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On the stochastic equation $\mathscr{L}(X) = \mathscr{L}[B(X + C)]$ and a property of gamma distributions

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This paper is concerned with the stochastic equation $X \overset{\mathscr{L}}{=} B(X+C)$, where B, X and C are independent. This equation has appeared in a number of contexts, notably in actuarial science. An apparently new property of gamma variables (Theorem 1) leads to the derivation of a new explicit example of solution of the stochastic equation (Theorem 2), where B is the product of two independent beta variables, C is gamma and X is the product of independent beta and gamma variables. Also, a number of previously known explicit examples are seen to be direct algebraic consequences of a well-known property of gamma variables.

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1. Introduction

Suppose $\{B_n, n \geq 1\}$ and $\{C_n, n \geq 0\}$ are two independent i.i.d. sequences, and consider the stochastic difference equation

$$X_{n+1} = B_{n+1}(X_n + C_n), \tag{1}$$

where $X_0 = x_0$ is a constant. Iterating (1) we get

$$X_n = x_0 B_1 \ldots B_n + \sum_{k=0}^{n-1} C_k B_{k+1} \ldots B_n. \tag{2}$$

$\{X_n\}$ is a homogeneous Markov chain. A related process is

$$Y_n = \sum_{k=1}^{n} C_k B_1 \ldots B_k. \tag{3}$$

$\{Y_n\}$ is not a Markov chain, but it can be seen that, given $x_0 = 0$, X_n and Y_n have the same distribution for any fixed $n \geq 1$ (just reverse the order of the indices of the Bs and Cs, and use the independence assumption).

Equations such as (1), (2) or (3) arise in a number of contexts (see Vervaat 1979, for some examples). In actuarial science, X_n might represent the accumulated value of amounts

$\{C_0, C_1, \ldots, C_{n-1}\}$, when the accumulating factors (i.e. one plus the rate of return) are $\{B_1, B_2, \ldots, B_n\}$. Dufresne (1990) describes the actuarial applications and also gives formulae for the moments X_n and Y_n.

Vervaat (1979) states the following sufficient conditions for the existence and uniqueness of the limit distribution of X_n as $n \to \infty$:

$$\mathrm{E}(\log B_1) < 0, \qquad \mathrm{E}(\log |C_1|)_+ < \infty. \tag{4}$$

The same conditions ensure the almost sure convergence of Y_n. When X_n converges in law the limit X must satisfy

$$X \stackrel{\mathscr{L}}{=} B(X + C), \qquad B, X \text{ and } C \text{ independent.} \tag{5}$$

A number of explicit examples of solutions of (5) have been found; see Vervaat (1979) and Chamayou and Letac (1991). Embrechts and Goldie (1994) provide further results on the convergence of X_n and Y_n.

Theorem 2 is a new explicit solution of (5), based on a certain property of gamma variables (Theorem 1). The law of X turns out to be the product of independent beta and gamma distributions.

It is necessary to make some brief observations on notation. The variable G_a has a $\Gamma(a, 1)$ distribution, that is to say, it has density

$$f(x) = \Gamma(a)^{-1} x^{a-1} \mathrm{e}^{-x} \mathbf{1}_{(0,\infty)}(x).$$

Primes and numerals will be used to indicate that two or more gamma variables are independent. B has a beta distribution of the first kind with parameters a and b, denoted $B \sim \beta^{(1)}_{a,b}$, if its density is

$$f_B(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}\mathbf{1}_{(0,1)}(x), \qquad a, b > 0.$$

X has a beta distribution of the second kind with parameters a and b, denoted $X \sim \beta^{(2)}_{a,b}$, if its density is

$$f_X(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1+x)^{-a-b}\mathbf{1}_{(0,\infty)}(x), \qquad a, b > 0.$$

If $V_i \sim \mathscr{L}_i$, $i = 1, 2$, are independent, then the distribution of their product $U = V_1 V_2$ will be denoted $\mathscr{L}_1 \odot \mathscr{L}_2$.

Remark 1. Letting $b \to 0$ in Theorem 1 we obtain: for any $a, b > 0$

$$\frac{G_a}{G_a + G'_b} \cdot (G''_a + G'''_b) \stackrel{\mathscr{L}}{=} G_a. \tag{6}$$

(This also results from the familiar independence of $Y_1 = G_a/(G_a + G_b)$ and $Y_2 = G_a + G_b$.) The following (known) explicit examples of (5) – the first taken from Letac (1986), the second and third from Chamayou and Letac (1991) – may be given simple algebraic proofs based on (6). This is in contrast with earlier proofs, which used *ad hoc* differential equation or Mellin transform arguments.

$$B \sim \beta^{(1)}_{a,b}, \;\; C \sim \Gamma(b,1), \;\; X \sim \Gamma(a,1).$$

$$B \sim \beta^{(2)}_{a,a+b}, \;\; C \equiv 1, \;\; X \sim \beta^{(2)}_{a,b}.$$

$$-B \sim \beta^{(1)}_{a,b}, \;\; C \equiv -1, \;\; X \sim \beta^{(1)}_{a,a+b}.$$

Detailed calculations may be found in Dufresne (1995).

2. A new explicit result

Theorem 1. *For any* $a, b, c > 0$,

$$\frac{G_a}{G_a + G'_{b+c}} \cdot G''_b + G'''_c \stackrel{\mathscr{L}}{=} \frac{G_{b+c}}{G'_a + G_{b+c}} \cdot G''_{a+c}. \tag{7}$$

Proof. Suppose $X \sim \beta^{(1)}_{a,b} \odot \Gamma(c,1)$. Then (letting $B \sim \beta^{(1)}_{a,b}$)

$$\mathrm{E}e^{tX} = \mathrm{E}(1 - tB)^{-c} = F(a, c; a + b; t), \qquad t < 1,$$

where ($z \in \mathbb{C}, \operatorname{Re}\zeta > \operatorname{Re}\gamma > 0$)

$$F(\alpha,\gamma;\zeta;z) = \int_0^1 \frac{\Gamma(\zeta)}{\Gamma(\gamma)\Gamma(\zeta-\gamma)} t^{\gamma-1}(1-t)^{\zeta-\gamma-1}(1-tz)^{-\alpha}\,\mathrm{d}t, \qquad |\arg(1-z)| < \pi.$$

$F(\alpha,\gamma;\zeta;z)$ is known as the hypergeometric function (see Chapter 9 of Lebedev 1972). Thus the moment generating function of the variable on the right of (7) is $F(b + c, a + c; a + b + c; t)$, $t < 1$. Using the identity

$$F(\alpha,\gamma;\zeta;z) = (1-z)^{\zeta-\alpha-\gamma}F(\zeta-\alpha,\zeta-\gamma;\zeta;z), \qquad |\arg(1-z)| < \pi$$

(Lebedev 1972, p. 248), we get

$$F(b+c, a+c; a+b+c; t) = (1-t)^{-c}F(a, b; a+b+c; t), \qquad t < 1. \qquad \square$$

Lemma. *For any* $a, b, c > 0$, $\beta^{(1)}_{a,b+c} \odot \Gamma(b,1) = \beta^{(1)}_{b,a+c} \odot \Gamma(a,1)$.

Proof. The lemma results from the well-known property $F(\alpha,\gamma;\zeta;z) = F(\gamma,\alpha;\zeta;z)$. $\square$

Theorem 2. *Suppose* $B \sim \beta^{(1)}_{a,c} \odot \beta^{(1)}_{b,c}$ *and* $C \sim \Gamma(c,1)$. *Then (5) has unique solution*

$$X \sim \beta^{(1)}_{a,b+c} \odot \Gamma(b,1).$$

Proof. Conditions (4) are obviously satisfied. Theorem 1 says that

$$X + C \overset{\mathscr{L}}{=} \frac{G_b + G'_c}{G''_a + G_b + G'_c} \cdot G'''_{a+c}, \tag{8}$$

and so

$$B(X + C) \overset{\mathscr{L}}{=} \frac{G_a^{(4)}}{G_a^{(4)} + G_c^{(5)}} \cdot \frac{G_b^{(6)}}{G_b^{(6)} + G_c^{(7)}} \cdot \frac{G_b + G'_c}{G''_a + G_b + G'_c} \cdot G'''_{a+c}.$$

There are four factors in the expression on the right. By (6), the first and fourth factors may be replaced by $G_a^{(8)}$. As to the second and third factors, define $f_1(x, y) = x/(x + y)$, $f_2(x + y) = x + y$, $U = (G_b, G'_c)$, $U' = (G_b^{(6)}, G_c^{(7)})$, and $g(f_1, f_2, v) = f_1 f_2/(v + f_2)$. The variables $\{f_1(U), f_2(U), G''_a\}$ are independent and so

$$g(f_1(U'), f_2(U), G''_a) \overset{\mathscr{L}}{=} g(f_1(U), f_2(U), G''_a) = \frac{G_b}{G''_a + G_b + G'_c}. \tag{9}$$

Finally, the lemma implies

$$B(X + C) \overset{\mathscr{L}}{=} \frac{G''_a}{G''_a + G_b + G'_c} \cdot G'''_b \overset{\mathscr{L}}{=} X. \qquad \square$$

Remark 2. Given (8), the proof of Theorem 2 may also be completed using the Mellin transform $X \mapsto \mathrm{E}X^t$. The above proof shows that the underlying 'algebraic structure' (given in Theorem 1) is nearly sufficient to obtain Theorem 2; the only other fact needed is the lemma.

Remark 3. As pointed out in the proof, the law of X may also be expressed as $\beta^{(1)}_{b,a+c} \odot \Gamma(a, 1)$. The Mellin transform of $A \sim \beta^{(1)}_{a,b}$ being

$$\mathrm{E}A^t = \frac{\Gamma(a + b)}{\Gamma(a + b + t)} \frac{\Gamma(a + t)}{\Gamma(a)},$$

it can be seen that the law of B is also $\beta^{(1)}_{a,b+c-a} \odot \beta^{(1)}_{b,a+c-b}$.

Corollary. *Suppose $B \sim \beta^{(1)}_{a,2c}$ and $C \sim \Gamma(c, 1)$. Then (5) has unique solution*

$$X \sim \beta^{(1)}_{a+c,a+c} \odot \Gamma(a, 1) = \beta^{(1)}_{a,a+2c} \odot \Gamma(a + c, 1).$$

Proof. Let $b = a'$ and $a = a' + c$ in Theorem 2, then proceed as in (9) to verify that

$$B \sim \beta^{(1)}_{a+c,c} \odot \beta^{(1)}_{a,c} = \beta^{(1)}_{a,2c}. \qquad \square$$

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