} {A_4^\top}^k \mathbb{1}_{F^c} = \sum_{k=0}^{\infty} {A_4^\top}^k \mathbb{1}_{F^c} - \sum_{k=1}^{\infty} {A_4^\top}^k \mathbb{1}_{F^c} = \mathbb{1}_{F^c}$$

coordinatewise, so (2.7a) holds. Moreover, (2.7b) follows from the convergence of the other series, since coordinatewise

$$A_3^\top h = A_3^\top \sum_{k=0}^{\infty} {A_4^\top}^k \mathbb{1}_{F^c} = \sum_{k=0}^{\infty} A_3^\top {A_4^\top}^k \mathbb{1}_{F^c} < \infty\,.$$

The implication (i') $\Rightarrow$ (i) is obvious. Q.E.D.

Remark 2.9. The condition consisting of (2.6a) and (2.6b) appears in an equivalent form in a stochastic result known as *Foster's Theorem* (e.g., [4, Chapter 5, Theorem 1.1] or [11, 2.2.3] (there stated as a characterization)). Foster's motivation was an application in queueing theory [12].

In fact, considering (2.6) coordinatewise, one directly obtains the following equivalent condition (note that $A_3^\top h_1 \geq 0$ can be dropped, and that $A_1^\top h_1$ is finite anyway).

Definition 2.10. For some infinite, column-stochastic $\mathbb{A} = (p_{ji})_{i,j\in\mathbb{N}}$ *Foster's condition* is

$$\exists\, F \subset \mathbb{N} \text{ finite, } \varepsilon > 0 \text{ and } h : \mathbb{N} \to [0,\infty) \text{ such that}$$

$$h(i) - \sum_{j\in\mathbb{N}} p_{ij} h(j) \geq \varepsilon \qquad \text{for all } i \notin F, \tag{2.11a}$$

$$\sum_{j\in\mathbb{N}} p_{ij} h(j) < \infty \qquad \text{for all } i \in F. \tag{2.11b}$$

The benefit of Foster's condition or the Lyapunov inequality consists in the use of the one-step transition probabilities only, i.e., one can work directly with the entries of the matrix $\mathbb{A}$. The difficulty is to find a suitable function h, sometimes called *Lyapunov function* (e.g., in [4, 11]) or *Foster-Lyapunov function.*

Foster's condition is a *drift condition* if h is interpreted as a function measuring the distance from the finite center F. Then (2.11a) means that after one random step the expected distance strictly decreases by a uniform portion ε, while (2.11b) limits the drift away from F.

For a discussion of examples of positive recurrent irreducible Markov chains we refer to [8], where they appear, with corresponding illustrations, in the context of networks.

2.5 The final characterization

We now collect the previous arguments in our characterization of almost periodicity.

Theorem 2.11. Suppose $\mathbb{A} \in \mathcal{L}(\ell^1)$ is an infinite, column-stochastic matrix. If $\mathbb{A}$ is irreducible, then for one (hence for any) decomposition of $\mathbb{A}$ as in (2.1), each of the following assertions characterizes almost periodicity of $\mathbb{A}$.

(i) The Foster conditions (2.11a) and (2.11b) hold.

(ii) There exist $\varepsilon > 0$ and a sequence $0 \leq h = (h_i)_{i\in\mathbb{N}}$ with $h_{|F} = 0$ satisfying (2.6a) and (2.6b).

(ii') There exists a sequence $0 \le h = (h_i)_{i\in\mathbb{N}}$ with $h_{|F} = 0$ satisfying (2.7a) and (2.7b).

(iii) The series $\sum_{k=0}^{\infty} A_4^k e_j$ and $\sum_{k=0}^{\infty} A_4^k A_3 e_j$ converge (absolutely) in ℓ^1 for all basis vectors e_j, $j \in \mathbb{N}$.

(iii') The associated Markov chain is positive recurrent.

(iv) 1 is an eigenvalue of $\mathbb{A}$.

(iv') There exists a strictly positive fixed vector $0 \ll x \in \ell^1$ of $\mathbb{A}$.

Without irreducibility, for every fixed decomposition of $\mathbb{A}$ the implications (i) $\iff$ (ii) $\iff$ (ii') $\Rightarrow$ (iii) $\Rightarrow$ $\mathbb{A}$ almost periodic $\Rightarrow$ (iv) hold (while the other implications do not hold in general).

In the stochastic literature it has also been observed, e.g., by [27], that Foster's condition implies positive recurrence of all indices without any (additional) irreducibility assumption.

Remark 2.12. Since for irreducible $\mathbb{A}$ the concrete choice of F is irrelevant, we can always achieve $|F| = 1$ (i.e., $F = \{j\}$ for a single $j \in \mathbb{N}$). It is also possible to take $\varepsilon = 1$ with equality in (2.6a) or (2.11a) (as stated in (ii')), if we take h defined by (2.10), which is by (2.8) the *minimal* solution (with $\varepsilon = 1$).

For the sake of completeness, we mention another classical criterion for the stronger compactness condition of quasi-compactness (for this property, cf. [18, Chapter 2, Definition 2.3], [9, § VIII.8]) and a corresponding list of equivalences.

Theorem 2.13. Suppose $\mathbb{A} \in \mathcal{L}(\ell^1)$ is a positive, column-stochastic infinite matrix. If $\mathbb{A}$ is irreducible, then the following assertions are equivalent.

(i) *Doeblin's Condition* holds, i.e.,

$$\begin{aligned}&\exists\, F \subset \mathbb{N} \text{ finite}, \delta > 0, \text{ and } L \in \mathbb{N} \text{ such that} \\ &\forall\, j \in \mathbb{N} \text{ the probability of reaching } F \text{ in exactly } L \text{ steps is } \ge \delta.\end{aligned} \tag{2.12}$$

(ii) There exist $F \subset \mathbb{N}$ finite, $\varepsilon > 0$, and a *bounded* sequence $0 \le h = (h_i)_{i\in\mathbb{N}}$ satisfying (2.11a) and (2.11b) (or (2.6a) and (2.6b)).

(iii) For $\mathbb{A}$ decomposed as in (2.1) the series $\sum_{k=0}^{\infty} A_4^k$ converges in operator norm.

(iv) $\mathbb{A}$ is quasi-compact.

Proof. By the argument in [7, Proposition 4.8], quasi-compactness of $\mathbb{A}$ is equivalent to the existence of some $L \in \mathbb{N}$, $0 < \delta < 1$ and a finite set F, such that for $\mathbb{A}$ splitted accordingly as in (2.1) one has $\|A_4^L\| \leq 1 - \delta < 1$.

This implies first that (iv) is equivalent to (i) if we use the stochastic meaning of $\|A_4^L e_j\|$ as the probability of not reaching F for the first L steps when starting in $j \in F^c$.

Second, (iv) means $\|A_4^L\| \leq 1 - \delta < 1$, thus $\|A_4^{nL}\| \leq (1-\delta)^n$ for all $n \in \mathbb{N}$, and the series $\sum_{k=0}^\infty A_4^k$ converges in operator norm, i.e., (iii) holds.

By (iii), $\sum_{k=0}^\infty A_4^k e_j$ and $\sum_{k=0}^\infty A_4^k A_3 e_j$ converge uniformly for all $j \in \mathbb{N}$. Hence the minimal solution h with $\varepsilon = 1$ of (2.11a) and (2.11b), defined by (2.10), satisfies for some $c > 0$

$$\langle e_j, h\rangle = \langle e_j, \sum_{k=0}^\infty {A_4^\top}^k \mathbb{1}_{F^c}\rangle \leq c \quad \text{for all } j \in \mathbb{N}.$$

This yields a bounded solution to (2.6a) and (2.6b), hence (ii) holds.

If we suppose (ii), then, a forteriori, the *minimal* solution (2.10) is bounded, and $\sum_{k=0}^\infty {A_4^\top}^k \mathbb{1}_{F^c}$ defines an element in ℓ^∞. Considering this coordinatewise, we see that $\sum_{k=0}^\infty \|A_4^k e_j\|$ is *uniformly* bounded for all j. Since the summands $\|A_4^k e_j\|$ also decrease monotonically with k, for some (any) fixed $0 < \delta < 1$ there is $L \in \mathbb{N}$ with

$$\|A_4^L e_j\| \leq 1 - \delta \quad \text{for all } j \in \mathbb{N},$$

which yields (iv). Q.E.D.

3 The general case

Based on the irreducible case above, we now characterize almost periodicity of $\mathbb{A}$ in general. The starting point is to bring the infinite matrix into a block matrix form similar to the normal form of positive finite matrices given in [22, Proposition II.8.8].

As a first step we assume that $\mathbb{A}$ is a direct sum of (countably many) irreducible blocks. Then we can apply Theorem 2.11.

Corollary 3.1. Assume that up to relabelling $\mathbb{A}$ is of the form

$$\mathbb{A} = \begin{pmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \end{pmatrix}$$

with irreducible blocks $A_j := \mathbb{A}_{|I_j}$ acting on pairwise disjoint $\mathbb{A}$-invariant ideals I_j, $j \in J$, whose 1-sum equals ℓ^1. Then the following are equivalent.

(i) $\mathbb{A}$ is almost periodic.

(ii) $\{A_j^n \mid n \in \mathbb{N}_0\}$ is relatively strongly compact for each $j \in J$.

(iii) $1 \in \mathrm{P}\sigma(A_j)$ for each $j \in J$.

(iv) Foster's condition (2.11a) and (2.11b) is satisfied for each A_j, $j \in J$.

(v) There exist a set $F \subset \mathbb{N}$ with $F \cap N_j$ finite for all $j \in J$, $\varepsilon > 0$ and a sequence $0 \leq h = (h_i)_{i\in\mathbb{N}}$ with $h_{|F} = 0$ such that (2.11a) and (2.11b) hold.

Proof. If we assume (ii), then we have relative strong compactness of $\{\mathbb{A}^n \mid n \in \mathbb{N}_0\}$ on the dense subset $\mathrm{lin}\{I_j \mid j \in J\}$ of ℓ^1. The uniform boundedness of $\{\mathbb{A}^n \mid n \in \mathbb{N}_0\}$ yields (i). Conversely, the relative strong compactness of $\{\mathbb{A}^n \mid n \in \mathbb{N}_0\}$ is inherited by each block by the $\mathbb{A}$-invariance of I_j. Theorem 2.11 applied independently to the irreducible blocks proves the rest. For (iv) $\Rightarrow$ (v) note that we can always achieve $\varepsilon = 1$ in (2.11a). Q.E.D.

A general almost periodic $\mathbb{A}$ can always be decomposed into the following block form (compare [22, Proposition II.8.8]).

Lemma 3.2. Assume that $\mathbb{A}$ is an infinite, column-stochastic matrix which is almost periodic.

Then there are $\mathbb{A}$-invariant ideals I_j with disjoint supports $N_j \subset \mathbb{N}$, $j \in J$, with $A_j := \mathbb{A}_{|I_j}$, and a band projection $I - \mathbb{P}$ onto $\ell^1(N_\infty)$, $N_\infty := \mathbb{N} \setminus \bigcup_{j\in J} N_j$, such that the resulting block matrix (after a suitable relabelling)

$$\mathbb{A} = \left(\begin{array}{ccc|c} A_1 & & & \\ & A_2 & & \mathbb{P}\mathbb{A}(I-\mathbb{P}) \\ & & \ddots & \\ \hline & 0 & & (I-\mathbb{P})\mathbb{A}(I-\mathbb{P}) \end{array}\right) =: \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \tag{3.1}$$

has the following properties.

(i) I_j is a minimal $\mathbb{A}$-invariant closed ideal for every $j \in J$, i.e., A_j is an irreducible stochastic matrix,

(ii) $A = \mathrm{diag}(A_j)_{j\in J}$ satisfies one of the equivalent conditions of Corollary 3.1,

(iii) $N_\infty = \varnothing$ (i.e., $\mathbb{A}$ is completely reducible) or $D^n \to 0$ strongly.

The form (3.1) with properties (i)-(iii) is unique up to permutations.

In addition, if Foster's condition (2.11a) and (2.11b) holds, then there are only finitely many irreducible blocks $A_1, \ldots, A_n$.

Proof. We start from the Jacobs-Glicksberg-deLeeuw decomposition

$$Y = Y_r \oplus Y_s = Q\ell^1 \oplus \ker Q$$

as stated in Theorem 1.3.

By, e.g., [8, Lemma 5], the sublattice Y_r is atomic with normalized atoms $a_j = Qe_j$, $j \in \mathrm{supp}Y_r$. The lattice isomorphism $R := \mathbb{A}_{|Y_r}$ on Y_r maps atoms to atoms (compare [8, Proposition 6]), and the group action of $(R^n)_{n\in\mathbb{Z}}$ on Y_r decomposes these atoms into disjoint orbits B_{a_j} indexed by some countable set J.

Each orbit B_{a_j} is finite: otherwise, as an infinite sequence of disjoint, normalized atoms, it is not relatively compact.

Clearly, the closed ideals I_j generated by B_{a_j} are $\mathbb{A}$-invariant and pairwise disjoint, with restriction $A_j := \mathbb{A}_{|I_j}$, and define N_j as the support of the ideal $I_j = \ell^1(N_j)$.

We now show that each A_j is irreducible. Fix arbitrary $i, p \in \mathrm{supp}B_{a_j}$. We already know that B_{a_j} is finite and consists of m_j atoms $R^\ell a_j$ cyclically permuted by R with period m_j. The element Qe_p is one of these atoms and equals $R^\ell a_j$ for some ℓ. Pick $k \in \mathbb{N}$ such that i is in the support of the atom $R^k(R^\ell a_j)$. Hence the representation of Y_s in Theorem 1.3 yields

$$\|A_j^{nm_j+k} e_p - R^k(R^\ell a_j)\| = \|A_j^{nm_j+k} e_p - A_j^{nm_j+k} Qe_p\| L^1([0,1],\ell^1) \xrightarrow{n\to\infty} 0.$$

Thus for n large enough $(A_j^{nm_j+k})_{i,p} = \langle A_j^{nm_j+k} e_p, e_i\rangle > 0$. Hence A_j is irreducible and (i) holds.

For (ii), we note that every I_j is $\mathbb{A}$-invariant, hence A_j inherits almost periodicity from $\mathbb{A}$. Note also that the support of every unimodular eigenvector is in $\bigcup_{j\in J} N_j$ by our construction via the Jacobs-Glicksberg-deLeeuw decomposition.

To verify (iii), we assume $N_\infty \neq \varnothing$ and use the domination

$$0 \le D^n = (I-\mathbb{P})\mathbb{A}^n(I-\mathbb{P}) \le (I-\mathbb{P})\mathbb{A}^n \quad \text{for all } n \in \mathbb{N}.$$

By the relative compactness of $\{\mathbb{A}^n \mid n \in \mathbb{N}_0\}$ each orbit contains a subsequence that approaches an element in $Q\ell^1$. Since $(I-\mathbb{P})Q = 0$ by definition, we obtain that $(I-\mathbb{P})\mathbb{A}^n(I-\mathbb{P})$ converges strongly to 0. Hence (iii) holds.

Assume that there is another such decomposition with properties (i)-(iii), indicated by the tilde sign, $\widetilde{\ }$. If for $j \in J$ we have

$$I_j \cap \widetilde{I}_i = \{0\} \quad \text{for all } i \in \widetilde{J},$$

then $I_j \subset \ell^1(\widetilde{N}_\infty)$. But this means that $\mathbb{A}$ acts as $\widetilde{D}$ on I_j. Since $\widetilde{D}^n$ converges strongly to 0 by (iii), this contradicts the stochasticity of $\mathbb{A}$.

Hence for every $j \in J$ there is $i \in \widetilde{J}$ such that $I_j \cap \widetilde{I}_i \neq \{0\}$. Then $I_j = \widetilde{I}_i$, because both I_j and $\widetilde{I}_i$ are minimal $\mathbb{A}$-invariant closed ideals. The same arguments applied to $i \in \widetilde{J}$ show that there is $j \in J$ such that $\widetilde{I}_i = I_j$, and we obtain uniqueness.

It remains to show that under Foster's condition there are only finitely many irreducible invariant blocks. In fact, for every $j \in \mathbb{N}$ there are finitely many successive states leading from j to F; since all intermediary states must belong to the same invariant set, it intersects F in at least one state. Since F is finite, we conclude that there are only finitely many minimal invariant ideals which are disjoint. Q.E.D.

Before showing the converse of Lemma 3.2, we first give some remarks on the previous result.

Remark 3.3. In the language of stochastics, by the argument of Proposition 2.6, the quantity

$$\lim_{n\to\infty} 1 - \|(I-\mathbb{P})\mathbb{A}^n(I-\mathbb{P})e_j\| = 1$$

expresses the positive probability to reach $\bigcup N_j$ from $j \in N_\infty$. In particular, all indices in N_∞ are *inessential*, i.e., there is a state i ($\in \bigcup N_j$) which can be reached from j but there is no return to j.

Thus Lemma 3.2 shows that for almost periodicity of $\{\mathbb{A}^n \mid n \in \mathbb{N}_0\}$ all indices are positive recurrent or inessential.

The matrix decomposition (2.1) is a special form of a *Doeblin decomposition* (e.g., [20]).

From Lemma 3.2 we draw the following conclusion on the long-term behavior of $\mathbb{A}^n$ (compare Theorem 4.2 below).

Theorem 3.4. An infinite, column-stochastic matrix $\mathbb{A}$ is almost periodic if and only if there are positive, contractive projections Q_j, $j \in J$, which have pairwise disjoint ranges $Q_j\ell^1$ on which the restrictions A_j of $\mathbb{A}$ are periodic and

$$\left\| \mathbb{A}^n y - \Big(\sum_{j\in J} A_j^n Q_j y\Big) \right\| L^1([0,1],\ell^1) \xrightarrow{n\to\infty} 0 \quad \text{for all } y \in \ell^1.$$

Now we show that the block matrix decomposition (3.1) with properties (i)-(iii) obtained in Lemma 3.2 for almost periodic $\mathbb{A}$ conversely implies almost periodicity of $\mathbb{A}$. This will need a more technical proof.

Theorem 3.5. Let $\mathbb{A}$ be an infinite, column-stochastic matrix. Then $\mathbb{A}$ is almost periodic if and only if $\mathbb{A}$ is (up to relabelling) of the form (3.1) as in Lemma 3.2 with properties (i)-(iii).

Proof. By Lemma 3.2, we only have to prove the converse direction. We adopt the notation of Lemma 3.2 and denote by P_N the band projection onto $\overline{\mathrm{lin}}\{I_j \mid j > N\}$.

By Corollary 3.1 it clearly suffices to find for fixed $x \in \ell^1(N_\infty) = \ker \mathbb{P}$, $\|x\| \le 1$, and each sequence $n_k \uparrow \infty$, a subsequence $(n_{k_\ell})_{\ell\in\mathbb{N}}$ such that $\mathbb{P}\mathbb{A}^{n_{k_\ell}} x$ converges. We construct $(n_{k_\ell})_{\ell\in\mathbb{N}}$ recursively.

For $\ell = 1$ we choose first a subsequence $\big(n_k^{(1)}\big)_{k\in\mathbb{N}}$ with $n_k^{(1)} - n_{k_0}^{(1)} \in m_1\mathbb{Z}$ for all k_0 (where m_1 for A_1 as in Corollary 2.2). This is possible because at least one of the m_1 many sets

$$m_1\mathbb{N},\ 1 + m_1\mathbb{N}, \dots, (m_1 - 1) + m_1\mathbb{N},$$

must contain a subsequence $(n_k^{(1)})_{k\in\mathbb{N}}$ of n_k, and $n_k^{(1)} - n_{k_0}^{(1)} \in m_1\mathbb{N}$. (This is needed in the next step in order to keep $E_{1,1}$ fixed.)

Then we choose $n_{k_1} := n_{k_0}^{(1)} \in \{n_k^{(1)} \mid k \in \mathbb{N}\}$ such that $\|(I - \mathbb{P})\mathbb{A}^{n_{k_1}} x\| \le \frac{2}{3} = \frac{1}{3\cdot 2^{-1}}$, which is possible by (iii) for n_{k_1} large enough. Define

$$E_1 := \mathbb{P}\mathbb{A}^{n_{k_1}} x \in \operatorname{ran}\mathbb{P}, \quad R_0 := R_1 := 0.$$

Decomposing $E_1 = E_{1,1} + E_{1,2}$ disjointly with $E_{1,1} \in I_1$, we have that

$$\mathbb{P}\mathbb{A}^{n_{k_1}} x = R_1 + E_{1,1} + E_{1,2},$$

with

$$E_{1,1} \in I_1,\ \|E_{1,1}\| \le 1 \le \frac{1}{2^{1-4}}, \quad \|E_{1,2}\| \le 1 \le \frac{1}{3\cdot 2^{1-3}}.$$

Set $N_1 := 1$, and take $N_2 \ge N_1$ such that $\|P_{N_2} E_{1,2}\| \le \frac{1}{3^2\cdot 2^{2-3}}$ (to reduce the “error” in the next step we cut $E_{1,2}$ and keep only $P_{N_2}E_{1,2}$ in the error term).

In the $\ell + 1$-st step we start with given

- $N_\ell \le N_{\ell+1} \in \mathbb{N}$,
- a subsequence $(n_k^{(\ell)})_{k\in\mathbb{N}} - n_{k_\ell} \subset M_{N_\ell}\mathbb{Z}$ where

  $$M_N := \operatorname{lcm}\{m_1, \dots, m_N\} \in \mathbb{N},$$

  m_j for A_j as in Corollary 2.2, and
- $E_\ell = E_{\ell,2} + E_{\ell,2}$ and R_ℓ such that

  $$\mathbb{P}\mathbb{A}^{n_{k_\ell}} x = R_\ell + E_{\ell,1} + E_{\ell,2} \tag{3.2}$$

  with $E_{\ell,1} \in \mathrm{lin}\{I_j \mid 1 \le j \le N_\ell\}$, $\|E_{\ell,1}\| \le \frac{1}{2^{\ell-4}}$, $E_{\ell,2} \in \overline{\mathrm{lin}}\{I_j \mid j > N_\ell\}$, $\|E_{\ell,2}\| \le \frac{1}{3\cdot 2^{\ell-3}}$, $\|P_{N_{\ell+1}} E_{\ell,2}\| \le \frac{1}{3^2\cdot 2^{\ell+1-3}}$, $R_\ell \in \mathrm{lin}\{Q_j I_j \mid 1 \le j \le N_\ell\}$, and $\|R_\ell - R_{\ell-1}\| \le \|E_{\ell-1}\|$.

For the *finitely* many ideals I_j, $1 \leq j \leq N_\ell$, we can apply Corollary 2.2 simultaneously. Since by construction $(n_k^{(\ell)})_{k\in\mathbb{N}} - n_{k_\ell} \subset m_j\mathbb{Z}$ for all $1 \leq j \leq N_\ell$, $\mathbb{A}^{n_{k_{\ell+1}}-n_{k_\ell}}$ acts as the identiy on Q_jI_j, $1 \leq j \leq N_\ell$, and thus leaves R_ℓ fixed. We also obtain for $E_{\ell,1} \in \text{lin}\{I_j \mid 1 \leq j \leq N_\ell\}$

$$\|\mathbb{A}^{n_k^{(\ell)}-n_{k_\ell}}E_{\ell,1} - QE_{\ell,1}\|L^1([0,\,1],\ell^1) \xrightarrow{k\to\infty} 0.$$

In the $\ell+1$-st step we first pick a new subsequence $(n_k^{(\ell+1)})_{k\in\mathbb{N}}$ of $(n_k^{(\ell)})_{k\in\mathbb{N}}$ with

$$n_k^{(\ell+1)} - n_{k_\ell} \in M_{N_{\ell+1}}\mathbb{N} + p \quad \text{for some } p \in \mathbb{N},$$

using again the same argument as for $\ell = 1$.

Then we choose $n_{k_{\ell+1}} > n_{k_\ell}$ large enough with $n_{k_{\ell+1}} \in \{n_k^{(\ell+1)} \mid k \in \mathbb{N}\}$ such that

$$\|\mathbb{A}^{n_{k_{\ell+1}}-n_{k_\ell}}E_{\ell,1} - QE_{\ell,1}\| \leq \frac{1}{3^2\cdot 2^{\ell+1-3}}.$$

By Lemma 3.2(iii), we can simultaneously achieve (by possibly increasing $n_{k_{\ell+1}}$)

$$\|(I-\mathbb{P})\mathbb{A}^{n_{k_{\ell+1}}}x\| \leq \frac{1}{3^2\cdot 2^{\ell+1-3}}.$$

We note that (as needed for the $\ell+2$nd step)

$$n_k^{(\ell+1)} - n_{k_{\ell+1}} = n_k^{(\ell+1)} - n_{k_\ell} - (n_{k_{\ell+1}} - n_{k_\ell}) \in M_{N_{\ell+1}}\mathbb{N} + p - p = M_{N_{\ell+1}}\mathbb{N}.$$

We use the splitting

$$\begin{aligned}
\mathbb{P}\mathbb{A}^{n_{k_{\ell+1}}}x &= \mathbb{P}\mathbb{A}^{n_{k_{\ell+1}}-n_{k_\ell}}\mathbb{A}^{n_{k_\ell}}x \\
&= \mathbb{A}^{n_{k_{\ell+1}}-n_{k_\ell}}\mathbb{P}\mathbb{A}^{n_{k_\ell}}x + \mathbb{P}\mathbb{A}^{n_{k_{\ell+1}}-n_{k_\ell}}(I-\mathbb{P})\mathbb{A}^{n_{k_\ell}}x
\end{aligned}$$

to see how $\mathbb{P}\mathbb{A}^{n_{k_{\ell+1}}}x$ depends on $\mathbb{P}\mathbb{A}^{n_{k_\ell}}x$. Inserting $\mathbb{P}\mathbb{A}^{n_{k_\ell}}x = R_\ell + E_{\ell,1} + E_{\ell,2}$ and using that R_ℓ is kept fixed, we obtain

$$\begin{aligned}
\mathbb{P}\mathbb{A}^{n_{k_{\ell+1}}}x &= \mathbb{A}^{n_{k_{\ell+1}}-n_{k_\ell}}E_{\ell,2} + \mathbb{A}^{n_{k_{\ell+1}}-n_{k_\ell}}E_{\ell,1} - QE_{\ell,1} + \\
&\quad + \mathbb{P}\mathbb{A}^{n_{k_{\ell+1}}-n_{k_\ell}}(I-\mathbb{P})\mathbb{A}^{n_{k_\ell}}x + QE_{\ell,1} + R_l.
\end{aligned}$$

So we obtain (3.2) for $\ell+1$ if we define the new "error" and "fixed" terms as

$$\begin{aligned}
E_{\ell+1} &:= \mathbb{A}^{n_{k_{\ell+1}}-n_{k_\ell}}E_{\ell,1} - QE_{\ell,1} + \mathbb{A}^{n_{k_{\ell+1}}-n_{k_\ell}}E_{\ell,2} + \\
&\quad + \mathbb{P}\mathbb{A}^{n_{k_{\ell+1}}-n_{k_\ell}}(I-\mathbb{P})\mathbb{A}^{n_{k_\ell}}x, \\
E_{\ell+1,2} &:= P_{N_{\ell+1}}E_{\ell+1}, \\
E_{\ell+1,1} &:= E_{\ell+1} - E_{\ell+1,2} \in \text{lin}\{I_j \mid 1 \leq j \leq N_{\ell+1}\}, \\
N_{\ell+1} \leq N_{\ell+2} \quad \text{s.t.} & \quad \|P_{N_{\ell+2}}E_{\ell,2}\| \leq \frac{1}{3^2\cdot 2^{\ell+2-3}}, \\
R_{\ell+1} &:= QE_{\ell,1} + R_\ell.
\end{aligned}$$

The estimate $\|R_{\ell+1} - R_\ell\| = \|QE_{\ell,1}\| \leq \|E_{\ell,1}\|$ is obvious. By construction we have reduced the "error" because

$$\begin{aligned} \|E_{\ell+1}\| &\leq \|\mathbb{A}^{n_{k_{\ell+1}}-n_{k_\ell}} E_{\ell,1} - QE_{\ell,1}\| + \|E_{\ell,2}\| + \|(I-\mathbb{P})\mathbb{A}^{n_{k_\ell}} x\| \\ &\leq \frac{1}{3^2 \cdot 2^{\ell+1-3}} + \frac{1}{3 \cdot 2^{\ell+1-4}} + \frac{1}{3^2 \cdot 2^{\ell+1-3}} \leq \frac{1}{2^{\ell+1-4}} \end{aligned}$$

as claimed in (3.2), with even smaller "tail" (outside I_j, $1 \leq j \leq N_{\ell+1}$)

$$\begin{aligned} \|E_{\ell+1,2}\| &\leq \|\mathbb{A}^{n_{k_{\ell+1}}-n_{k_\ell}} E_{\ell,1} - QE_{\ell,1}\| + \|P_{N_{\ell+1}} E_{\ell,2}\| + \\ &\quad \frac{1}{3^2 \cdot 2^{\ell+1-3}} \|(I-\mathbb{P})\mathbb{A}^{n_{k_\ell}} x\| \\ &\leq \frac{1}{3^2 \cdot 2^{\ell+1-3}} + \frac{1}{3^2 \cdot 2^{\ell+1-3}} + \frac{1}{3^2 \cdot 2^{\ell+1-3}} \leq \frac{1}{3 \cdot 2^{\ell+1-3}}. \end{aligned}$$

This closes the recursion.

The properties given in (3.2) for all ℓ imply that R_ℓ converges to some R, and $E_{\ell,1} + E_{\ell,2} \to 0$ as $\ell \to \infty$. Hence

$$\mathbb{P}\mathbb{A}^{n_{k_\ell}} x = R_\ell + E_{\ell,1} + E_{\ell,2} \to R$$

converges in norm, as desired. Q.E.D.

In analogy to (and using the) result in the irreducible case, we now look for a generalized matrix-vector inequality of Lyapunov type, or a generalized drift condition corresponding to almost periodicity of $\mathbb{A}$. To this aim we quote the following result from stochastics (cf. [11, Theorem 2.2.1] or [19, Theorem 1] with proofs (and [24, Theorem 5.2])).

Theorem 3.6. Suppose $\mathbb{A} = (p_{ji})_{i,j\in\mathbb{N}}$ is irreducible. Then one/all indices are recurrent if and only if for one (and then for any) finite $F \subset \mathbb{N}$ there is $h : \mathbb{N} \to [0,\infty)$, $h_{|F} = 0$, with

$$\begin{aligned} & h(j) \to \infty, && (\text{for } j \to \infty) \\ \text{such that} \quad & h(i) - \sum_{j\in\mathbb{N}} p_{ij} h(j) \geq 0 && \text{for all } i \notin F. \end{aligned} \tag{3.3}$$

Thus we can add a drift condition to our final characterization.

Theorem 3.7. Let $\mathbb{A} = (p_{ji})_{i,j\in\mathbb{N}}$ be an infinite, column-stochastic matrix. Then the following are equivalent.

(i) $\mathbb{A}$ is almost periodic, i.e., $\{\mathbb{A}^n \mid n \in \mathbb{N}_0\}$ is relatively weakly/strongly compact.

(ii) $\mathbb{A}$ is (up to relabelling) of the form

$$\mathbb{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \tag{3.4}$$

where $A = \operatorname{diag}(A_j)_{j \in J}$ with A_j irreducible and satisfying one of the equivalent conditions of Corollary 3.1, and $D^n \to 0$ strongly.

(iii) $\mathbb{A}$ is in block form (3.4), with

(a) $A = \operatorname{diag}(A_j)$ where each A_j satisfies (2.6a) and (2.6b) for some splitting of $\mathbb{A}$ as in (2.1),

(b) D (with support N_∞) can be extended (adding one column and row) to an irreducible stochastic matrix, and there is $h_\infty : N_\infty \to [0, \infty)$, $h_\infty(n) \to \infty$ as $N_\infty \ni n \to \infty$, such that

$$(1 - D^\top) h_\infty \geq 0.$$

(iv) There are $\mathbb{A}$-invariant sets $N_j \subset \mathbb{N}$, $j \in J$, such that the following drift condition holds:

(a) Foster's condition (2.11a) and (2.11b) holds on each N_j, and

(b) For all $k \in N_\infty := \mathbb{N} \setminus \bigcup_{j \in J} N_j$ there is a positive probability to eventually reach $\bigcup_{j \in J} N_j$, and there is $h_\infty : N_\infty \to [0, \infty)$, $h_\infty(n) \to \infty$ as $N_\infty \ni n \to \infty$, such that

$$h_\infty(i) - \sum_{k \in N_\infty} p_{ik} h_\infty(k) \geq 0 \quad \text{for all } i \in N_\infty.$$

Proof. Based on Theorem 3.5 it remains to show (ii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) (the equivalence of (iv) and (iii) follows from their proof and the reformulation as matrix-vector inequality as done before). We consider the auxiliary matrix

$$\widetilde{\mathbb{A}} := \left(\begin{array}{c|c} 0 & p_1 \quad p_2 \qquad \cdots \\ \hline q_1 & \\ q_2 & (I - \mathbb{P})\mathbb{A}(I - \mathbb{P}) \\ \vdots & \end{array} \right), \tag{3.5}$$

where (q_n) is any strictly positive vector in ℓ^1 of norm 1 and $p_j \geq 0$ are chosen such that $\widetilde{\mathbb{A}}$ becomes column-stochastic. This effectuates a decomposition of $\widetilde{\mathbb{A}}$ as in (2.1) with $(I - \mathbb{P})\mathbb{A}(I - \mathbb{P})$ in the role of A_4.

This means that we merge all the states corresponding to the irreducible invariant blocks into one absorbing state $\{0\}$, and then introduce an artificial feedback to N_∞.

In each implication, we obtain that $\widetilde{\mathbb{A}}$ is irreducible and then use the characterization of recurrence of Theorem 3.6.

Starting from (ii), as in Remark 3.3 (which shows that the states in N_∞ are inessential) we conclude from

$$\|(I-\mathbb{P})\mathbb{A}^n(I-\mathbb{P})e_j\| \to 0 \quad \text{for all } j \in N_\infty$$

that for all j there is an n such that $\widetilde{\mathbb{A}}^n$ has a non-zero entry in its first row (i.e., j leads to the added state $\{0\}$ with probability > 0). On the other hand, each ideal whose support includes $\{0\}$ also includes N_∞, because the first column of $\widetilde{\mathbb{A}}$ has been chosen strictly positive. Hence $\widetilde{\mathbb{A}}$ is irreducible.

Moreover, the strong stability of $(I-\mathbb{P})\mathbb{A}^n(I-\mathbb{P})$ implies, by the proof of Proposition 2.6, that $\{0\}$ is a recurrent index of $\widetilde{\mathbb{A}}$. Consequently, the characterization of Theorem 3.6 (with $F = \{0\}$) yields the asserted drift on N_∞. The irreducible blocks A_j are almost periodic by assumption, hence satisfy (mutually independent) Foster conditions on the irreducible blocks. We obtain (iv).

Conversely, assume that (iv) holds. By (iv)(b), for any $k \in N_\infty$ it is possible to escape to the state $\{0\}$ after finitely many steps. Arguing as before, this implies irreducibility of the matrix $\widetilde{\mathbb{A}}$. So Theorem 3.6 can be applied, and yields recurrence of $\{0\}$. Thus, together with the proof of Proposition 2.6 (and strict positivity of (q_j)), $(I-\mathbb{P})\mathbb{A}^n(I-\mathbb{P}) \to 0$ strongly.

Moreover, Foster's condition (iv)(a) for the stochastic blocks A_j corresponding to the invariant sets N_j implies by Theorem 2.11 relative strong compactness of $\{A_j^n \mid n \in \mathbb{N}\}$. Applying Lemma 3.2 to A_j, we obtain, after a suitable relabelling of the indices, a block matrix as in Lemma 3.2. So without loss of generality, the given decomposition of $\mathbb{N}$ yields irreducible blocks A_j each satisfying Foster's condition. This is (ii). Q.E.D.

4 Application

Almost periodicity of an irreducible infinite stochastic matrix $\mathbb{B} \in \mathcal{L}(\ell^1)$ has been used in [8] in the context of flows in infinite networks.

By [7], such a flow can be described, on an abstract level, on the Banach space $X := L^1([0,1],\ell^1)$ by the C_0-semigroup $(T(t))_{t\geq 0}$ given by

$$T(t)f(s) = \mathbb{B}^n f(t+s-n) \ \text{ if } \ n \leq t+s < n+1, \ n \in \mathbb{N}_0, \tag{4.1}$$

for all $t \geq 0$, $f \in X$ and almost all $s \in [0,1]$. This semigroup has generator

$$A := \frac{d}{ds}, \quad D(A) := \{f \in W^{1,1}([0,1],\ell^1) \mid f(1) = \mathbb{B}f(0)\}. \tag{4.2}$$

In the network interpretation, $\mathbb{B}$ is the weighted, transposed adjacency matrix of the line graph and column stochastic. Cf. [7] for details.

Under the assumption of irreducibility (of $\mathbb{B}$ or $(T(t))_{t\geq 0}$ equivalently), the long-term behavior of the semigroup orbits $\{T(t)x \mid t \geq 0\}$ as $t \to \infty$ has been studied in [7] with respect to the uniform and in [8] with respect to the strong operator topology. The aim in each case was to show asymptotic periodicity, i.e., convergence to a periodic rotation group on $L^1(\Gamma)$, Γ the unit circle, in the sense of [7, Theorem 4.10] and [8, Theorem 1], respectively.

It turned out that in each case the asymptotic behavior is governed by properties of the infinite matrix $\mathbb{B}$. In fact, the necessary and sufficient condition for asymptotic periodicity was $1 \in \mathrm{P}\sigma(\mathbb{B})$, which, since $\mathbb{B}$ was assumed to be irreducible, means almost periodicity of $\mathbb{B}$.

Lemma 4.1. The semigroup $(T(t))_{t\geq 0}$ is irreducible if and only if $\mathbb{B}$ is irreducible; the semigroup $(T(t))_{t\geq 0}$ is relatively strongly compact if and only if $\{\mathbb{B}^n \mid n \in \mathbb{N}_0\}$ is relatively (strongly) compact, i.e., $\mathbb{B}$ is almost periodic.

The proof can be found in [7, Proposition 4.9] and [14], respectively.

We drop the irreducibility assumption and work with the form (3.4) of $\mathbb{B}$ from the previous section, and use the results in the irreducible case from [8] (in particular [8, Theorem 16]). Compare the approach of [17, Theorem 4.10] for finite, non-strongly connected networks.

We obtain the following abstract characterization of (generalized) asymptotic periodicity of $(T(t))_{t\geq 0}$.

Theorem 4.2. For an infinite, column-stochastic matrix $\mathbb{B}$ the following are equivalent.

(i) $\mathbb{B}$ satisfies one of the equivalent assertions of Theorem 3.7.

(ii) $\{\mathbb{B}^n \mid n \in \mathbb{N}_0\}$ is relatively (strongly) compact.

(iii) $(T(t))_{t\geq 0}$ given by (4.1) is relatively strongly compact.

(iv) $(T(t))_{t\geq 0}$ given by (4.1) converges strongly to a sum of irreducible periodic semigroups in the following sense: There are a positive projection P and pairwise disjoint positive projections P_j, $j \in J \subset \mathbb{N}$, all commuting with $(T(t))_{t\geq 0}$, such that

(iv.1) P_jX, $j \in J$, is a closed sublattice of X isomorphic to $L^1(\Gamma)$;

(iv.2) the semigroups $(R_j(t)) := (T_j(t)P_j)$ restricted to P_jX are periodic and irreducible and similar to the rotation group on $L^1(\Gamma)$ of the same period;

(iv.3) for all $f \in L^1([0, 1], \ell^1)$ we have

$$\|T(t)f - T(t)Pf\| = \Big\| T(t)f - \Big(\sum_{j\in J} R_j(t)P_jPf\Big)\Big\| \xrightarrow{t\to\infty} 0. \quad (4.3)$$

Proof. (i) $\iff$ (ii) is contained in Theorem 3.7 while (ii) $\iff$ (iii) holds by Lemma 4.1.

If we assume (iii), then we have the Jacobs-Glicksberg-deLeeuw decomposition $X = X_r \oplus X_s = PX \oplus \ker P$ for $(T(t))_{t\geq 0}$ with properties as described in [10, Theorem V.2.14] (compare [8, Lemma 11]). In particular,

$$PX = \overline{\text{lin}}\{f \in D(A) \mid Af = i\alpha f,\, \alpha \in \mathbb{R}\},$$

and the difference $T(t) - T(t)P$ converges strongly to 0.

Furthermore, by the already established equivalence of (i) and (iii) we may assume that $\mathbb{B}$ has the form as in Lemma 3.2, with irreducible, almost periodic restrictions B_j of $\mathbb{B}$ to the $\mathbb{B}$-invariant ideals I_j. The $\mathbb{B}$-invariant ideals $I_j = \ell^1(N_j)$ with support sets $N_j \subset \mathbb{N}$ lead (by (4.1)) to $(T(t))_{t\geq 0}$-invariance of the closed ideals

$$\mathcal{I}_j := L^1([0,1] \times N_j)$$

of $X = L^1\left([0,1], \ell^1\right) \cong L^1([0,1] \times \mathbb{N})$. We denote the restricted semigroup by $(T_j(t))_{t\geq 0}$. Since the restriction B_j of $\mathbb{B}$ to $\ell^1(N_j)$ is irreducible, so is $(T_j(t))_{t\geq 0}$ (by [7, Proposition 4.9]).

By Proposition 2.1 we have $1 \in \mathrm{P}\sigma(B_j)$, thus [8, Theorem 16] implies strong asymptotic periodicity of $(T_j(t))_{t\geq 0}$ and (iv.1) and (iv.2) are satisfied with Jacobs-Glicksberg-deLeeuw projection P_j. We still write P_j for the composition of P_j with the band projection onto $\mathcal{I}_j$.

Using that P and P_j are the Jacobs-Glicksberg-deLeeuw projections with respect to $(T(t))_{t\geq 0}$ on X and $(T_j(t))_{t\geq 0}$ on $\mathcal{I}_j$, respectively, we now show that

$$Pf = \sum_{j\in J} P_j P f \quad \text{for all } f \in X. \tag{4.4}$$

Suppose $f = Pf$ is an eigenvector to a purely imaginary eigenvalue $i\alpha$ of A. Then $T(t)f = e^{i\alpha t} f$ for all $t \geq 0$, in particular $T(n)f = e^{i\alpha n} f$ for all $n \in \mathbb{N}$. By (4.1), $f(s) \in \ell^1$ and $\mathbb{B}f(s) = e^{i\alpha n} f(s)$ for almost all $s \in [0,1]$. Since by Lemma 3.2 all eigenvectors of $\mathbb{B}$ corresponding to unimodular eigenvalues have their support in $\bigcup_{j\in J} N_j$, we obtain $f(s) = \sum_{j\in J} f_j(s)$ with $f_j(s) \in I_j$ for almost all $s \in [0,1]$, i.e., $f = \sum_{j\in J} f_j$.

By $(T(t))_{t\geq 0}$-invariance and disjointness of the $\mathcal{I}_j$, every f_j is an eigenvector of A, hence

$$f_j \in P_j X = \overline{\text{lin}}\{f \in D(A) \cap \mathcal{I}_j \mid Af = i\alpha f,\, \alpha \in \mathbb{R}\} = PX \cap \mathcal{I}_j$$

and thus $f_j = P_j f_j = P_j P f$ for all $j \in J$. The claim (4.4) follows by totality of these eigenvectors in PX.

Hence we have

$$T(t)Pf - \sum_{j\in J} (T(t)P_j)Pf = T(t)\Big(Pf - \sum_{j\in J} P_j P f\Big) = 0 \quad \text{for all } f \in X$$

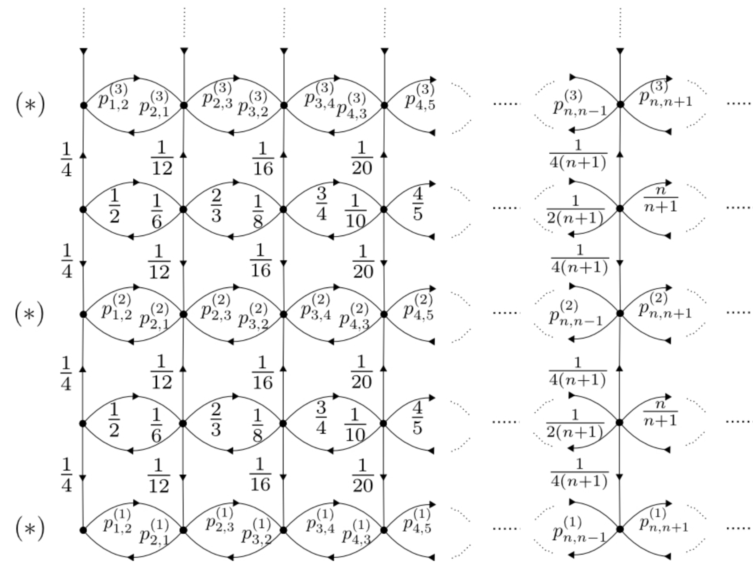

FIGURE 1.

and obtain the assertion from $T(t) - T(t)P \to 0$ strongly.

(iv) $\Rightarrow$ (iii): Since $(T_j(t))_{t\geq 0}$ is periodic on P_jX, it is relatively strongly compact for every $j \in J$. Hence by linearity and density, every

$$f \in \operatorname{ran} P = \overline{\operatorname{lin}}\Big(\bigcup_{j\in J} P_j X\Big)$$

has relatively compact orbit. Then the orbit of $f \in X$ is contained in the sum of the relatively strongly compact orbits $T(\cdot)Pf$ and $T(\cdot)(1-P)f = T(\cdot)f - T(\cdot)Pf$, which is strongly stable, hence $T(\cdot)f$ is relatively strongly compact. Q.E.D.

In the line of the network papers [7, 8] we obtain the following interpretations.

Remark 4.3. The transport process described by (4.1) can be considered as a flow on an infinite, directed, weighted graph $G = (V, E)$ where mass is transported along the edges $\mathsf{e}_j \in E$ in the given directions, and is redistributed into the vertices $\mathsf{v}_i \in V$ to the outgoing edges according to their weights. Then $\mathbb{B}$ is the weighted, transposed adjacency matrix of the line graph, and its entries are the weights, where non-zero entries express connections between the edges. Cf. [7] and [8] for details.

We call a weighted graph *essentially positive recurrent* if its adjacency matrix $\mathbb{B}$ satisfies one of the equivalent conditions of Theorem 3.7. Thus, by Theorem 4.2, the graph is essentially positive recurrent if and only if $(T(t))_{t\geq 0}$ converges strongly to a sum of irreducible periodic groups $(R_j(t))_{t\geq 0}$ where $j \in J$.

We show that distinct invariant strongly connected components (V_j, E_j) of the underlying graph lead to disjoint minimal $(T(t))_{t\geq 0}$-invariant ideals and vice versa.

First, every invariant strongly connected component (V_j, E_j) of the graph figures as support set of a minimal $\mathbb{B}$-invariant ideal $\ell^1(N_j)$ where

$$N_j := \{k \in \mathbb{N} \mid \mathsf{e}_k \in E_j\}.$$

As in the proof of Theorem 4.2, $\mathcal{I}_j = L^1([0,1] \times N_j)$ yields a minimal $(T(t))_{t\geq 0}$-invariant ideal. Since distinct invariant strongly connected components do not intersect, the ideals are disjoint.

Second, a minimal $(T(t))_{t\geq 0}$-invariant ideal $\mathcal{I}_j = L^1([0,1]\times N_j)$, $N_j \subset \mathbb{N}$, is in particular invariant under $T(1)$. Thus (4.1) shows that $\ell^1(N_j)$ is invariant under the adjacency matrix $\mathbb{B}$, hence the subgraph (V_j, E_j), with $E_j := \{\mathsf{e}_k \mid k \in N_j\}$ and the needed vertices, has no outgoing edges. By irreducibility of $(T(t))_{t\geq 0}$ on $\mathcal{I}_j$, the restriction of the adjacency matrix to $\ell^1(N_j)$ is irreducible and the subgraph is strongly connected (by [7, Proposition 4.9]).

Thus for any essentially positive recurrent graph $(T(t))_{t\geq 0}$ converges strongly to a sum of irreducible periodic groups $(R_j(t))_{t\geq 0}$, $j \in J$, each supported by an invariant strongly connected component (V_j, E_j) of the graph. These groups can be identified with the rotation groups acting on $|J|$ disjoint polygons.

Furthermore, using the results from the irreducible case in [8, Theorem 16], the period of each rotation group $(R_j(t))$ is determined by the greatest common divisor of all cycle lengths occurring in the strongly connected component $G_j = (V_j, E_j)$. More precisely, the period of the rotation group $(R_j(t))$ is

$$p_j = \gcd\{\, l \mid \text{ there is a directed cycle of length } l \text{ in } G_j\},$$

and $(T(t))_{t\geq 0}$ converges strongly to a *periodic* group of period

$$p = \operatorname{lcm}\{\, p_j \mid j \in J\} \tag{4.5}$$

if and only if (4.5) is finite.

We now give an example of an infinite, weighted, directed graph which is essentially positive recurrrent with an adjacency matrix $\mathbb{B}$ exhausting the most general form (Theorem 3.7).

Example 4.4. In Figure 1 the horizontal strings (∗) are the invariant strongly connected components.

On the weights of the jth string we impose a drift condition which is well-known in the theory of *birth-and-death processes* (cf., e.g., [11, Theorem 1.4.1]). Foster's condition holds on the jth string if we require

$$\left(\frac{p_{1,2}^{(j)}\cdot\ldots\cdot p_{n,n+1}^{(j)}}{p_{2,1}^{(j)}\cdot\ldots\cdot p_{n+1,n}^{(j)}}\right)_{n\in\mathbb{N}} \in \ell^1 \tag{4.6}$$

(compare [8, Example 1]).

All remaining vertices have paths leading to the strongly connected components, with weights as indicated in the picture. It is easy to see that the generalized drift condition Theorem 3.7(iii) is satisfied, while Foster's condition does not hold (not even in the version defined in Corollary 3.1(v)).

By Theorem 4.2 and (4.5) in the above remarks, the flow semigroup given by (4.1) is asymptotically periodic with asymptotic period 2, since in the strongly connected components the directed cycles have length $2, 4, \ldots$ as depicted in Figure 1.

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