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# Hyperbolic distributions in finance

ERNST EBERLEIN and ULRICH KELLER

*Institut für Mathematische Stochastik, Universität Freiburg, Hebelstrasse 27, 79104 Freiburg, Germany*

Distributional assumptions for the returns on the underlying assets play a key role in valuation theories for derivative securities. Based on a data set consisting of daily prices of the 30 DAX shares over a three-year period, we investigate the distributional form of compound returns. After performing a number of statistical tests, it becomes clear that some of the standard assumptions cannot be justified. Instead, we introduce the class of hyperbolic distributions which can be fitted to the empirical returns with high accuracy. Two models based on hyperbolic Lévy motion are discussed. By studying the Esscher transform of the process with hyperbolic returns, we derive a valuation formula for derivative securities. The result suggests a correction of standard Black–Scholes pricing, especially for options close to expiration.

**Keywords:** absolute continuous change of measure; hyperbolic distributions; hyperbolic Lévy motion; option pricing; statistical analysis of stock price data

## 1. Introduction

In valuation theories for derivative securities as well as in other questions in finance the distributional form of the returns on the underlying assets plays a key role. In this paper, after investigating classical assumptions, in particular the normality hypothesis, we introduce a model which fits the data with high accuracy and draw some conclusions concerning option pricing.

Let $(P_t)_{t\geq 0}$ denote the price process of a security, in particular of a stock. In order to allow comparison of investments in different securities, we shall investigate the rates of return defined by

$$X_t = \log P_t - \log P_{t-1}. \tag{1}$$

Like most authors, we prefer these rates, which correspond to continuous compounding, to the alternative

$$Y_t = (P_t - P_{t-1})/P_{t-1}. \tag{2}$$

The reason for this is that the return over $n$ periods, for example $n$ days, is then just the sum

$$X_t + \ldots + X_{t+n-1} = \log P_{t+n-1} - \log P_{t-1}. \tag{3}$$

This does not hold for $Y_t$. Another aspect we have in mind is that the underlying price process is a continuous-time process from which discrete time series are drawn at equidistant time-points. But for continuous-time processes returns with continuous compounding are the natural choice. The fact that the underlying process is a continuous-time process led us to use $t$ both as a continuous and as a discrete parameter. What is actually meant should be clear from the context. Numerically the difference between $X_t$ and $Y_t$ is negligible since $Y_t - X_t = \frac{1}{2}X_t^2 + \frac{1}{6}X_t^3 + \ldots$ and $X_t$ is typically of the order $10^{-2}$.

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The standard continuous-time model for stock prices is the geometric Brownian motion:

$$P_t = P_0 \exp\{(\mu - \sigma^2/2)t + \sigma B_t\} \tag{4}$$

which solves the stochastic differential equation

$$\mathrm{d}P_t = \mu P_t \mathrm{d}t + \sigma P_t \mathrm{d}B_t, \tag{5}$$

where $\sigma > 0$ and $\mu$ are constant coefficients denoting volatility and drift and $(B_t)_{t\geq 0}$ is a standard Brownian motion. It is the model underlying the Black–Scholes formula (Black and Scholes 1973). Often the model itself is called the Black–Scholes model, although it goes back to Samuelson (1965) who improved on Bachelier's (1900) ingenious introduction of Brownian motion. Its key properties are that it is multiplicative and complete. The latter allows duplication of the cash flow of derivative securities and thus the valuation of these products by arbitrage (see, for example, Harrison and Pliska 1981). We do not discuss the vast literature where this model has been generalized (see, for example, the discussion in Aase 1984). Instead, we will concentrate on an empirical study and try to identify the correct distributions for the returns. Among the many models which have been investigated besides the normal distribution, let us mention in particular the stable Pareto distribution (to be discussed later), the Student distribution (see, for example, Blattberg and Gonedes 1974) and finite discrete mixtures of normals (see, for example, Kon 1984). But it is the class of hyperbolic distributions which will turn out to be an excellent candidate and which will provide a more realistic model. This class of distributions was introduced by Barndorff-Nielsen (1977), and we are indebted to him for the hint to explore this class after we had presented results on stock returns. Hyperbolic distributions have been used in various scientific fields. One area is the modelling of the distribution of particle size from aeolian sand deposits. An excellent reference for this project is Barndorff-Nielsen *et al.* (1985). Other areas to be mentioned are the modelling of turbulence (see, for example Barndorff-Nielsen *et al.* 1989) and the use of hyperbolic distributions in statistical physics.

Figure 1 shows a typical path of geometric Brownian motion, where the parameters are $P_0 = 100$, $\mu = 0.5$ and $\sigma = 0.08$. Due to the self-similarity of Brownian motion $(B_t)_{t\geq 0}$ which enters as the source of randomness, the qualitative picture does not change if we change the time-scale. In contrast to this, real stock-price paths change drastically if we look at them on different time-scales. Figure 2 shows daily stock prices of five major companies over a period of three years, while Fig. 3 shows a path if one goes down to the level of price changes during a single day. The picture shows the price at which Siemens shares were traded on the Frankfurt Stock Exchange on 2 March 1992 from the opening at 10.30 a.m. to the close at 1.30 p.m. Comparing the model (Fig. 1) with reality (Fig. 3), it becomes obvious that its paths are too erratic. One could say that at least locally model (4) is too random. This justifies the introduction of discrete models, by which we mean models with price changes at equidistant discrete time-points only. Their paths can be considered as a first approximation of reality, where price changes occur at random time-points as can be seen in Fig. 3. The interplay between discrete- and continuous-time models was investigated in Eberlein (1992) by comparing them pathwise. Again for discrete models the question of the correct return distributions arises.

The returns resulting from the geometric Brownian motion are increments of a Brownian motion process, thus are independent and normally distributed. Tests applied to real data in the following show the degree to which the assumption of normality fails. In contrast to this, hyperbolic

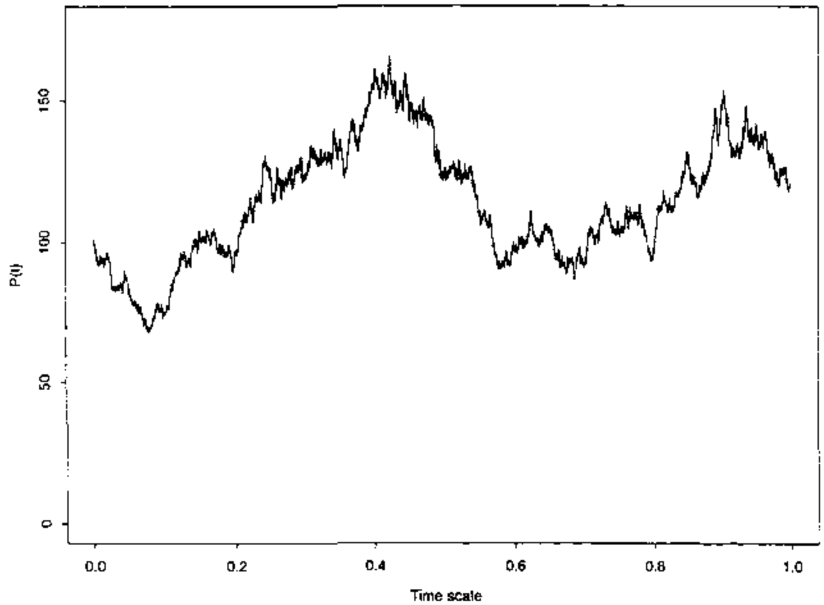


**Figure 1.** Simulation of geometric Brownian motion

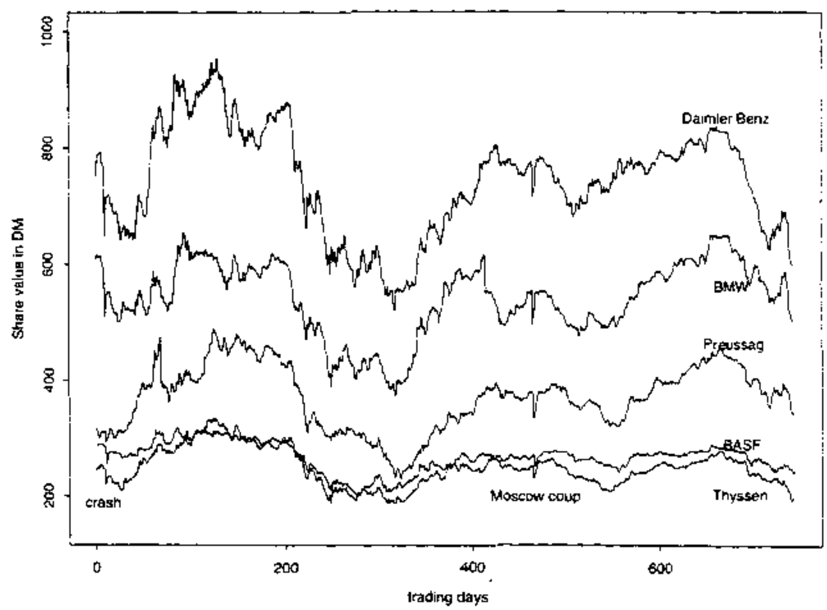


**Figure 2.** Daily stock prices from October 1989 to September 1992

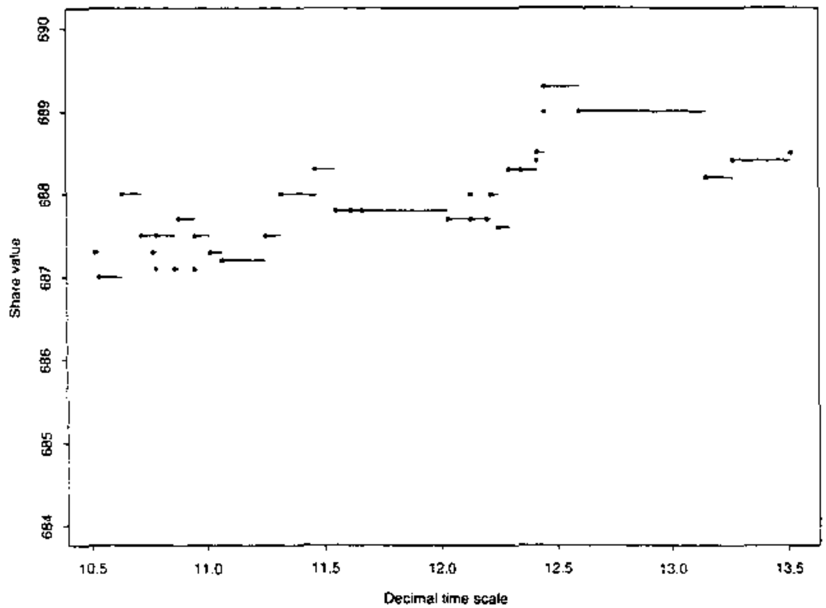


**Figure 3.** Intraday value of Siemens shares, 2 March 1992

distributions can be fitted to the empirical distributions with high accuracy. The data we are exploring are daily KASSA prices of 10 of the 30 stocks which compose the DAX, the German stock index, during the three-year period from 2 October 1989 to 30 September 1992. This gives time series of 745 data-points each for the returns. The data are corrected for dividend payouts, that is to say, the returns on ex-dividend days are defined by

$$X_t = \log(P_t + d_t) - \log P_{t-1}, \tag{6}$$

where $d_t$ is the amount paid in dividend on day $t$. Note that dividends are paid only once a year for German stocks. Two unusual price changes occurred in this period: the crash on 16 October 1989 and the drop as a consequence of the coup in Moscow on 19 August 1991. The latter can be seen as a deep notch in Fig. 2. The 10 stocks considered here were chosen on account of their large trading volume and also because of the specific activity of the company in order to get a reasonable representation of the market. This choice has no influence on the conclusions.

## 2. Testing for classical assumptions

It is well known that the normal distribution is a poor model for stock returns. In this section we show explicitly how large the discrepancy actually is. A qualitative yet very powerful method for testing the goodness of fit are quantile–quantile plots. Figure 4 shows normal QQ plots for the returns of BASF and Deutsche Bank. The deviation from a straight line and thus from normality is obvious. Also shown are the corresponding empirical densities and the normal density. It is evident

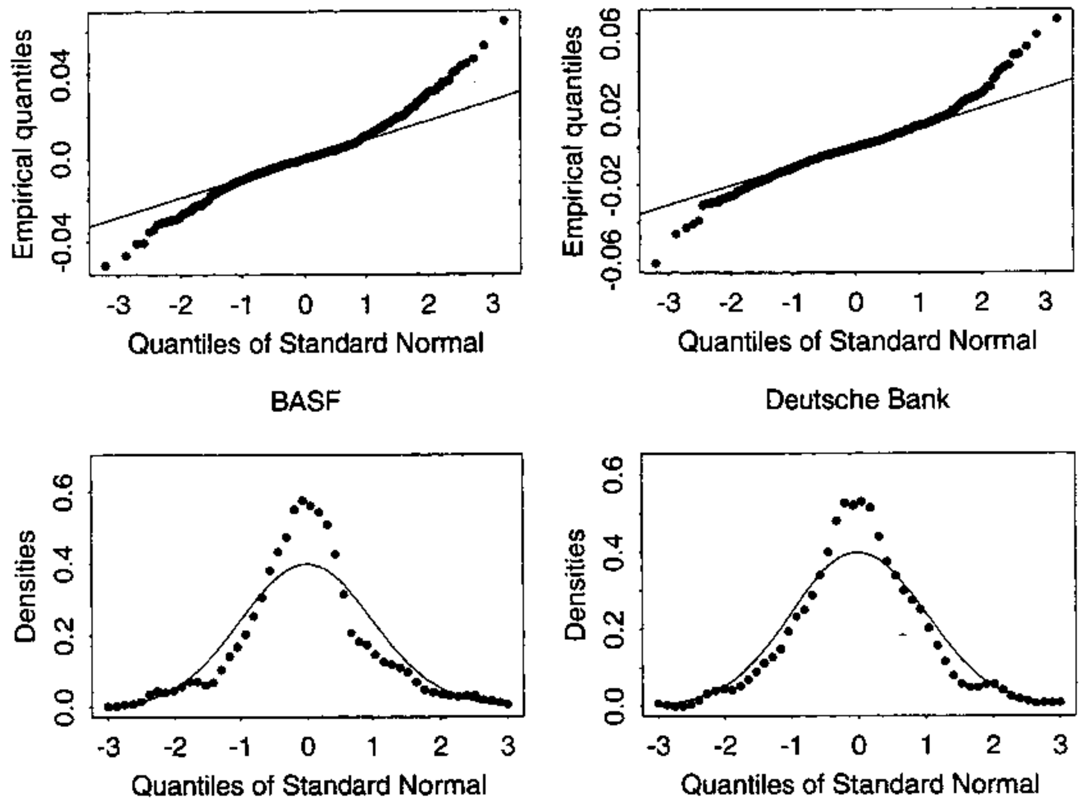


**Figure 4.** Normal QQ plots and density plots

that there is considerably more mass around the origin and in the tails than the standard normal distribution can provide.

The order of magnitude of the deviation can be seen from Table 3 below. Counting the number of returns in the interval $(-k\sigma, k\sigma)$, for $k \leq 5$, we compare the relative frequencies which are given in the first line for each stock with the expected normal frequencies which are given in the first line of the table. The values for the 10 companies considered are quite uniform and differ from the normal at the origin, that is to say, in the interval $(-\sigma, \sigma)$ by 0.1. In particular, note that the empirical distributions have roughly 5000 times the mass of the normal distribution in the tails starting at $5\sigma$.

We continue with $\chi^2$ tests for normality. To avoid any problems arising from partition sensitivity, three different estimation procedures were considered. Let $\hat{\chi}_1^2$ denote the test statistic computed with cells of equal probability $(1/k)$, while $\hat{\chi}_2^2$ is used for cells of equal width. The second cell structure was modified by collapsing outer cells, such that the expected value of observations becomes greater than 5. The third procedure is very much the same as the second, but starting with $k = 40$ instead of $k = 22$. For all stocks the null hypothesis is rejected at the level $\alpha = 0.01$. As an example we cite the corresponding values for the two stocks considered above. Full-length tables are available in Eberlein and Keller (1994). $\chi^2_{k-1;0.99}$ denotes the 0.99 quantile of the $\chi^2$ distribution with $k-1$ degrees of freedom.

| | $\hat{\chi}_1^2$ | $\chi^2_{k-1;0.99}$ | $\hat{\chi}_2^2$ | $\chi^2_{k-1;0.99}$ | $\hat{\chi}_3^2$ | $\chi^2_{k-1;0.99}$ |
|---|---|---|---|---|---|---|
| BASF | 104.02 | 38.93 | 62.54 | 18.48 | 93.74 | 27.69 |
| Deutsche Bank | 88.02 | 38.93 | 55.88 | 18.48 | 87.92 | 30.58 |

Another standard method of testing for normality is to compute certain functions of the moments of the sample data and to compare them with the expected values for a normal population. We use two such tests which, moreover, have the favourable feature of scale and location invariance, so we are able to test the composite hypothesis by means of these tests. If we denote by $m_k = n^{-1}\sum_{i=1}^{n}(x_i - \bar{x})^k$ the sample moment of order $k$, then the test statistics are given by $\hat{K} = m_4/m_2^2 - 3$ and $\hat{S} = m_3/m_2^{3/2}$, measuring kurtosis and skewness of the sample, which should both be zero under the assumption of normality. Again for brevity's sake we mention the results for BASF and Deutsche Bank. For the former the skewness is 0.52 and the kurtosis 7.40, while for the latter we got 1.40 and 16.88. For all stocks the hypothesis is rejected at the 1% level. Finally, because of the long tails which we observe for financial data, the range $x_{\max} - x_{\min}$ of the sample should be a good indicator for non-normality. Indeed the studentized range test turned out to be another useful tool. The corresponding statistic is given by

$$\hat{u} = \frac{x_{\max} - x_{\min}}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}}.$$

$\hat{u}$ is 12.59 and 14.56 for the two stocks considered above, which means rejection at the smallest level $\alpha = 0.005$ which can be found in the tables in David *et al.* (1954).

We will now discuss briefly another class of distributions proposed by Mandelbrot (1963), the stable Pareto distributions with characteristic exponent $\alpha$, denoted by $SP(\alpha)$. In the symmetric case stable Pareto distributions are defined by the log-characteristic function

$$\log\phi(t) = \mathrm{i}\delta t - (c|t|)^{\alpha} \tag{7}$$

where $\delta$ denotes the location parameter, $c$ the scale parameter and $\alpha$ the characteristic exponent, defined in the interval $0 < \alpha \leq 2$. For $\alpha = 2$ it coincides with the normal distribution, for $\alpha = 1$ it gives the Cauchy distribution. As an immediate consequence, one sees that if $(X_i)_{1\leq i\leq n}$ are independent $SP(\alpha)$-distributed variables, then $X = \sum_{i=1}^{n} a_i X_i$ is again $SP(\alpha)$-distributed. This means that the class is stable with respect to building portfolios of independent components, which was also a desirable property of models with normal components. For $\alpha < 2$ stable distributions are more peaked around the centre than the normal ones and have arbitrarily heavy tails. In some sense this rules them out from the beginning: for $\alpha < 2$ the variance is infinite and for $\alpha \leq 1$ even the first moment does not exist. It is obvious that models of stock returns at least for blue chips, that is, the major stocks traded at the exchange, should have finite moments. The price changes observed from one day to the next are less than 20% for these stocks. Therefore the variables are bounded.

Several authors have so far rejected the stable hypothesis for American stocks (see, for example, Barnea and Downes 1973; Hagerman 1978). Because of the analytic difficulties with this class of distributions the stability-under-addition property is often used to test the stable hypothesis. Assuming independence of the returns of a security, one should again get $SP(\alpha)$-distributed variables – with the same $\alpha$ – if one considers sums. A technique for estimating the characteristic exponent $\alpha$ has been developed by Fama and Roll (1968; 1971). They propose a fractile estimator given by

$$\hat{f}_{SP(\alpha)} = 0.827 \frac{\hat{x}_f - \hat{x}_{1-f}}{\hat{x}_{f=0.72} - \hat{x}_{f=0.28}}, \qquad 0.95 \le f \le 0.97, \tag{8}$$

where $\hat{x}_f$ denotes the $f$-quantile of the sample data and the corresponding $SP(\alpha)$-fractile is given in their paper. For the test the return values are split into groups of increasing size and each group is summed. Then the characteristic exponent is estimated for each resulting distribution. If the value of $\hat{\alpha}$ increases with increasing sum size we have to reject the stable hypothesis. To overcome the problem of serial correlation between successive returns, which has been discussed by many authors, we are following a method proposed by Fielitz and Rozelle (1983). For this data are randomized before building groups. Table 1 gives the results of the estimation procedure for our data set and leads to rejection of the stable hypothesis. We do not report the results for the original data, which show a similar behaviour for $\alpha$, but the tendency to approach 2 is not as strong as in Table 1. So we conclude that the presence of serial correlation induces a higher kurtosis of monthly returns. As Barnea and Downes (1973) showed, a finite mixture of non-Gaussian stable laws with varying scale parameters will also exhibit the tendency for $\alpha$ to increase, but with a limiting value of $\alpha$ less than 2. Since in our case the value of $\hat{\alpha}$ reaches 2 or at least comes close to 2 for most of the shares, we are additionally led to conclude that for monthly returns a Gaussian distribution is appropriate. This was indeed suggested earlier by many authors (see, for example, Officer 1972; Kon 1984) and supports our thesis that for different time-scales different classes of distributions have to be considered. See also Fig. 7 in Section 3 in this context.

To demonstrate the extent of serial correlation Fig. 5 shows autocorrelations for Deutsche Bank for time-lags (days) up to 30. Autocorrelation is present but of only moderate size. Because of the current interest in autocorrelations of squared increments we added these for our data as well. As

**Table 1.** Estimates of the characteristic exponent $\alpha$ for randomized data series

| | *Sum length* | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 5 | 10 | 15 | 20 | 25 |
| BASF | 1.44 | 1.56 | 1.58 | 1.47 | 1.60 | 2* | 2* | 2* |
| BMW | 1.48 | 1.71 | 1.84 | 1.61 | 2* | 1.92 | 2* | 2* |
| Daimler Benz | 1.57 | 1.60 | 1.81 | 1.76 | 1.72 | 2* | 2* | 2* |
| Deutsche Bank | 1.60 | 1.60 | 1.98 | 1.98 | 1.92 | 1.78 | 2* | 1.72 |
| Dresdner Bank | 1.50 | 1.82 | 1.92 | 1.95 | 1.75 | 2* | 2* | 2* |
| Hoechst | 1.48 | 1.65 | 1.56 | 2* | 1.63 | 2* | 2* | 2* |
| Preussag | 1.62 | 1.74 | 1.68 | 2* | 2* | 1.83 | 1.59 | 1.89 |
| Siemens | 1.60 | 1.66 | 1.76 | 1.68 | 1.77 | 2* | 2* | 2* |
| Thyssen | 1.66 | 1.77 | 1.90 | 1.96 | 1.40 | 1.97 | 2* | 1.99 |
| VW | 1.71 | 1.60 | 1.65 | 2* | 1.43 | 1.94 | 2* | 2* |

*These values are corrected to 2, because the estimation procedure resulted in a value $\alpha > 2$ which is not in the permitted domain. For sums of length 25 the resulting series have a length of only $n_{25} = 29$ with our series of $n = 745$ data-points for each company. This may lead to considerable sampling errors.

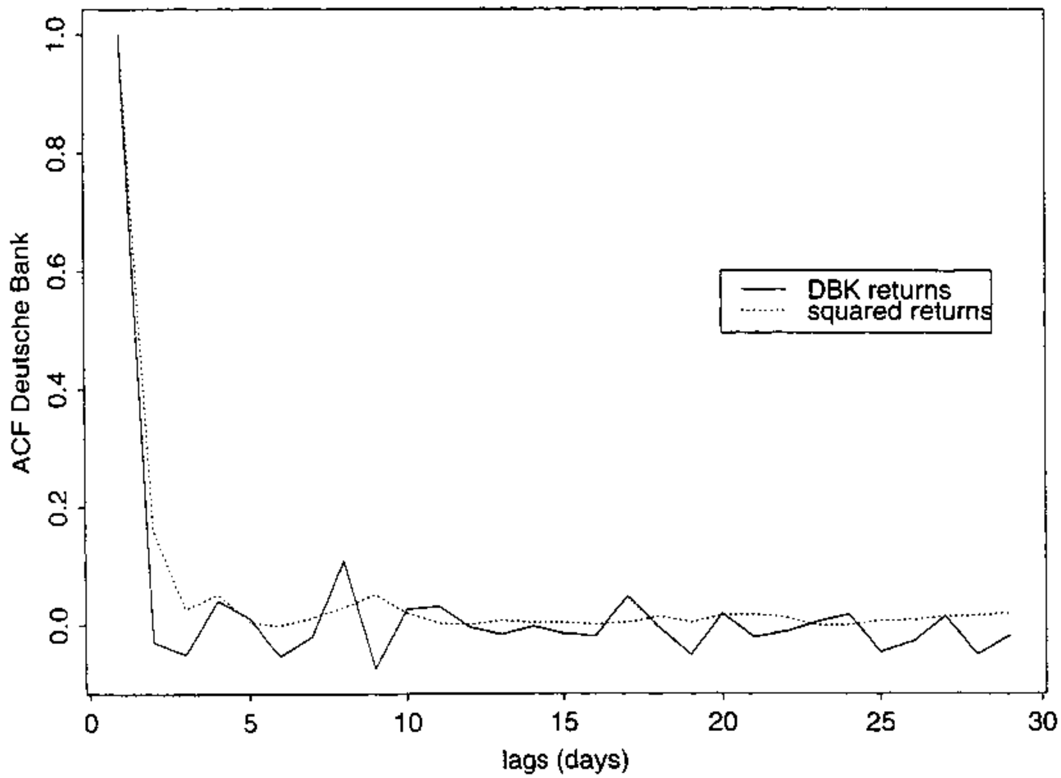


**Figure 5.** Autocorrelations

the picture shows, the autocorrelations of squared returns are not higher than those for the returns themselves.

## 3. Hyperbolic distributions and financial data

Hyperbolic distributions are characterized by their log-density being a hyperbola. Recall that for normal distributions the log-density is a parabola, so one can expect to obtain a reasonable alternative for heavy tail distributions. The parametrization of the hyperbolic density, which we will use later, is given by

$$\operatorname{hyp}(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2 - \beta^2})} \exp(-\alpha\sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu)), \tag{9}$$

where $K_1$ denotes the modified Bessel function of the third kind with index 1. The first two of the four parameters, namely $\alpha$ and $\beta$ with $\alpha > 0$ and $0 \leq |\beta| < \alpha$, determine the shape of the distribution, while the other two, $\delta$ and $\mu$, are scale and location parameters. With $\xi = (1 + \delta\sqrt{\alpha^2 - \beta^2})^{-1/2}$ and $\chi = \xi\beta/\alpha$ one gets a different parametrization $\operatorname{hyp}(x; \chi, \xi, \delta, \mu)$,

**Table 2.** Estimated parameters for the hyperbolic distribution

| | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\delta}$ | $\hat{\mu}$ |
|---|---|---|---|---|
| BASF | 108.82 | 1.3550 | 0.0014 | −0.0005 |
| BMW | 89.72 | 4.7184 | 0.0009 | −0.0015 |
| Daimler Benz | 88.19 | 4.1713 | 0.0019 | −0.0014 |
| Deutsche Bank | 108.60 | 0.3972 | 0.0030 | −0.00001 |
| Dresdner Bank | 110.94 | −0.1123 | 0.0016 | 0.0002 |
| Hoechst | 104.78 | 2.8899 | 0.0020 | −0.0007 |
| Preussag | 83.51 | 3.0702 | 0.0031 | −0.0008 |
| Siemens | 112.65 | 6.4287 | 0.0033 | −0.0011 |
| Thyssen | 94.27 | 0.0792 | 0.0073 | −0.0003 |
| VW | 83.85 | −4.0652 | 0.0077 | 0.0009 |

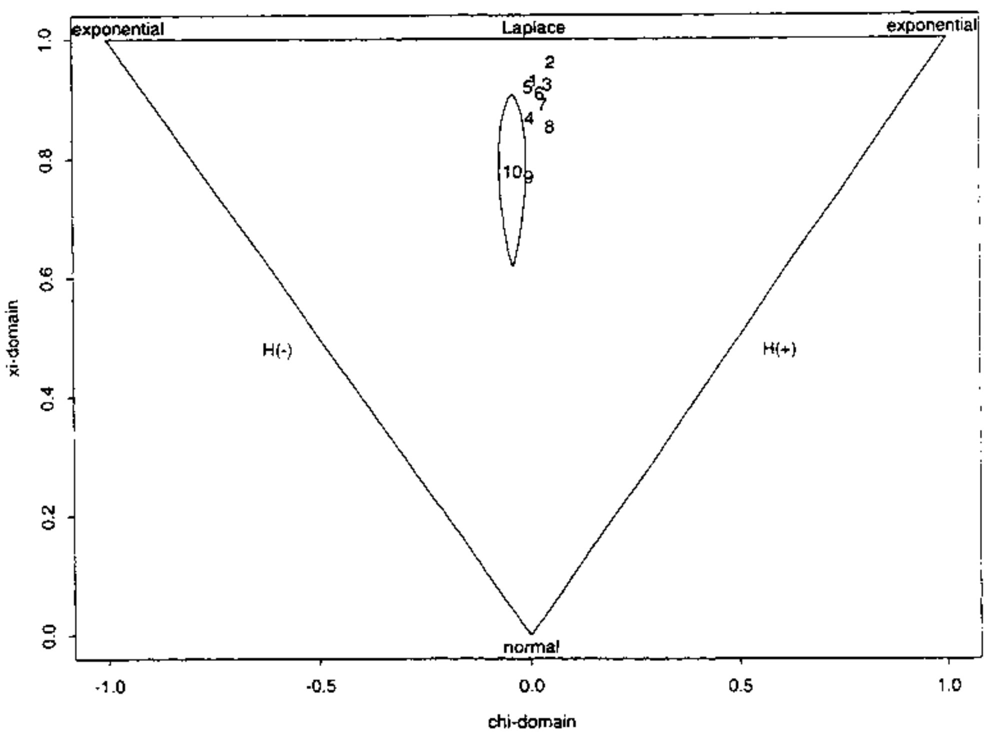


**Figure 6.** The hyperbolic shape triangle. The location of the estimates for the invariant parameters $(\hat{\xi}, \hat{\chi})$ is indicated by the numbers 1 to 10, where the numbers correspond to the order of the shares in Table 2. The locations of the limits of hyperbolic distributions are indicated, where H(−) (or H(+)) means a generalized inverse Gaussian distribution. The solid line marks the 95%-confidence region for the estimates of VW (number 10)

**Table 3.** Frequency distributions

| | $< 1\sigma$ | $< 2\sigma$ | $< 3\sigma$ | $< 4\sigma$ | $< 5\sigma$ | $> 5\sigma$ | |
|---|---|---|---|---|---|---|---|
| Standard normal | 0.683 | 0.955 | 0.997 | 1.000 | 1.000 | 0.000 000 6 | |
| BASF | 0.774 | 0.944 | 0.987 | 0.996 | 0.997 | 0.002 684 6 | (2) |
| hyp ($\alpha = 108.82, \beta = 1.36, \delta = 0.0014, \mu = 0$) | 0.769 | 0.948 | 0.988 | 0.997 | 0.999 | 0.000 638 3 | (0.5) |
| BMW | 0.796 | 0.949 | 0.987 | 0.996 | 0.996 | 0.004 026 8 | (3) |
| hyp ($\alpha = 89.72, \beta = 4.72, \delta = 0.0009, \mu = 0$) | 0.779 | 0.951 | 0.989 | 0.998 | 0.999 | 0.000 534 2 | (0.4) |
| Daimler Benz | 0.785 | 0.954 | 0.987 | 0.995 | 0.999 | 0.001 342 3 | (1) |
| hyp ($\alpha = 88.19, \beta = 4.17, \delta = 0.0019, \mu = 0$) | 0.774 | 0.950 | 0.989 | 0.998 | 0.999 | 0.000 582 6 | (0.4) |
| Deutsche Bank | 0.797 | 0.964 | 0.988 | 0.993 | 0.997 | 0.002 684 6 | (2) |
| hyp ($\alpha = 108.60, \beta = 0.38, \delta = 0.003, \mu = 0$) | 0.777 | 0.954 | 0.990 | 0.998 | 1.000 | 0.000 426 7 | (0.3) |
| Dresdner Bank | 0.807 | 0.950 | 0.988 | 0.995 | 0.997 | 0.002 684 6 | (2) |
| hyp ($\alpha = 110.94, \beta = -0.11, \delta = 0.0016, \mu = 0$) | 0.781 | 0.953 | 0.990 | 0.998 | 1.000 | 0.000 449 7 | (0.3) |
| Hoechst | 0.785 | 0.944 | 0.989 | 0.996 | 0.997 | 0.002 684 6 | (2) |
| hyp ($\alpha = 104.78, \beta = 2.89, \delta = 0.002, \mu = 0$) | 0.771 | 0.950 | 0.989 | 0.998 | 0.999 | 0.000 543 9 | (0.4) |
| Preussag | 0.792 | 0.957 | 0.991 | 0.995 | 0.997 | 0.002 684 6 | (2) |
| hyp ($\alpha = 83.51, \beta = 3.07, \delta = 0.0031, \mu = 0$) | 0.778 | 0.953 | 0.990 | 0.998 | 1.000 | 0.000 447 2 | (0.3) |
| Siemens | 0.792 | 0.953 | 0.989 | 0.995 | 0.997 | 0.002 684 6 | (2) |
| hyp ($\alpha = 112.65, \beta = 6.43, \delta = 0.0033, \mu = 0$) | 0.768 | 0.950 | 0.989 | 0.998 | 0.999 | 0.000 513 9 | (0.4) |
| Thyssen | 0.762 | 0.960 | 0.989 | 0.993 | 0.996 | 0.004 026 8 | (3) |
| hyp ($\alpha = 94.27, \beta = 0.07, \delta = 0.0073, \mu = 0$) | 0.788 | 0.957 | 0.991 | 0.998 | 1.000 | 0.000 354 7 | (0.3) |
| VW | 0.772 | 0.962 | 0.991 | 0.995 | 0.997 | 0.002 684 6 | (2) |
| hyp ($\alpha = 83.85, \beta = -4.07, \delta = 0.0077, \mu = 0$) | 0.741 | 0.938 | 0.984 | 0.996 | 0.999 | 0.001 079 0 | (0.8) |

In the columns the relative frequencies of the returns in the interval $(-k\sigma, k\sigma)$ are compared with the probabilities of the fitted hyperbolic distributions. The tails starting at $5\sigma$ are given in the column labelled by $> 5\sigma$, where the numbers in brackets give the absolute frequencies found in this region and the expected values for the corresponding hyperbolic distribution.

which has the advantage that $\xi$ and $\chi$ are invariant under transformations of scale and location. The new invariant shape parameters vary in the triangle $0 \leq |\chi| < \xi < 1$, which was therefore called the shape triangle by Barndorff-Nielsen *et al.* (1985). For $\xi \to 0$ the normal distribution is obtained as a limiting case; for $\xi \to 1$ one gets the symmetric and asymmetric Laplace distribution; for $\chi \to \pm\xi$ it is a generalized inverse Gaussian distribution (in fact, a distribution having density of the form (10)); and, finally, for $|\chi| \to 1$ we will end up with an exponential distribution.

It was pointed out by Barndorff-Nielsen (1977) that the hyperbolic distribution can be represented as a normal variance–mean mixture where the mixing distribution is generalized

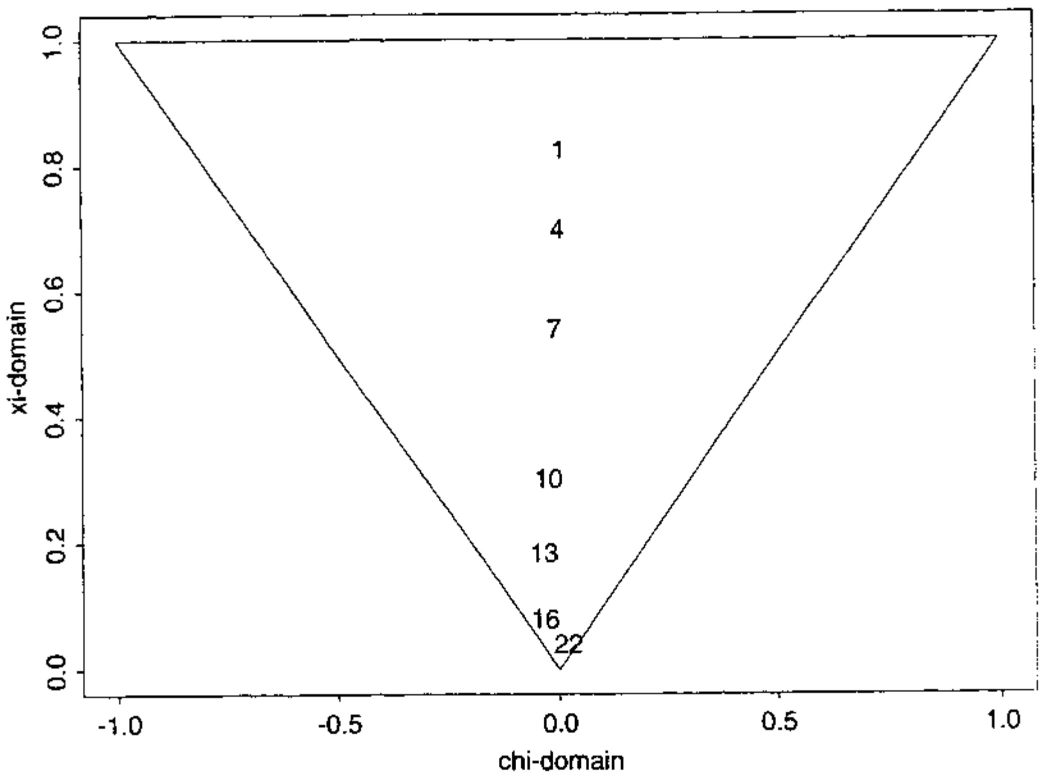


**Figure 7.** Parameter estimates for the returns of Commerzbank as time-lags increase

inverse Gaussian with density

$$d_{\mathrm{IG}}(x) = \frac{\sqrt{\psi/\gamma}}{2K_1(\sqrt{\psi\gamma})}\exp\left\{-\frac{1}{2}(\gamma x^{-1} + \psi x)\right\}, \qquad x > 0. \tag{10}$$

Let $\gamma = \delta^2$ and $\psi = \alpha^2 - \beta^2$ and consider a normal distribution with mean $\mu + \beta\sigma^2$ and variance $\sigma^2$ such that $\sigma^2$ is endowed with the distribution (10). Then the mixture is a hyperbolic distribution with density (9).

To estimate the parameters of the hyperbolic distribution given by (9) we used a computer program described in Blæsild and Sørensen (1992). Assuming independent and identically distributed variables, a maximum likelihood estimation is performed by the program and the results for our data set are given in Table 2.

Figure 6 shows the estimates $(\hat{\xi}, \hat{\chi})$ which are all towards the top of the triangle, thus far from normality. They are well centred, which means that the distributions are nearly symmetric. Moreover, the indicated 95% confidence region for the estimates of VW gives first evidence that hyperbolic distributions are appropriate for daily returns. To assess the goodness of fit, in Table 3 we provide the frequency distributions in the same form as in Section 2 (see also Fama 1965).

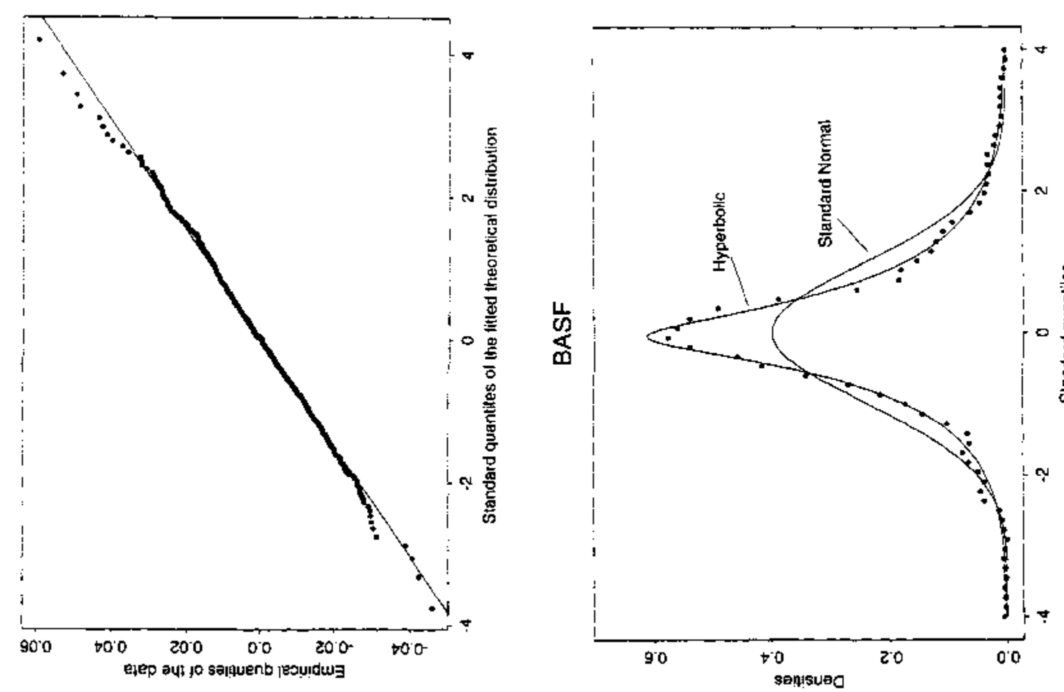


**Figure 8.** QQ and density plots for BASF

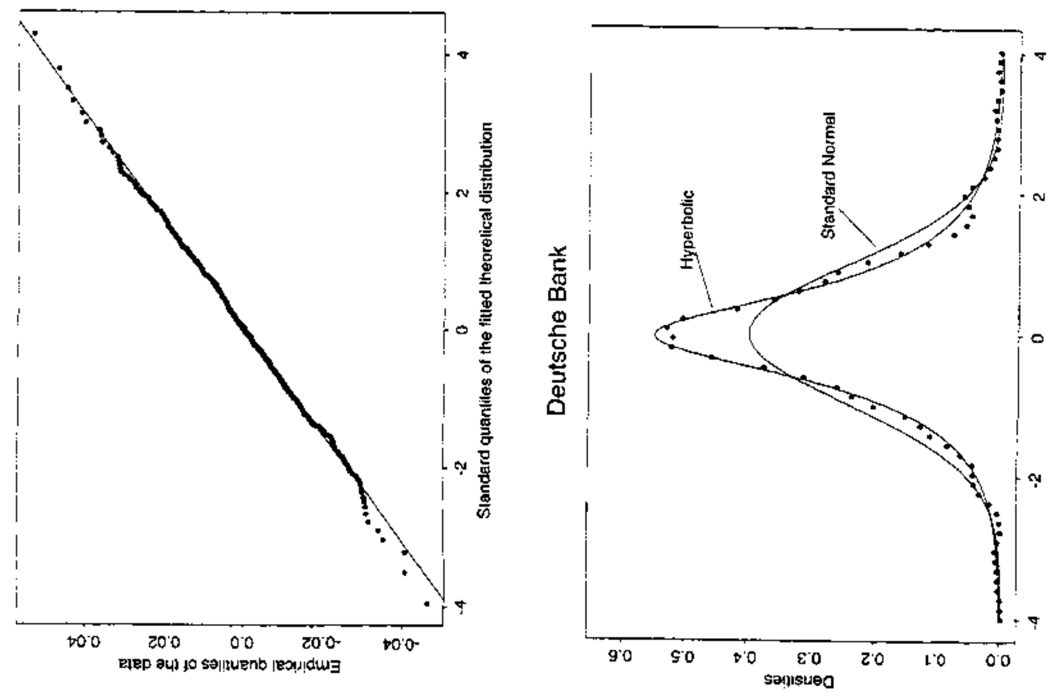


**Figure 9.** QQ and density plots for Deutsche Bank

For a quantitative analysis we will again make use of $\chi^2$ tests with the same estimation procedures mentioned above (see Section 2 for a closer description). In this case it is correct to accept the hypothesis of a hyperbolic distribution at significance level $\alpha$ whenever $\hat{\chi}_i^2 \leq \chi^2_{k-1-4;1-\alpha}$, $i = 1,2,3$, since we were estimating the four parameters of the distributions. For all stocks the hypothesis of a hyperbolic distribution is accepted at the level $\alpha = 0.01$. We give the explicit values for two cases.

| | $\hat{\chi}_1^2$ | $\chi^2_{k-5;0.99}$ | $\hat{\chi}_2^2$ | $\chi^2_{k-5;0.99}$ | $\hat{\chi}_3^2$ | $\chi^2_{k-5;0.99}$ |
|---|---|---|---|---|---|---|
| BASF | 26 | 33.41 | 5.45 | 13.28 | 10.7 | 26.22 |
| Deutsche Bank | 17.5 | 33.41 | 9.27 | 13.28 | 15.94 | 26.22 |

Kolmogorov–Smirnov test values vary between 0.70 and 1.20 for all shares and are well below 1.63, representing a lower bound for the 1% level for this number of observations. Let us also note that the values for the normal distribution are, with one exception, all above 1.8.

Using a larger data set, namely stock prices from 1 January 1988 to 24 May 1994, we also applied the fitting procedure to returns for time-lags of 1, 4, 7, ..., 22 trading days. The resulting estimates for $\chi$ and $\xi$ for the data for Commerzbank are given in the shape triangle in Fig. 7. It is striking how the parameters tend to the normal distribution limit as the lags increase.

Finally, we return to the graphical methods used above to underline our conclusion. In Figs 8 and 9 we see the QQ plots for two of the shares. Because of the strong effect outliers have on QQ plots, for these plots the two outliers produced by the crash in October 1989 and by the Moscow coup were removed. To stay consistent we did the same for the density pictures and the plots of Fig. 4 in Section 2, although there is no effect as far as the density plots are concerned.

Figures 8 and 9 show an excellent fit. Comparison with Fig. 4 suggests a strong preference for this model over the classical one. So we arrive at the conclusion that daily stock returns should be modelled by hyperbolic distributions.

## 4. Hyperbolic Lévy motion

Hyperbolic distributions are infinitely divisible. This was shown by Barndorff-Nielsen and Halgreen (1977) by proving infinite divisibility of the generalized inverse Gaussian distribution, which is used in the representation of the hyperbolic distribution as a mixture of normals. Given the empirical findings on stock returns in the previous section we concentrate now on the symmetric centred case, that is, where $\beta = \mu = 0$. This means also $\chi = 0$ in the second parametrization mentioned above. Consequently in the symmetric centred case, using $\zeta = \xi^{-2} - 1$ for notational convenience, the density (9) can be written as

$$\text{hyp}_{\zeta,\delta}(x) = \frac{1}{2\delta K_1(\zeta)} \exp\left(-\zeta\sqrt{1 + \left(\frac{x}{\delta}\right)^2}\right). \tag{11}$$

Let $(Z_t^{\zeta,\delta})_{t\geq 0}$ be the Lévy process defined by the infinitely divisible hyperbolic law with density $\text{hyp}_{\zeta,\delta}$, that is, the process with stationary independent increments such that $Z_0^{\zeta,\delta} = 0$ and $\mathscr{L}(Z_1^{\zeta,\delta})$ has

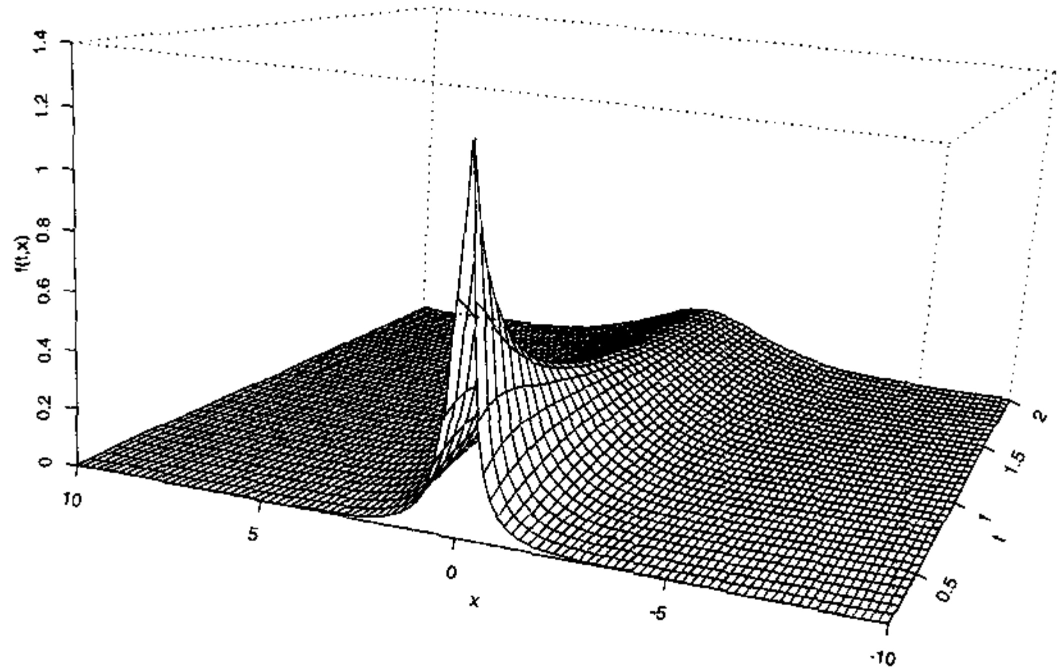


**Figure 10.** Convolution semigroup densities

density $\text{hyp}_{\zeta,\delta}$. We call $(Z_t^{\zeta,\delta})_{t\geq 0}$ *hyperbolic Lévy motion*. Since $(Z_t^{\zeta,\delta})_{t\geq 0}$ is a process with centred independent increments it is a martingale. By stationarity and independence the second moments are given by $\mathrm{E}\{(Z_t^{\zeta,\delta})^2\} = t\mathrm{E}\{(Z_1^{\zeta,\delta})^2\} < \infty$ with

$$\mathrm{E}\{(Z_1^{\zeta,\delta})^2\} = \delta^2 \frac{1}{\zeta}\frac{K_2(\zeta)}{K_1(\zeta)}. \tag{12}$$

As we will see later, all moments of $(Z_t^{\zeta,\delta})_{t\geq 0}$ exist, therefore $(Z_t^{\zeta,\delta})_{t\geq 0}$ is in fact an $L^p$-martingale $(p \geq 1)$.

Using the mixture representation, one can easily compute the characteristic function of the hyperbolic distribution given by $\text{hyp}_{\zeta,\delta}$, namely

$$\phi(u;\zeta,\delta) = \frac{\zeta}{K_1(\zeta)} \frac{K_1(\sqrt{\zeta^2+\delta^2u^2})}{\sqrt{\zeta^2+\delta^2u^2}}, \tag{13}$$

so one can see that hyperbolic distributions are not closed under convolution. By construction it follows that $\mathrm{E}(\mathrm{e}^{iuZ_t^{\zeta,\delta}}) = \phi_t(u;\zeta,\delta) = \{\phi(u;\zeta,\delta)\}^t$ for $t \geq 0$. Therefore the density of $\mathscr{L}(Z_t^{\zeta,\delta})$ is given by the Fourier inversion formula

$$f_t^{\zeta,\delta}(x) = \frac{1}{\pi}\int_0^\infty \cos(ux)\phi_t(u;\zeta,\delta)\mathrm{d}u. \tag{14}$$

The integral on the right-hand side can be computed numerically. The results are shown in Fig. 10, where the densities of the convolution semigroup are represented for values of $t$ varying between 0.2 and 2. The shape and the scale parameters were chosen as $\zeta = \delta = 1$.

From the explicit form of $\phi$ one can derive the fact that the process $(Z_t^{\zeta,\delta})_{t\geq 0}$ does not have a Gaussian part, since the first cumulant of the characteristic function given by (13) is asymptotically linear in $u$. Therefore it is a purely discontinuous process. We can choose a cadlag version. The process can be represented in the form

$$Z_t^{\zeta,\delta} = \int_0^t \int_{\mathbb{R}\setminus\{0\}} x(\mu^Z(\cdot, \mathrm{d}u, \mathrm{d}x) - \mathrm{d}u\ \nu(\mathrm{d}x)), \tag{15}$$

where $\mu^Z$ is the random measure of jumps

$$\mu^Z(\omega, \mathrm{d}t, \mathrm{d}x) = \sum_{s>0} 1_{\{\Delta Z_s(\omega)\neq 0\}} \epsilon_{(s, \Delta Z_s(\omega))}(\mathrm{d}t, \mathrm{d}x) \tag{16}$$

associated with the process $(Z_t^{\zeta,\delta})_{t\geq 0}$. Its compensator $\mathrm{d}u\ \nu(\mathrm{d}x)$ contains the Lévy measure $\nu$ of the distribution given by $\mathrm{hyp}_{\zeta,\delta}$. Note that the compensator is deterministic, because of the independence of the increments. Deriving the Lévy–Khinchine formula for $(Z_t^{\zeta,\delta})_{t\geq 0}$, the density $g(x;\zeta,\delta)$ of the Lévy measure $\nu$ can be computed as follows. Again from the representation of $\mathrm{hyp}_{\zeta,\delta}$ as a mixture of normals one gets

$$\phi_t(u) = \exp\left\{ t\kappa\left(\frac{u^2}{2}\right)\right\} \tag{17}$$

where $\kappa(s) = \log \mathrm{E}\{\exp(-sY)\}$ and $Y$ is an inverse Gaussian variable with density given by (10). Using the integral representation (5) given in Halgreen (1979) and a modified expansion for $\log(1 + u^2/2)$ given on p. 16 of the same paper, which in our case is written

$$-\log\left(1 + \frac{u^2}{2}\right) = \int_{-\infty}^{\infty} \frac{\mathrm{e}^{iux} - 1 - iux}{|x|} \mathrm{e}^{-\sqrt{2}|x|}\mathrm{d}x \tag{18}$$

one gets

$$\kappa\left(\frac{u^2}{2}\right) = \int_{-\infty}^{\infty} (\mathrm{e}^{iux} - 1 - iux) g(x;\zeta,\delta)\mathrm{d}x. \tag{19}$$

This is in view of (17) the Lévy–Khinchine representation for the integrable process (15) with

$$g(x;\zeta,\delta) = \frac{1}{\pi^2|x|}\int_0^{\infty} \frac{\exp\left(-|x|\sqrt{2y + (\zeta/\delta)^2}\right)}{y(J_1^2(\delta\sqrt{2y}) + Y_1^2(\delta\sqrt{2y}))}\mathrm{d}y + \frac{\exp(-|x|)}{|x|}. \tag{20}$$

Here $J_1$ and $Y_1$ are the Bessel functions of the first and second kind, respectively. Using well-known asymptotic relations about $J_1$ and $Y_1$ (see formulae 9.1.7, 9 and 9.2.1, 2 in Abramowitz and Stegun 1968) the denominator of the integrand is asymptotically equivalent to a constant for $y \to 0$ and to $y^{-\frac{1}{2}}$ for $y \to \infty$. Therefore, one can deduce that $g(x)$ behaves like $1/x^2$ at the origin, which means that there is an infinite number of small jumps in every finite time-interval.

As a model for stock prices the natural candidate is now the process driven by a hyperbolic Lévy motion $(Z_t^{\zeta,\delta})_{t\geq 0}$

$$\mathrm{d}Y_t = \rho Y_{t-}\mathrm{d}t + Y_{t-}\mathrm{d}Z_t^{\zeta,\delta}. \tag{21}$$

This can also be written in the form $\mathrm{d}Y_t = Y_{t-}\mathrm{d}X_t$ with $X_t = Z_t^{\zeta,\delta} + \rho t$. The solution of this equation is the Doléans–Dade exponential

$$Y_t = Y_0 \exp(Z_t^{\zeta,\delta} + \rho t) \prod_{0<s\leq t} (1 + \Delta Z_s^{\zeta,\delta}) \mathrm{e}^{-\Delta Z_s^{\zeta,\delta}}. \tag{22}$$

If $[X]^c$ denotes the continuous part of the quadratic variation $[X]$ of $X$, note that $[X]_t^c = [Z^{\zeta,\delta}]_t^c = 0$ since $(Z_t^{\zeta,\delta})_{t\geq 0}$ is a purely discontinuous process. Therefore the term $[X]^c$ does not appear in the exponential above and the quadratic variation process is given by

$$[Z^{\zeta,\delta}]_t = \sum_{s\leq t} (\Delta Z_s^{\zeta,\delta})^2 = \int_0^t \int_{\mathbb{R}\setminus\{0\}} x^2 \mu^Z(\cdot, \mathrm{d}u, \mathrm{d}x). \tag{23}$$

Note also that $Y_t > 0$ only for $\Delta Z_t^{\zeta,\delta} > -1$, so bankruptcy by means of crash is included in this model. To avoid problems, we have to consider the process $\tilde{Y}_t = Y_t 1_{t<T}$ with a stopping time $T$ given by $T = \inf\{t > 0 | \Delta Z_t^{\zeta,\delta} < -1\}$.

Looking at the returns of this price process, one has not only the hyperbolic increment $(Z_t^{\zeta,\delta} - Z_{t-1}^{\zeta,\delta})$ coming from the exponent in (22), but in addition the log of the product. Regarding only small jumps, this is approximately given by $-\frac{1}{2}\sum_{t-1<s\leq t}(\Delta Z_s^{\zeta,\delta})^2 = -\frac{1}{2}([Z^{\zeta,\delta}]_t - [Z^{\zeta,\delta}]_{t-1})$.

The model which produces exactly hyperbolic returns along time-intervals of length 1 is

$$S_t = S_0 \exp(Z_t^{\zeta,\delta}). \tag{24}$$

Like (22) this model is not complete. There is no unique equivalent martingale measure, that is, an equivalent measure such that the discounted process $(\mathrm{e}^{-rt}S_t)_{t\geq 0}$ is a martingale. Here $r$ denotes the (daily) interest rate. But following an idea of Gerber and Shiu (1995), for $(S_t)_{t\geq 0}$ it is easy to compute explicitly at least one equivalent martingale measure. This can be used for the valuation of derivative securities. One must be aware that this valuation cannot be justified as in the case of unicity.

Let $f_t^{\zeta,\delta}$ be the density of $\mathscr{L}(Z_t^{\zeta,\delta})$. For some real number $\theta$ we can define a new density

$$f_t^{\zeta,\delta}(x;\theta) = \frac{\mathrm{e}^{\theta x} f_t^{\zeta,\delta}(x)}{\int_{-\infty}^{\infty} \mathrm{e}^{\theta y} f_t^{\zeta,\delta}(y)\mathrm{d}y}. \tag{25}$$

Under the corresponding probability $P^\theta$ the process is again a Lévy process, which is called the *Esscher transform* of the original process. From another point of view, all processes that are Lévy processes under a certain $P^\theta$ for $\theta \in \Theta = \{\theta \in \mathbb{R} | \int \mathrm{e}^{\theta x}\mathrm{d}P_t(x) < \infty,\ \forall t \geq 0\}$ form an exponential class of processes with

$$\mathrm{d}P_t^\theta(x) = \frac{\exp(\theta x)}{N(\theta)^t} \mathrm{d}P_t(x) \tag{26}$$

where $N(\theta) = \int \mathrm{e}^{\theta x}\mathrm{d}P_1(x)$. It turns out that $\Theta = (-\alpha, \alpha)$ for the hyperbolic Lévy motion and $P_1^\theta$ is a member of the skewed hyperbolic distributions, where $\theta$ corresponds to the skewness parameter.

**Table 4.** Option prices for Deutsche Bank

| *Time to expiration* | *Share value* | *Hyperbolic price* | *Black–Scholes price* |
|---|---|---|---|
| $\tau = 2$ | 650 | 0.01 | 0.00 |
| | 700 | 5.28 | 5.56 |
| | 750 | 50.46 | 50.45 |
| $\tau = 5$ | 650 | 0.10 | 0.06 |
| | 700 | 8.82 | 9.00 |
| | 750 | 51.24 | 51.20 |
| $\tau = 10$ | 650 | 0.65 | 0.58 |
| | 700 | 12.94 | 13.07 |
| | 750 | 52.87 | 52.82 |
| $\tau = 30$ | 650 | 5.25 | 5.26 |
| | 700 | 24.00 | 24.09 |
| | 750 | 60.63 | 60.65 |

These are the prices for a European call option with an assumed strike of $\Gamma = 700$ and an assumed interest rate of $r = 0.08$ (p.a.). The values for $\zeta$ and the volatility $\sigma$ are the estimates described above.

Now we choose $\theta$ by

$$S_0 = e^{-rt}E^{\theta}(S_t). \tag{27}$$

such that $(e^{-rt}S_t)_{t\geq 0}$ is a martingale.

Let $M^{\zeta,\delta}(u,t) = E\{\exp(uZ_t^{\zeta,\delta})\}$ denote the moment generating function of the hyperbolic Lévy motion indexed by $\zeta$ and $\delta$ and consider

$$M^{\zeta,\delta}(u,t;\theta) = \int_{-\infty}^{\infty} e^{ux} f_t^{\zeta,\delta}(x;\theta)\mathrm{d}x, \tag{28}$$

the corresponding function under $P^{\theta}$. Since by stationarity $M^{\zeta,\delta}(u,t;\theta) = M^{\zeta,\delta}(u,1;\theta)^t$, from (27) we get

$$e^r = M^{\zeta,\delta}(1,1;\theta) = \frac{M^{\zeta,\delta}(\theta+1,1)}{M^{\zeta,\delta}(\theta,1)}. \tag{29}$$

As the characteristic function (13), $M^{\zeta,\delta}(u,1)$ is easy to compute:

$$M^{\zeta,\delta}(u,1) = \frac{\zeta}{K_1(\zeta)} \frac{K_1(\sqrt{\zeta^2-\delta^2u^2})}{\sqrt{\zeta^2-\delta^2u^2}}, \qquad |u| < \frac{\zeta}{\delta}. \tag{30}$$

Introducing this in (29) we get the value $\theta^*$ which defines the martingale measure as the solution of

$$r = \ln \frac{K_1\left(\sqrt{\zeta^2-\delta^2(\theta+1)^2}\right)}{K_1(\sqrt{\zeta^2-\delta^2\theta^2})} - \frac{1}{2}\ln\frac{\zeta^2-\delta^2(\theta+1)^2}{\zeta^2-\delta^2\theta^2}. \tag{31}$$

By numerical methods we can find a solution for $\theta$ given the (daily) interest rate $r$ and the parameters

$\delta$ and $\zeta$. If the derivative security is, for example, a European call option with exercise price $\Gamma$ and time to expiration $\tau$, the value at time 0 is given by

$$\mathrm{E}^{\theta^*}[\mathrm{e}^{-r\tau}(S_\tau - \Gamma)_+] \tag{32}$$

where $x_+$ denotes $\max(x, 0)$. This expectation under $P^{\theta^*}$ can be written as

$$S_0 \int_c^\infty f_\tau^{\zeta,\delta}(x;\theta^* + 1)\mathrm{d}x - \mathrm{e}^{-r\tau}\Gamma \int_c^\infty f_\tau^{\zeta,\delta}(x;\theta^*)\mathrm{d}x \tag{33}$$

where $c = \ln(\Gamma/S_0)$. Since $f_t^{\zeta,\delta}(x;\theta^*)$ is related to the original density $f_t^{\zeta,\delta}(x)$ by (25), where $f_t^{\zeta,\delta}(x)$ is given by (14), this value can be computed numerically (see Table 4).

Some practitioners might prefer to see the volatility $\sigma$ as an explicit parameter, although $\delta$ is the more natural parameter here. By (12), $\sigma^2 = \delta^2 K_2(\zeta)/\zeta K_1(\zeta)$. Thus for

$$\delta =: \delta_\zeta = \left(\zeta \frac{K_1(\zeta)}{K_2(\zeta)}\right)^{1/2} \tag{34}$$

we get a process $(Z_t^\zeta)_{t\geq 0} := (Z_t^{\zeta,\delta_\zeta})_{t\geq 0}$ with $\mathrm{E}\{(Z_1^\zeta)^2\} = 1$. Now one can replace $(Z_t^{\zeta,\delta})_{t\geq 0}$ by the process $(\sigma Z_t^\zeta)_{t\geq 0}$. For example, equation (21) reads

$$\mathrm{d}Y_t = \rho Y_{t-}\mathrm{d}t + \sigma Y_{t-}\mathrm{d}Z_t^\zeta, \tag{35}$$

where $\sigma$ denotes the daily volatility. In order to get $\theta^*$ from (31) one has to replace $\delta$ by $\sigma\delta_\zeta$ in that formula, and the same remark holds for the pricing formula (33).

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