On the number of normal measures $\aleph_1$ and $\aleph_2$ can carry

Arthur W. Apter*

Department of Mathematics, Baruch College of CUNY, New York, New York 10010, United States of America

The CUNY Graduate Center, Mathematics, 365 Fifth Avenue, New York, New York 10016, United States of America

E-mail: awapter@alum.mit.edu

Abstract

We show that assuming the consistency of certain large cardinals (namely a supercompact cardinal with a measurable cardinal above it), it is possible to force and construct choiceless universes of ZF in which the first two uncountable cardinals $\aleph_1$ and $\aleph_2$ are both measurable and carry certain fixed numbers of normal measures. Specifically, in the models constructed, $\aleph_1$ will carry exactly one normal measure, namely $\mu_\omega = \{x \subseteq \aleph_1 \mid x \text{ contains a club set}\}$, and $\aleph_2$ will carry exactly τ normal measures, where $\tau \geq \aleph_3$ is any regular cardinal. This contrasts with the well-known facts that assuming $\mathsf{AD} + \mathsf{DC}$, $\aleph_1$ is measurable and carries exactly one normal measure, and $\aleph_2$ is measurable and carries exactly two normal measures.

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We begin with a brief introduction to our terminology, notation, and conventions. We will primarily be discussing the construction of models of Zermelo-Fraenkel (ZF) set theory without the Axiom of Choice (AC). These universes, however, will satisfy the *Axiom of Dependent Choice* (DC), a weakened form of AC which says, roughly speaking, that it is possible to make countably many arbitrary choices, each dependent on the preceding. A more precise statement of DC may be found in [3, p. 50]. We will also mention an axiom that contradicts AC, the *Axiom of Determinacy* (AD). This axiom, roughly speaking, says that certain infinite two-person games of perfect information are *determined*, i.e., one of the players must have a winning strategy for the game. A more precise statement of AD may be found in [3, p. 627].

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We will be working with different *large cardinal axioms*, i.e., axioms asserting the existence of cardinal numbers not provable in either ZF or $\mathsf{ZF} + \mathsf{AC}$ alone. Large cardinals, both with and without AC, have played a significant role in modern set theory, as can be seen by consulting [3]. A large cardinal axiom that will be of particular importance to us is the axiom asserting the existence of a *measurable cardinal.* The cardinal κ is *measurable* if κ carries a κ-additive, nonprincipal ultrafilter μ, which is frequently referred to as a *measure.* The measure μ over the measurable cardinal κ is *normal* if for every function $f : \kappa \to \kappa$ such that $\{\alpha < \kappa \mid f(\alpha) < \alpha\} \in \mu$, there is some $\alpha_0 < \kappa$ with $\{\alpha < \kappa \mid f(\alpha) = \alpha_0\} \in \mu$. In addition, there is a generalization of the notion of measurable cardinal which is important for the purposes of this paper. For $\kappa < \lambda$ two cardinals, κ *is* λ *supercompact* if the set $P_\kappa(\lambda) = \{p \subseteq \lambda \mid |p| < \kappa\}$ carries a κ-additive, fine, normal ultrafilter $\mathcal{U}$. (The ultrafilter $\mathcal{U}$ over $P_\kappa(\lambda)$ is *fine* if for every ordinal $\alpha < \lambda$, $\{p \mid \alpha \in p\} \in \mathcal{U}$. The ultrafilter $\mathcal{U}$ is *normal* if for every function $f : P_\kappa(\lambda) \to \lambda$, there is an ordinal $\alpha < \lambda$ such that $\{p \in P_\kappa(\lambda) \mid f(p) = \alpha\} \in \mathcal{U}$.) The existence of a λ supercompact cardinal for a cardinal $\lambda > \kappa$ is much stronger in consistency strength than the existence of a measurable cardinal.

There are some combinatorial notions which are also relevant to the forthcoming discussion. Suppose κ is a regular uncountable cardinal. The set $C \subseteq \kappa$ is called *closed unbounded* or *club* if for every $\alpha < \kappa$, there is some $\beta \geq \alpha$, $\beta \in C$ (unbounded), and for every increasing sequence $\langle \beta_\alpha \mid \alpha < \gamma\rangle$ of elements of C, of any length $\gamma < \kappa$, $\sup(\langle \beta_\alpha \mid \alpha < \gamma\rangle) \in C$ (closed). If the set $C \subseteq \kappa$ is unbounded and closed under suprema of increasing γ sequences for $\gamma < \kappa$ a regular cardinal, then C is called γ *closed unbounded* or γ *club.* Finally, for ordinals α, β, γ with $\gamma \leq \beta \leq \alpha$, the *partition property* $\alpha \to (\beta)^\gamma$ means that for every $F : [\alpha]^\gamma \to 2$, there is some $X \subseteq \alpha$ having order type β such that $|F''[X]^\gamma| = 1$. The Axiom of Choice contradicts all such partition properties with γ infinite.

We continue now with the main narrative. It is a consequence of $\mathsf{AD}+\mathsf{DC}$ that $\aleph_1$ and $\aleph_2$ are measurable cardinals, $\aleph_1$ carries exactly one normal measure (namely $\mu_\omega = \{x \subseteq \aleph_1 \mid x \text{ contains a club set}\}$), and $\aleph_2$ carries exactly two normal measures. This follows since assuming $\mathsf{AD} + \mathsf{DC}$, $\aleph_1 \to (\aleph_1)^{\aleph_1}$, $\forall \delta < \aleph_2[\aleph_2 \to (\aleph_2)^\delta]$, and if a successor cardinal κ satisfies the *weak partition property* $\forall \delta < \kappa[\kappa \to (\kappa)^\delta]$, then κ is measurable and carries exactly the same number of normal measures as regular cardinals below κ. The proofs of these first two facts (along with a historical discussion) can be found in [5, pp. 1–7, 39–45, 67], and the proof of this last fact can be found in [4, § 2, pp. 416–420].[1]

[1] In fact, if a successor cardinal κ satisfies the weak partition property, then any normal measure κ carries must be of the form $\mu_\delta = \{x \subseteq \kappa \mid x \text{ contains a set which is } \delta \text{ club}\}$,

When the Axiom of Determinacy is not assumed, however, the number of normal measures that $\aleph_1$ and $\aleph_2$ can carry if both of these cardinals are measurable is not so clear. This motivates the purpose of this note, which is to shed new light on this situation and construct, via forcing over a ground model of ZFC containing large cardinals, models of $\mathsf{ZF}+\mathsf{DC}$ in which both $\aleph_1$ and $\aleph_2$ are measurable, $\aleph_1$ carries exactly one normal measure (specifically, μ_ω), and $\aleph_2$ carries exactly τ normal measures, where $\tau \geq \aleph_3$ is any regular cardinal. More explicitly, we will prove the following two theorems.

Theorem 1. Let $V^* \models \mathsf{ZFC}+\mathsf{GCH}+$"$\kappa < \lambda$ are such that κ is supercompact and λ is the least measurable cardinal above κ" + "$\tau > \lambda^+$ is a fixed but arbitrary regular cardinal". There are then a generic extension V of V^*, a partial ordering $\mathbb{P} \in V$, and a symmetric submodel $N \subseteq V^{\mathbb{P}}$ such that $N \models \mathsf{ZF} + \mathsf{DC}+$"$\kappa = \aleph_1$ and $\lambda = \aleph_2$ are measurable cardinals". In N, the regular cardinals greater than or equal to λ are the same as in V (which has the same cardinal and cofinality structure at and above λ as V^*), $\aleph_1$ carries exactly one normal measure (namely μ_ω), and $\aleph_2$ carries exactly τ normal measures.

Theorem 2. Let $V^* \models \mathsf{ZFC}+\mathsf{GCH}+$"$\kappa < \lambda$ are such that κ is supercompact and λ is the least measurable cardinal above κ". There are then a generic extension V of V^*, a partial ordering $\mathbb{P} \in V$, and a symmetric submodel $N \subseteq V^{\mathbb{P}}$ such that $N \models \mathsf{ZF} + \mathsf{DC}+$"$\kappa = \aleph_1$ and $\lambda = \aleph_2$ are measurable cardinals". In N, the regular cardinals greater than or equal to λ are the same as in V (so $\aleph_3$ is regular), $\aleph_1$ carries exactly one normal measure (namely μ_ω), and $\aleph_2$ carries exactly $\aleph_3$ normal measures.

We note that of necessity, the models constructed witnessing the conclusions of Theorems 1 and 2 must violate the Axiom of Choice. This is since a successor cardinal which is measurable can only exist in a choiceless model of ZF. Also, as Schindler's results of [7] show, the consistency strength of two successive measurable cardinals is quite large, thereby requiring strong assumptions for the proofs of Theorems 1 and 2.

Our method of proof will hinge on a use of Woodin's technique exposited in [2, Theorem 1] for forcing both $\aleph_1$ and $\aleph_2$ to be simultaneously measurable. Since this method will ensure that μ_ω is a normal measure over $\aleph_1$, μ_ω is in fact the *unique* normal measure over $\aleph_1$. We encapsulate this in the following easy proposition.

Proposition 3. $\mathsf{ZF} \vdash$"If μ_ω is a normal measure over $\aleph_1$, then it is the *unique* normal measure over $\aleph_1$".

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where $\delta < \kappa$ is a regular cardinal. (When $\kappa = \aleph_1$, this definition of μ_ω coincides with the one given earlier.) From this and the preceding, we may immediately infer that assuming AD + DC, $\aleph_2$ carries exactly two normal measures, which are given by μ_ω and $\mu_{\aleph_1}$.

Proof. By standard arguments (see [3, Exercise 8.8, p. 104 and Lemma 8.11, p. 96]), in ZF alone, any normal measure over a measurable cardinal must contain all club sets. Thus, if μ_ω is a normal measure, and in particular, an ultrafilter over $\aleph_1$, it is automatically the case that for any other normal measure μ over $\aleph_1$, $\mu_\omega \subseteq \mu$. It then immediately follows as usual that $\mu = \mu_\omega$. Q.E.D.

Having completed our introductory comments, we turn now to the proofs of Theorems 1 and 2. We stress that we will be presuming henceforth a reasonably good understanding of large cardinals and forcing. Some knowledge of the proof of [2, Theorem 1] will be helpful as well.

Proof. Our presentation is similar in spirit to that given in [1]. As in the proofs of [1, Theorems 1 & 2], we present a unified proof of the results in question. We begin by noting that by [1, Lemma 2.1] and the remarks immediately preceding, we may assume without loss of generality that V^* has been generically extended to a model V having certain additional key properties. For Theorem 1, V has the same cardinal and cofinality structure as V^*, and $V \vDash \mathsf{ZFC}$+"$\kappa < \lambda$ are such that κ is λ supercompact and λ is the least measurable cardinal above λ"+"λ carries exactly τ normal measures". Here, τ is as in the statement of Theorem 1. For Theorem 2, $V \vDash \mathsf{ZFC}$+"$\kappa < \lambda$ are such that κ is λ supercompact and λ is the least measurable cardinal above λ"+"λ carries exactly λ^+ normal measures".

We are now able to describe the symmetric inner model N which will witness the conclusions of either Theorem 1 or Theorem 2. What we are about to present is almost completely dependent on the discussion of the proof of [2, Theorem 1]. Since this material is quite complicated, we will not duplicate it here, but will refer readers to [2] for any missing details.

The forcing conditions $\mathbb{P}$ to be used are $\mathrm{SC}(\kappa, \lambda) \times \mathrm{Coll}(\omega, <\kappa)$, where $\mathrm{SC}(\kappa, \lambda)$ is supercompact Radin forcing as described in [2], and $\mathrm{Coll}(\omega, <\kappa)$ is the usual Lévy collapse of κ to $\aleph_1$. Let G be V-generic over $\mathbb{P}$. Take $\mathcal{G}$ as the set of restrictions of G described in [2, p. 595], which code collapses of cardinals in the open interval (ω, κ) to $\aleph_1$ and collapses of cardinals in the open interval (κ, λ) to κ^+. N is then given by $\mathrm{HVD}^{V[G]}(\mathcal{G})$, the class of all sets hereditarily V-definable in $V[G]$ from an element of the set $\mathcal{G}$.

Standard arguments now show that $N \vDash \mathsf{ZF}$. By [2, Lemmas 1.1–1.5] and the intervening remarks, $N \vDash \mathsf{DC}$ + "$\kappa = \aleph_1$" + "$\lambda = \kappa^+ = \aleph_2$"+"For any normal measure $\mathcal{U} \in V$ over λ, $\mathcal{U}' = \{x \subseteq \lambda \mid \exists y \subseteq x[y \in \mathcal{U}]\}$ is a normal measure over λ"+"$\aleph_1$ is measurable via μ_ω". In addition, [2, Lemmas 1.2 & 1.3] and their proofs provide us with the following proposition.

Proposition 4. For every set $r = \{r_1, r_2\}$ for which $r_1, r_2 \in \mathcal{G}$, there is a term $\dot{r}$ such that any formula mentioning only (canonical terms for ground model sets and) $\dot{r}$ may be decided in $V[r]$ the same way as in $V[G]$. Further, $V[r]$ is obtained by forcing with a partial ordering having size less than λ. In particular, any set of ordinals in N is actually a member of $V[r]$ for the appropriate r.

Proposition 4 will be critical in the proof of Theorems 1 and 2 and the following two lemmas.

Lemma 5. Suppose $\mathcal{U}^* \in N$ is a normal measure over λ. Then for some normal measure $\mathcal{U} \in V$ over λ, $\mathcal{U}^* = \{x \subseteq \lambda \mid \exists y \subseteq x[y \in \mathcal{U}]\}$.

Proof. Our proof is almost identical to the proof of [1, Lemma 2.2]. Let τ be a term for $\mathcal{U}^*$. Since $\mathcal{U}^* \in N$, we may choose $r = \{r_1, r_2\}$ with $r_1, r_2 \in \mathcal{G}$ such that τ mentions only $\dot{r}$ and canonical terms for sets in V. By Proposition 4, the set $\mathcal{U}^* \restriction r = \mathcal{U}^* \cap V[r] \in V[r]$, which immediately implies that $\mathcal{U}^* \restriction r$ is in $V[r]$ a normal measure over λ. Again by Proposition 4 and the Lévy-Solovay results [6], it must consequently be the case that for some $\mathcal{U} \in V$ a normal measure over λ, $\mathcal{U}^* \restriction r$ is definable in $V[r]$ as $\{x \subseteq \lambda \mid \exists y \subseteq x[y \in \mathcal{U}]\}$. Therefore, since $\mathcal{U} \subseteq \mathcal{U}^* \restriction r \subseteq \mathcal{U}^*$ and $\mathcal{U}' = \{x \subseteq \lambda \mid \exists y \subseteq x[y \in \mathcal{U}]\}$ as defined in N is an ultrafilter over λ, $\mathcal{U}' = \mathcal{U}^*$. This completes the proof of Lemma 5. Q.E.D.

Lemma 6. In N, the regular cardinals greater than or equal to λ are the same as in V.

Proof. We mimic the proof of [1, Lemma 2.3]. Let β and γ be arbitrary ordinals, and suppose $N \vDash$ "$f : \beta \to \gamma$ is a function". Since f may be coded by a set of ordinals, by Proposition 4, $f \in V[r]$ for some $r = \{r_1, r_2\}$ where $r_1, r_2 \in \mathcal{G}$. Since $V[r]$ is obtained by forcing with a partial ordering having size less than λ, f cannot witness that any V-regular cardinal greater than or equal to λ has a different cardinality or cofinality. This completes the proof of Lemma 6. Q.E.D.

By Lemmas 5 and 6 and our earlier remarks, if V^* and V are as in the proof of Theorem 1, then N witnesses the conclusions of Theorem 1. Similarly, Lemmas 5 and 6 and our earlier remarks imply that if V^* and V are as in the proof of Theorem 2, then N witnesses the conclusions of Theorem 2. This completes the proofs of Theorems 1 and 2. Q.E.D.

We conclude by asking the general question of how many normal measures each of $\aleph_1$ and $\aleph_2$ can carry when both $\aleph_1$ and $\aleph_2$ are simultaneously measurable. Because of the present state of set theoretic technology, the results under $\mathsf{AD} + \mathsf{DC}$ and of this paper seem to be all that can be currently established. These theorems paint what appears to be a rather incomplete picture of what we conjecture the general situation most likely is, i.e., that

$\aleph_1$ and $\aleph_2$ can carry any number of normal measures when both are measurable.

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