Bernoulli **12**(4), 2006, 713–735

Power variation of some integral fractional processes

JOSÉ MANUEL CORCUERA[1,*], DAVID NUALART[1,**] and JEANNETTE H.C. WOERNER[2]

[1]*Universitat de Barcelona, Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain. E-mail: *jmcorcuera@ub.edu; **dnualart@ub.edu*
[2]*Institut für Mathematische Stochastik, Universität Göttingen, Maschmühlenweg 8–10, 37073 Göttingen, Germany. E-mail: woerner@math.uni-goettingen.de*

We consider the asymptotic behaviour of the realized power variation of processes of the form $\int_0^t u_s \, dB_s^H$, where B^H is a fractional Brownian motion with Hurst parameter $H \in (0, 1)$, and u is a process with finite q-variation, $q < 1/(1-H)$. We establish the stable convergence of the corresponding fluctuations. These results provide new statistical tools to study and detect the long-memory effect and the Hurst parameter.

Keywords: central and non-central limit theorems; fractional Brownian motion; long memory; p-variation; realized power variation

1. Introduction

In this paper we determine the limit of realized power variation of certain integral fractional processes. The realized quadratic variation has been widely used in statistics of random processes. Its generalization, the realized power variation of order $p > 0$, is defined as

$$\sum_{i=1}^{[nt]} |X_{i/n} - X_{(i-1)/n}|^p \tag{1}$$

where $\{X_t, t \geqslant 0\}$ is a stochastic process. It was introduced in Barndorff-Nielsen and Shephard (2002, 2003, 2004a, 2004b) to estimate the integrated volatility in some stochastic volatility models used in quantitative finance and also, under an appropriate modification, to estimate the jumps of the processes under analysis. The main interest in these papers is the asymptotic behaviour of the statistic (1), or some appropriate *renormalized* version of it, as n tends to infinity, when the process X_t is a stochastic integral with respect to a Brownian motion. Refinements of their results have been obtained in Woerner (2003, 2005), and further extensions can be found in Barndorff-Nielsen *et al.* (2006).

A fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$, $B^H = \{B_t^H, t \geqslant 0\}$, is a zero-mean Gaussian process with covariance function

$$E(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \qquad s, t \geqslant 0. \tag{2}$$

The fBm is a self-similar process, that is, for any constant $a > 0$, the processes $\{a^{-H} B_{at}^H, t \geqslant 0\}$ and $\{B_t^H, t \geqslant 0\}$ have the same distribution. For $H = \frac{1}{2}$, B^H coincides with the classical Brownian motion.

If we take $X_k := B_k^H - B_{k-1}^H$, then it is easy to see that the correlation function of the sequence $\{X_k\}_{k \geqslant 1}$ is given by

$$\rho_H(n) = \frac{1}{2}\left[(n+1)^{2H} + (n-1)^{2H} - 2n^{2H}\right] \sim cn^{2H-2},$$

as n tends to infinity. When $H > \frac{1}{2}$, $\sum_{n=0}^{\infty} \rho_H(n) = \infty$ and this property is taken as the definition of long memory.

For $0 < H < \frac{1}{2}$ the fBm has been used as a model of turbulence; see Shiryaev (1999) and references therein.

In this paper we consider a process of the form $\int_0^t u_s \, \mathrm{d}B_s^H$, where B^H is an fBm with Hurst parameter $H \in (0, 1)$, and u is a stochastic process with paths of finite q-variation, $q < 1/(1-H)$. The integral is a pathwise Riemann–Stieltjes integral. We are interested in the asymptotic behaviour of the realized power variation,

$$\xi_t^{(n)} = n^{-1+pH} \sum_{i=1}^{[nt]} \left| \int_{(i-1)/n}^{i/n} u_s \, \mathrm{d}B_s^H \right|^p.$$

In Section 2 we establish the convergence in probability of the stochastic process $\xi_t^{(n)}$ to the stochastic process $\xi_t = \mathrm{E}(|B_1^H|^p) \int_0^t |u_s|^p \, \mathrm{d}s$.

Section 3 is devoted to the analysis of the convergence in distribution of the fluctuations $\sqrt{n}(\xi_t^{(n)} - \xi_t)$ to a process of the form $v_1 \int_0^t |u_s|^p \, \mathrm{d}W_s$, where W is a Brownian motion independent of the fBm B^H and v_1 is a constant. This result holds if $H \in (0, \frac{3}{4})$, and it is a stable convergence in $\mathcal{D}([0, T])$. For $H = \frac{3}{4}$ a similar result can be obtained but with an additional normalizing factor equal to $(\log n)^{-1/2}$. To prove these results we make use of a central limit theorem for multiple stochastic integrals proved in Nualart and Peccati (2005), Peccati and Tudor (2005) and Hu and Nualart (2005). Recent related results have been obtained by León and Ludeña (2004), who consider special cases of diffusions with respect to an fBm and where the function $|x|^p$ is replaced by a locally Lipschitz function $G(x)$ satisfying some additional conditions.

For $H > \frac{3}{4}$, the problem is more involved because non-central limit theorems are required. Here we have only considered the case where u_t is constant, and the limit theorem follows directly from the results of Taqqu (1979) or Dobrushin and Major (1979). The limit in this case will be a quadratic functional of the Brownian motion (Rosenblatt process). The first example of a non-central limit theorem with strong or long-range dependence was given by Rosenblatt (1961) and generalized by Taqqu (1975); see also the excellent review of the topic by Sun and Ho (1986).

2. Power variation for fractional stochastic integrals

Suppose that $B^H = \{B_t^H, t \geqslant 0\}$ is an fBm with Hurst parameter $H \in (0, 1)$ defined in a complete probability space $(\Omega, \mathcal{F}, P)$. That is, B^H is a zero-mean Gaussian process with covariance function (2). From the equality

$$\mathrm{E}(|B_t^H - B_s^H|^2) = |t-s|^{2H}$$

we deduce that the trajectories of B^H are $(H-\varepsilon)$-Hölder continuous on any finite interval, for any $\varepsilon > 0$.

For each $t \geqslant 0$ we denote by $\mathcal{F}_t^H$ the σ-field generated by the random variables $\{B_s^H, 0 \leqslant s \leqslant t\}$ and the null sets.

For any $p > 0$, the p-variation of a real-valued function f on an interval $[a, b]$ is defined as

$$\mathrm{var}_p(f; [a, b]) = \sup_{\pi} \left(\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p \right)^{1/p},$$

where the supremum runs over all partitions $\pi = \{a = t_0 < t_1 < \ldots < t_n = b\}$. Clearly, if f is α-Hölder continuous then it has finite $(1/\alpha)$-variation on any finite interval. We set

$$\|f\|_\alpha := \sup_{a \leqslant s < t \leqslant b} \frac{|f(t) - f(s)|}{|t-s|^\alpha}.$$

Young (1936) proved that the Riemann–Stieltjes integral $\int_a^b f \,\mathrm{d}g$ exists if f and g have finite p-variation and finite q-variation, respectively, in the interval $[a, b]$ and $1/p + 1/q > 1$. Moreover, the following inequality holds:

$$\left| \int_a^b f \,\mathrm{d}g - f(a)(g(b) - g(a)) \right| \leqslant c_{p,q} \,\mathrm{var}_p(f; [a, b]) \mathrm{var}_q(g; [a, b]), \tag{3}$$

where $c_{p,q} = \zeta(1/q + 1/p)$, with $\zeta(s) := \sum_{n \geqslant 1} n^{-s}$.

We are interested in stochastic processes of the form $\int_0^t u_s \,\mathrm{d}B_s^H$, where the stochastic integral is a pathwise Riemann–Stieltjes integral. By Young's results this integral will exist provided the trajectories of the process $u = \{u_t, t \geqslant 0\}$ have finite q-variation on any finite interval for some $q < 1/(1-H)$. In fact, the trajectories of B^H have finite $1/(H-\varepsilon)$-variation on any finite interval. Note that if we want to consider processes u of the form $u_t = g(B_t^H)$ we need $H > \frac{1}{2}$.

For any $p > 0$, a natural number $n \geqslant 1$, and for any stochastic process $Z = \{Z_t, t \geqslant 0\}$, we write

$$V_p^n(Z)_t = \sum_{i=1}^{[nt]} |Z_{i/n} - Z_{(i-1)/n}|^p.$$

Set

$$c_p = \mathrm{E}(|B_1^H|^p) = \frac{2^{p/2}\Gamma((p+1)/2)}{\Gamma(1/2)}.$$

Fix $T > 0$, and denote by u.c.p. the uniform convergence in probability in the time interval $[0, T]$ and by $\| \cdot \|_\infty$ the supremum norm on $[0, T]$. The main result of this section is the following theorem.

Theorem 1. *Suppose that $u = \{u_t,\ t \in [0, T]\}$ is a stochastic process with finite q-variation, where $q < 1/(1-H)$. Set*

$$Z_t = \int_0^t u_s \, \mathrm{d}B_s^H.$$

Then,

$$n^{-1+pH} V_p^n(Z)_t \overset{u.c.p}{\to} c_p \int_0^t |u_s|^p \, \mathrm{d}s,$$

as n tends to infinity.

Proof. Consider first the case $p \leqslant 1$. We obtain, for any $m \geqslant n$,

$$\begin{aligned}
&m^{-1+pH} V_p^m(Z)_t - c_p \int_0^t |u_s|^p \, \mathrm{d}s \\
&\quad = m^{-1+pH} \sum_{j=1}^{[mt]} \left(\left| \int_{(j-1)/m}^{(j/m)} u_s \, \mathrm{d}B_s^H \right|^p - |u_{(j-1)/m}(B_{j/m}^H - B_{(j-1)/m}^H)|^p \right) \\
&\qquad + m^{-1+pH} \left(\sum_{j=1}^{[mt]} |u_{(j-1)/m}(B_{j/m}^H - B_{(j-1)/m}^H)|^p - \sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p \sum_{j \in I_n(i)} |B_{j/m}^H - B_{(j-1)/m}^H|^p \right) \\
&\qquad + m^{-1+pH} \sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p \sum_{j \in I_n(i)} |B_{j/m}^H - B_{(j-1)/m}^H|^p - c_p n^{-1} \sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p \\
&\qquad + c_p \left(n^{-1} \sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p - \int_0^t |u_s|^p \, \mathrm{d}s \right) \\
&\quad = A_t^{(m)} + B_t^{(n,m)} + C_t^{(n,m)} + D_t^{(n)},
\end{aligned}$$

where

$$I_n(i) = \left\{ j : \frac{j}{m} \in \left(\frac{i-1}{n}, \frac{i}{n} \right] \right\}, \qquad 1 \leqslant i \leqslant [nt].$$

For any fixed n, $C_t^{(n,m)}$ converges in probability to zero, uniformly in t, as m tends to infinity. In fact,

$$\|C^{(n,m)}\|_\infty \leqslant \sum_{i=1}^{[nT]} |u_{(i-1)/n}|^p \left| m^{-1+pH} \sum_{j \in I_n(i)} |B_{j/m}^H - B_{(j-1)/m}^H|^p - c_p n^{-1} \right|$$

and by the self-similarity of the fBm, the term

$$\left| m^{-1+pH} \sum_{j \in I_n(i)} |B^H_{j/m} - B^H_{(j-1)/m}|^p - c_p n^{-1} \right|$$

has the same distribution as

$$\left| \frac{1}{m} \sum_{j \in I_n(i)} |B^H_j - B^H_{j-1}|^p - c_p n^{-1} \right|,$$

which by the ergodic theorem converges to zero as m tends to infinity.

On the other hand, we have

$$\begin{aligned} \|B^{(n,m)}\|_\infty \leqslant\ & m^{-1+pH} \sum_{i=1}^{[nT]} \sum_{j \in I_n(i)} \big| |u_{(i-1)/n}|^p - |u_{(j-1)/m}|^p \big| \, |B^H_{j/m} - B^H_{(j-1)/m}|^p \\ & + \| \, |u|^p \|_\infty \sup_{0 \leqslant t \leqslant T} m^{-1+pH} \sum_{mn^{-1}[nt] \leqslant j \leqslant mn^{-1}([nt]+1)} |B^H_{j/m} - B^H_{(j-1)/m}|^p \\ \leqslant\ & m^{-1+pH} \sum_{i=1}^{[nT]} \sup_{s \in \mathcal{I}_n(i) \cup \mathcal{I}_n(i-1)} \big| |u_{(i-1)/n}|^p - |u_s|^p \big| \sum_{j \in I_n(i)} |B^H_{j/m} - B^H_{(j-1)/m}|^p \\ & + \sup_{0 \leqslant t \leqslant T} \| \, |u|^p \|_\infty m^{-1+pH} \sum_{mn^{-1}[nt] \leqslant j \leqslant mn^{-1}([nt]+1)} |B^H_{j/m} - B^H_{(j-1)/m}|^p, \end{aligned}$$

where we denote

$$\mathcal{I}_n(i) := \left(\frac{i-1}{n}, \frac{i}{n} \right], \ 1 \leqslant i \leqslant [nt].$$

As m tends to infinity, by the ergodic theorem, this converges in probability to

$$\mathrm{E}_n = \frac{c_p}{n} \left(\sum_{i=1}^{[nT]} \sup_{s \in \mathcal{I}_n(i) \cup \mathcal{I}_n(i-1)} \big| |u_{(i-1)/n}|^p - |u_s|^p \big| + \| \, |u|^p \|_\infty \right).$$

We claim that E_n tends to zero almost surely as n tends to infinity. In fact, since $|u|^p$ is *regulated* it has right and left limits at each point of the interval $[0, T]$. Hence, for any $\varepsilon > 0$, there exists n_0 such that, for all $n > n_0$,

$$\sup_{s \in \mathcal{I}_n(i) \cup \mathcal{I}_n(i-1)} \big| |u_{(i-1)/n}|^p - |u_s|^p \big| < \varepsilon + \big| |u_{(i-1)/n}|^p - |u_{((i-1)/n)-}|^p \big| + \big| |u_{(i-1)/n}|^p - |u_{((i-1)/n)+}|^p \big|,$$

$1 \leqslant i \leqslant [nT]$. Also because $|u|^p$ is regulated, by an application of the Bolzano–Weierstrass theorem, the number of its jumps bigger than ε is finite. Therefore,

$$E_n \leqslant c_p\left(3T\varepsilon + \frac{2}{n}\sum_{||u_{(i-1)/n}|^p - |u_{((i-1)/n)+}|^p|>\varepsilon} ||u_{(i-1)/n}|^p - |u_{((i-1)/n)-}|^p|\right.$$

$$\left. + \frac{2}{n}\sum_{||u_{(i-1)/n}|^p - |u_{((i-1)/n)+}|^p|>\varepsilon} ||u_{(i-1)/n}|^p - |u_{((i-1)/n)+}|^p| + \frac{\||u|^p\|_\infty}{n}\right),$$

which implies

$$\limsup_{n\to\infty} \mathrm{E}_n \leqslant 3c_pT\varepsilon,$$

and the result follows by letting ε tend to zero.

We have $\lim_{n\to\infty}\|D^{(n)}\|_\infty = 0$; in fact,

$$\|D^{(n)}\|_\infty \leqslant c_p n^{-1}\sum_{i=1}^{[nT]} \sup_{s\in\mathcal{I}_n(i)} ||u_{(i-1)/n}|^p - |u_s|^p| + c_p\frac{\||u|^p\|_\infty}{n}.$$

For the term $A_t^{(m)}$, and for $p \leqslant 1$, we can write, by the Young inequality (3),

$$|A_t^{(m)}| \leqslant m^{-1+pH}\left|\sum_{j=1}^{[mt]}\left(\left|\int_{(j-1)/m}^{j/m} u_s\,\mathrm{d}B_s^H\right|^p - |u_{(j-1)/m}(B_{j/m}^H - B_{(j-1)/m}^H)|^p\right)\right|$$

$$\leqslant m^{-1+pH}\sum_{j=1}^{[mt]}\left|\int_{(j-1)/m}^{j/m} u_s\,\mathrm{d}B_s^H - u_{(j-1)/m}(B_{j/m}^H - B_{(j-1)/m}^H)\right|^p$$

$$\leqslant c_{p^*,q}m^{-1+pH}\sum_{j=1}^{[mT]}\big(\mathrm{var}_q(u; \mathcal{I}_m(j))\mathrm{var}_{1/(H-\varepsilon)}(B^H; \mathcal{I}_m(j))\big)^p$$

$$= c_{p^*,q}F_m,$$

where $p^* = 1/(H-\varepsilon)$, $0 < \varepsilon < H$. Fix $\delta > 0$ and consider the decomposition

$$F_m \leqslant m^{-1+pH}\sum_{j:\mathrm{var}_q(u;\mathcal{I}_m(j))>\delta}\big(\mathrm{var}_q(u; \mathcal{I}_m(j))\mathrm{var}_{1/(H-\varepsilon)}(B^H; \mathcal{I}_m(j))\big)^p$$

$$+ \delta^p m^{-1+pH}\sum_{j=1}^{[mT]}\big(\mathrm{var}_{1/(H-\varepsilon)}(B^H; \mathcal{I}_m(j))\big)^p.$$

We have

$$\sum_{j=1}^{[mT]}(\mathrm{var}_q(u; \mathcal{I}_m(j)))^q \leqslant (\mathrm{var}_q(u; [0, T]))^q < \infty,$$

and, consequently, the number of indexes j for which $\mathrm{var}_q(u; \mathcal{I}_m(j)) > \delta$ is bounded by $(\mathrm{var}_q(u; [0, T]))^q/\delta^q = M$. Hence,

$$F_m \leqslant Mm^{-1+pH} \max_{1 \leqslant j \leqslant [mT]} \mathrm{var}_{1/(H-\varepsilon)}(B^H; \mathcal{I}_m(j))^p (\mathrm{var}_q(u; [0, T]))^p$$

$$+ \delta^p m^{-1+pH} \sum_{j=1}^{[mT]} \big(\mathrm{var}_{1/(H-\varepsilon)}(B^H; \mathcal{I}_m(j))\big)^p.$$

The first summand goes to zero when m goes to infinity if $\varepsilon < 1/p$:

$$m^{-1+pH} \mathrm{var}_{1/(H-\varepsilon)}(B^H; \mathcal{I}_m(j))^p \leqslant m^{-1+pH} \|B^H\|^p_{H-\varepsilon} m^{-p(H-\varepsilon)} = m^{-1+\varepsilon p} \|B^H\|^p_{H-\varepsilon}.$$

For the second summand we use the fact that it has the same law as

$$\delta^p m^{-1} \sum_{j=1}^{[mT]} \big(\mathrm{var}_{1/(H-\varepsilon)}(B^H; [j-1, j])\big)^p$$

which converges almost surely and in L^1 to $\delta^p T \mathrm{E}[(\mathrm{var}_{1/(H-\varepsilon)}(B^H; [0, 1]))^p] < \infty$ as m tends to infinity, by the ergodic theorem. In fact, the functional $\mathrm{var}_{1/(H-\varepsilon)}(B^H; [0, 1])$ is a seminorm on the trajectories of the fBm which is finite almost surely. Hence, we have that $\mathrm{E}[(\mathrm{var}_{1/(H-\varepsilon)}(B^H; [0, 1]))^p] < \infty$ for any $p > 0$ by Fernique's theorem (see Fernique 1975), and we can apply the ergodic theorem. Finally, it suffices to let δ tend to zero.

For $p > 1$, we can proceed similarly using Minkowski's inequality instead:

$$\left| \left(m^{-1+pH} V_p^m(Z)_t \right)^{1/p} - \left(c_p \int_0^t |u_s|^p \, \mathrm{d}s \right)^{1/p} \right|$$

$$\leqslant m^{-1/p+H} \left(\sum_{j=1}^{[mt]} \left| \int_{(j-1)/m}^{j/m} u_s \, \mathrm{d}B_s^H - u_{(j-1)/m} (B^H_{j/m} - B^H_{(j-1)/m}) \right|^p \right)^{1/p}$$

$$+ m^{-1/p+H} \left(\sum_{i=1}^{[nt]} \sum_{j \in I_n(i)} |(u_{(j-1)/m} - u_{(i-1)/n})(B^H_{j/m} - B^H_{(j-1)/m})|^p \right)^{1/p}$$

$$+ \left| m^{-1/p+H} \left(\sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p \sum_{j \in I_n(i)}^{m} |B^H_{j/m} - B^H_{(j-1)/m}|^p \right)^{1/p} \right.$$

$$\left. - \left(c_p n^{-1} \sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p \right)^{1/p} \right|$$

$$+ c_p^{1/p} \left| \left(n^{-1} \sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p \right)^{1/p} - \left(\int_0^t |u_s|^p \, \mathrm{d}s \right)^{1/p} \right|.$$

□

The previous theorem can be generalized as follows.

Corollary 2. *Assume the same conditions as in Theorem 1. Consider a stochastic process* $Y = \{Y_t,\ t \in [0,\ T]\}$ *such that*

$$n^{-1+pH} V_p^n(Y)_t \overset{u.c.p}{\to} 0 \tag{4}$$

as n *tends to infinity. Then*

$$n^{-1+pH} V_p^n(Z+Y)_t \overset{u.c.p}{\to} c_p \int_0^t |u_s|^p \, \mathrm{d}s,$$

as n *tends to infinity.*

Proof. First, we look at the case $p \leqslant 1$. Applying the triangular inequality, we obtain

$$\begin{aligned} &\left| n^{-1+pH} V_p^n(Z+Y)_t - c_p \int_0^t |u_s|^p \, \mathrm{d}s \right| \\ &\quad \leqslant |n^{-1+pH} V_p^n(Z+Y)_t - n^{-1+pH} V_p^n(Z)_t| \\ &\qquad + \left| n^{-1+pH} V_p^n(Z)_t - c_p \int_0^t |u_s|^p \, \mathrm{d}s \right| \\ &\quad \leqslant n^{-1+pH} V_p^n(Y)_t + \left| n^{-1+pH} V_p^n(Z)_t - c_p \int_0^t |u_s|^p \, \mathrm{d}s \right|. \end{aligned}$$

The first term tends to zero by the assumption and the second by Theorem 1.

For $p > 1$ we can proceed similarly using Minkowski's inequality instead. □

Condition (4) is obviously satisfied if Y is a process whose trajectories are γ-Hölder for some $\gamma \in (H,\ 1]$, that is, a process which possesses slightly more regularity than the fBm.

Under some further conditions, (4) is also satisfied for semimartingales with jumps; see Woerner (2005) for the case of $H = \frac{1}{2}$. Assume that the semimartingale Y has Blumenthal–Getoor index β and canonical representation $Y_t = Y_0 + B(h) + Y^c + h * (\mu - \nu) + (x - h(x)) * \mu$, where $B(h)$ is predictable of bounded variation, h is a truncation function, behaving like x at the origin, Y^c denotes the continuous local martingale part, μ the jump measure and ν its compensator. If $\beta \geqslant 1$ we assume that $\langle Y^c \rangle = 0$; if $\beta < 1$ we assume that $\langle Y^c \rangle = 0,\ B(h) + (x - h) * \nu = 0$. Now (4) is satisfied if $1/H > \beta$ and $1/H > p$. To see this we look at the case $p > \beta$ first. For these parameters we know that $V_p^n(Y)_t < \infty$ and the norming sequence tends to zero. For the case $p \leqslant \beta$ we have to use Hölder's inequality with $1/a + 1/b = 1$:

$$n^{-1+pH} V_p^n(Y)_t \leqslant n^{-1/a} (n^{-1+pbH} V_{pb}^n(Y)_t)^{1/b}.$$

Now we can always find some b such that $1/H > bp > \beta$, which implies the desired result as before.

3. Central limit theorem for the power variation

For $H \in (0, \frac{3}{4}]$ the fluctuations of the power variation, properly normalized, have Gaussian asymptotic distributions. In order to establish this result we first introduce some notation.

For any $p > 0$, we put

$$\delta_p = 2^p \left(\frac{1}{\sqrt{\pi}} \Gamma\left(p + \frac{1}{2}\right) - \frac{1}{\pi} \Gamma\left(\frac{p+1}{2}\right)^2 \right)$$

and

$$v_1^2 = \delta_p + 2 \sum_{j \geqslant 1} \left(\gamma_p(\rho_H(j)) - \gamma_p(0) \right),$$

where $\gamma_p(x)$ is given by (11) (see the Appendix), and

$$\rho_H(n) = \frac{1}{2} \left((n+1)^{2H} + (n-1)^{2H} - 2n^{2H} \right).$$

We will first show a functional limit theorem for the realized p-variation of the fBm.

Theorem 3. *Fix* $p > 0$. *Assume* $0 < H < \frac{3}{4}$. *Then*

$$(B_t^H, n^{-1/2+pH} V_p^n(B^H)_t - c_p t n^{1/2}) \xrightarrow{\mathcal{L}} (B_t^H, v_1 W_t), \tag{5}$$

as n tends to infinity, where $W = \{W_t, t \in [0, T]\}$ *is a Brownian motion independent of the process* B^H, *and the convergence is in the space* $\mathcal{D}([0, T])^2$ *equipped with the Skorohod topology.*

Proof. The proof will be done in two steps. Set

$$Z_t^{(n)} = n^{-1/2+pH} V_p^n(B^H)_t - c_p t n^{1/2}.$$

Step 1. We will first show the convergence of the finite-dimensional distributions. Let $J_k = (a_k, b_k]$, $k = 1, \dots, N$, be pairwise disjoint intervals contained in $[0, T]$. Define the random vectors $B = (B_{b_1}^H - B_{a_1}^H, \dots, B_{b_N}^H - B_{a_N}^H)$ and $X^{(n)} = (X_1^{(n)}, \dots, X_N^{(n)})$, where

$$X_k^{(n)} = n^{-1/2+pH} \sum_{[na_k] < j \leqslant [nb_k]} |B_{j/n}^H - B_{(j-1)/n}^H|^p - n^{1/2} c_p |J_k|,$$

$k = 1, \dots, N$ and $|J_k| = b_k - a_k$. We claim that

$$(B, X^{(n)}) \xrightarrow{\mathcal{L}} (B, V), \tag{6}$$

where B and V are independent and V is a Gaussian random vector with zero mean and independent components of variance $v_1^2 |J_k|$.

By the self-similarity of the fBm, the sequence $(n^{pH} |B_{j/n}^H - B_{(j-1)/n}^H|^p - c_p)_{1 \leqslant j \leqslant n}$ has the same law as $(|B_j^H - B_{j-1}^H|^p - c_p)_{1 \leqslant j \leqslant n}$. Set $X_j = B_j^H - B_{j-1}^H$ and $H(x) = |x|^p - c_p$.

Then $\{X_j, j \geqslant 1\}$ is a stationary Gaussian sequence with zero mean, unit variance and $\mathrm{E}(X_j X_{j+n}) = \rho_H(n)$.

Thus, the convergence (6) is equivalent to the convergence in distribution of $(B^{(n)}, Y^{(n)})$ to (B, V), where

$$B_k^{(n)} = n^{-H} \sum_{[na_k] < j \leqslant [nb_k]} X_j, \qquad 1 \leqslant k \leqslant N, \tag{7}$$

and

$$Y_k^{(n)} = \frac{1}{\sqrt{n}} \sum_{[na_k] < j \leqslant [nb_k]} H(X_j), \qquad 1 \leqslant k \leqslant N. \tag{8}$$

The convergence $(B^{(n)}, Y^{(n)}) \xrightarrow{\mathcal{L}} (B, V)$ is proved in Proposition 10 of the Appendix, by means of a direct argument based on a recent central limit theorem for stochastic integrals (see Nualart and Peccati 2005; Peccati and Tudor 2005; Hu and Nualart 2005).

We remark that, taking into account that $H(x)$ has Hermite rank 2, and we have

$$\sum_{n=1}^{\infty} \rho_H^2(n) < \infty,$$

because $\rho_H(n) = O(n^{2H-2})$, the convergence of the sequence of vectors $Y^{(n)}$ to the vector W would also follow from Breuer and Major (1983: Theorem 1) or Giraitis and Surgailis (1985: Theorem 5).

Step 2. We need to show that the sequence of processes $Z^{(n)}$ is tight in $\mathcal{D}([0, T])$. Let us compute, for $s < t$,

$$\mathrm{E}\left(|Z_t^{(n)} - Z_s^{(n)}|^4\right) = n^{-2}\mathrm{E}\left(\left|\sum_{j=[ns]+1}^{[nt]} H(X_j)\right|^4\right).$$

By Taqqu (1977: Proposition 4.2) we know that, for all $N \geqslant 1$,

$$\frac{1}{N^2}\mathrm{E}\left(\left|\sum_{j=1}^{N} H(X_j)\right|^4\right) \leqslant K\left(\sum_{u=0}^{\infty} \rho_H^2(u)\right)^2.$$

As a consequence,

$$\sup_n \mathrm{E}\left(|Z_t^{(n)} - Z_s^{(n)}|^4\right) \leqslant C|t-s|^2,$$

and by Billingsley (1968: Theorem 15.6) we obtain the desired tightness property. □

The convergence established in Theorem 3 can be also expressed in terms of the *stable convergence* (see Aldous and Eagleson 1978). In fact, for any bounded random variable X measurable with respect to the σ-field $\mathcal{F}_T^H$ and for any continuous and bounded function ϕ on the Skorohod space $\mathcal{D}([0, T])$, we have

$$\lim_{n\to\infty} \mathrm{E}(X\phi(Z^{(n)})) = \mathrm{E}(X)\mathrm{E}(\phi(W)).$$

If X is a continuous functional of $\{B_t^H, 0 \leqslant t \leqslant T\}$ this convergence is an immediate consequence of Theorem 3, and in the general case follows by an easy approximation argument. In this sense, Theorem 3 can also be obtained as an application of the general convergence result established by León and Ludeña (2004).

As a consequence of Theorem 3 we can derive the following central limit theorem for the realized power variation of the stochastic integrals studied in Section 2. Here a Hölder continuous condition on the trajectories of the process u is required.

Theorem 4. *Fix $p > 0$. Let B^H be an fBm with Hurst parameter $H \in (0, \frac{3}{4})$. Suppose that $u = \{u_t, t \in [0, T]\}$ is a stochastic process measurable with respect to $\mathcal{F}_T^H$, and with Hölder continuous trajectories of order $a > 1/(2(p \wedge 1))$. Set $Z_t = \int_0^t u_s \, \mathrm{d}B_s^H$. Then*

$$\left(B_t^H, n^{-1/2+pH} V_p^n(Z)_t - c_p \sqrt{n} \int_0^t |u_s|^p \, \mathrm{d}s \right) \xrightarrow{\mathcal{L}} \left(B_t^H, v_1 \int_0^t |u_s|^p \, \mathrm{d}W_s \right),$$

as n tends to infinity, where $W = \{W_t, t \in [0, T]\}$ is a Brownian motion independent of $\mathcal{F}_T^H$, and the convergence is in $\mathcal{D}([0, T])^2$.

Proof. The proof will be based on Theorem 3. For any $m \geqslant n$ and with the same notation as in Theorem 1, we can write

$$m^{-1/2+pH} V_p^m(Z)_t - \sqrt{m} c_p \int_0^t |u_s|^p \, \mathrm{d}s = A_t^{(m)} + B_t^{(n,m)} + C_t^{(n,m)} + D_t^{(m)},$$

where

$$A_t^{(m)} = m^{-1/2+pH} \sum_{j=1}^{[mt]} \left(\left| \int_{(j-1)/m}^{j/m} u_s \, \mathrm{d}B_s^H \right|^p - |u_{(j-1)/m}(B_{j/m}^H - B_{(j-1)/m}^H)|^p \right),$$

$$B_t^{(n,m)} = m^{-1/2+pH} \sum_{j=1}^{[mt]} |u_{(j-1)/m}(B_{j/m}^H - B_{(j-1)/m}^H)|^p - m^{-1/2} c_p \sum_{j=1}^{[mt]} |u_{(j-1)/m}|^p$$

$$- \sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p \sum_{j \in I_n(i)} m^{-1/2+pH} |B_{j/m}^H - B_{(j-1)/m}^H|^p + \frac{\sqrt{m}}{n} c_p \sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p,$$

$$C_t^{(n,m)} = \sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p \sum_{j \in I_n(i)} m^{-1/2+pH} |B_{j/m}^H - B_{(j-1)/m}^H|^p - \frac{\sqrt{m}}{n} c_p \sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p$$

and

$$D_t^{(m)} = m^{-1/2} c_p \sum_{j=1}^{[mt]} |u_{(j-1)/m}|^p - \sqrt{m} c_p \int_0^t |u_s|^p \, \mathrm{d}s.$$

First we show that $\|D^{(m)}\|_\infty \to 0$ almost surely as $m \to \infty$. Using the Hölder continuity of u, we can write

$$
\begin{aligned}
|D_t^{(m)}| &\leqslant c_p m^{-1/2} \sum_{j=1}^{[mt]} \big|\, |u_{(j-1)/m}|^p - |u_{\tilde{t}_{j-1}^m}|^p \big| + \frac{c_p}{\sqrt{m}} \|\, |u|^p \|_\infty \\
&\leqslant c_p m^{-1/2} (p \vee 1) \|u\|_\infty^{(p-1)+} \sum_{j=1}^{[mt]} |u_{(j-1)/m} - u_{\tilde{t}_{j-1}^m}|^{p\wedge 1} + \frac{c_p}{\sqrt{m}} \|\, |u|^p \|_\infty \\
&\leqslant c_p T (p \vee 1) \|u\|_a^{p\wedge 1} \|u\|_\infty^{(p-1)+} m^{1/2 - a(p\wedge 1)} + \frac{c_p}{\sqrt{m}} \|\, |u|^p \|_\infty,
\end{aligned}
$$

where $\tilde{t}_{j-1}^m \in \mathcal{I}_m(j)$. Hence $\|D^{(m)}\|_\infty \to 0$ because $a(p \wedge 1) > \frac{1}{2}$.

Let us now study the term $C_t^{(n,m)}$. Set

$$
Y_{n,m}^i := \sum_{j \in I_n(i)} m^{-1/2 + pH} |B_{j/m}^H - B_{(j-1)/m}^H|^p - \frac{\sqrt{m}}{n} c_p.
$$

By Theorem 3 and taking into account that it implies the stable convergence of $\{Y_{n,m}^1, Y_{n,m}^2, \ldots, Y_{n,m}^n\}_{m\geqslant 1}$ for any n (see the comment after Theorem 3 and Aldous and Eagleson 1978: Proposition 1), we have that for any $\mathcal{F}_T^H$-measurable random variable $|u_{(i-1)/n}|^p$, as $m \to \infty$,

$$
\left(|u_{(i-1)/n}|^p,\, Y_{n,m}^i \right)_{1\leqslant i \leqslant [nt]} \xrightarrow{\mathcal{L}} \left(|u_{(i-1)/n}|^p,\, v_1 \left(W_{i/n} - W_{(i-1)/n} \right) \right)_{1\leqslant i \leqslant [nt]},
$$

where W is a Brownian motion independent of $\mathcal{F}_T^H$. Hence,

$$
C_t^{(n,m)} \xrightarrow{\mathcal{L}} v_1 \sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p \left(W_{i/n} - W_{(i-1)/n} \right)
$$

as m tends to infinity, and this convergence is also stable (see Aldous and Eagleson 1978: Theorem 1′). On the other hand, $\sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p \left(W_{i/n} - W_{(i-1)/n} \right)$ converges uniformly in probability to $\int_0^t |u_s|^p \,\mathrm{d}W_s$, as n tends to infinity. This implies, by first letting first m and then n tend to infinity, that $C_t^{(n,m)}$ converges in distribution to $v_1 \int_0^t |u_s|^p \,\mathrm{d}W_s$ in $\mathcal{D}([0, T])$.

We show that $\|B^{(n,m)}\|_\infty \xrightarrow{P} 0$ as n and m tend to infinity. We can rewrite $B_t^{(n,m)}$ in the following way:

$$
\begin{aligned}
B_t^{(n,m)} = {} & m^{-1/2+pH} \sum_{j=1}^{[mt]} |u_{(j-1)/m} (B_{j/m}^H - B_{(j-1)/m}^H)|^p - m^{-1/2} c_p \sum_{j=1}^{[mt]} |u_{(j-1)/m}|^p \\
& - \sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p \sum_{j \in I_n(i)} m^{-1/2+pH} |B_{j/m}^H - B_{(j-1)/m}^H|^p + \frac{\sqrt{m}}{n} c_p \sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p.
\end{aligned}
$$

As a consequence, by the mean value theorem

$$
\begin{aligned}
|B_t^{(n,m)}| = \Bigg| & \sum_{i=1}^{[nt]} \sum_{j\in I_n(i)} |u_{(j-1)/m}|^p \Big(m^{-1/2+pH} |B^H_{j/m} - B^H_{(j-1)/m}|^p - m^{-1/2} c_p \Big) \\
& - \sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p \left(\sum_{j\in I_n(i)} m^{-1/2+pH} |B^H_{j/m} - B^H_{(j-1)/m}|^p - \frac{\sqrt{m}}{n} c_p \right) \\
& + \sum_{j\geqslant \frac{m}{n}[nt]}^{[mt]} |u_{(j-1)/m}|^p \Big(m^{-1/2+pH} |B^H_{j/m} - B^H_{(j-1)/m}|^p - m^{-1/2} c_p \Big) \Bigg| \\
\leqslant \Bigg| & \sum_{i=1}^{[nt]} |u_{\tilde{s}}|^p \sum_{j\in I_n(i)} \Big(m^{-1/2+pH} |B^H_{j/m} - B^H_{(j-1)/m}|^p - m^{-1/2} c_p \Big) \\
& - \sum_{i=1}^{[nt]} |u_{(i-1)/n}|^p \left(\sum_{j\in I_n(i)} m^{-1/2+pH} |B^H_{j/m} - B^H_{(j-1)/m}|^p - \frac{\sqrt{m}}{n} c_p \right) \Bigg| \\
& + \sup_{0\leqslant t\leqslant T} \sum_{\frac{m}{n}[nt]\leqslant j\leqslant [mt]} \big| |u_{j-1/m}|^p (m^{-1/2+pH} |B^H_{j/m} - B^H_{(j-1)/m}|^p - m^{-1/2} c_p) \big| \\
\leqslant & \sum_{i=1}^{[nT]} \sup_{s\in \mathcal{I}_n(i)\cup \mathcal{I}_n(i-1)} \big| |u_s|^p - |u_{(i-1)/n}|^p \big| \, |Y^i_{n,m}| + \frac{c_p}{\sqrt{m}} \| \, |u|^p \|_\infty \\
& + \sup_{0\leqslant t\leqslant T} \Bigg| \sum_{\frac{m}{n}[nt]\leqslant j\leqslant [mt]} |u_{(j-1)/m}|^p \Big(m^{-1/2+pH} |B^H_{j/m} - B^H_{(j-1)/m}|^p - m^{-1/2} c_p \Big) \Bigg|
\end{aligned}
$$

where $\tilde{s}(\omega) \in \mathcal{I}_n(i) \cup \mathcal{I}_n(i-1)$. Then, by Theorem 3, for any $\varepsilon > 0$, we obtain

$$
\begin{aligned}
\limsup_{m\to\infty} P\big(\|B^{(n,m)}\|_\infty > \varepsilon \big) \leqslant P\Bigg(& v_1 \sum_{i=1}^{[nT]} \sup_{s\in \mathcal{I}_n(i)\cup \mathcal{I}_n(i-1)} \big| |u_s|^p - |u_{(i-1)/n}|^p \big| \, |W_{i/n} - W_{(i-1)/n}| \\
& + v_1 \| \, |u|^p \|_\infty \frac{1}{n} \sup_{0\leqslant t\leqslant T} |W_t - W_{[nt]/n}| > \varepsilon \Bigg).
\end{aligned}
$$

The Hölder continuity of the trajectories of u and the condition $a(p \wedge 1) > \frac{1}{2}$ imply

$$\sum_{i=1}^{[nT]} \sup_{s\in\mathcal{I}_n(i)\cup\mathcal{I}_n(i-1)} \big| |u_s|^p - |u_{(i-1)/n}|^p \big| |W_{i/n} - W_{(i-1)/n}|$$

$$\leqslant (p\vee 1)T\|u\|_a^{p\wedge 1}\|u\|_\infty^{(p-1)+}2^{a(p\wedge 1)}n^{-a(p\wedge 1)+1/2-\varepsilon},$$

which converges to zero as n tends to infinity. Moreover

$$\frac{1}{n}\sup_{0\leqslant t\leqslant T}|W_t - W_{[nt]/n}| \underset{n\to\infty}{\overset{a.s.}{\longrightarrow}} 0,$$

and we deduce the desired result.

Finally, we have to show that $\|A^{(m)}\|_\infty \overset{P}{\to} 0$ as $m\to\infty$. Then

$$|A_t^{(m)}| \leqslant m^{-1/2+pH}(p\vee 1)2^{(p-2)_+}\sum_{j=1}^{[mt]} |u_{(j-1)/m}(B^H_{j/m} - B^H_{(j-1)/m})|^{(p-1)_+}$$

$$\times \left| \int_{(j-1)/m}^{j/m} u_s\,\mathrm{d}B_s^H - u_{(j-1)/m}(B^H_{j/m} - B^H_{(j-1)/m}) \right|^{p\wedge 1}$$

$$+ m^{-1/2+pH}(p\vee 1)2^{(p-2)_+}\sum_{j=1}^{[mt]} \left| \int_{(j-1)/m}^{j/m} u_s\,\mathrm{d}B_s^H - u_{(j-1)/m}(B^H_{j/m} - B^H_{(j-1)/m}) \right|^{p}$$

and using Young's inequality, as in Theorem 1, we obtain

$$|A_t^{(m)}| \leqslant (p\vee 1)2^{(p-2)_+}c^p_{1/(H-\varepsilon),1/a}\|B^H\|_{H-\varepsilon}^{(p-1)+}\|u\|_\infty^{(p-1)+}m^{-1/2+pH-(H-\varepsilon)(p-1)_+}$$

$$\times\sum_{j=1}^{[mT]}(\mathrm{var}_{1/a}(u;\mathcal{I}_m(j))\mathrm{var}_{1/(H-\varepsilon)}(B^H,\mathcal{I}_m(j)))^{p\wedge 1}$$

$$+ (p\vee 1)2^{(p-2)_+}c^p_{1/(H-\varepsilon),1/a}m^{-1/2+pH}$$

$$\times\sum_{j=1}^{[mT]}(\mathrm{var}_{1/a}(u;\mathcal{I}_m(j))\mathrm{var}_{1/(H-\varepsilon)}(B^H,\mathcal{I}_m(j)))^{p}$$

$$\leqslant (p\vee 1)2^{(p-2)+}c^p_{1/(H-\varepsilon),1/a}T\|B^H\|^p_{H-\varepsilon}\|u\|_a^{p\wedge 1}\|u\|_\infty^{(p-1)+}m^{1/2-a(p\wedge 1)+p\varepsilon}$$

$$+ (p\vee 1)2^{(p-2)+}c^p_{1/(H-\varepsilon),1/a}T\|B^H\|^p_{H-\varepsilon}\|u\|_a^p m^{1/2-ap+p\varepsilon},$$

which converges to zero as m tends to infinity, provided $\varepsilon < p^{-1}(a(p\wedge 1)-\frac{1}{2})$. This completes the proof. □

We remark that for a process of the form $u_t = g(B_t^H)$, where g is locally Lipschitz, we need $p > 1/(2H)$ and $H > \frac{1}{2}$ for the validity of the Theorem 4.

We can also deduce the convergence stated in Theorem 4 under different conditions on u, which include the case of a jump process. Assume that u has trajectories locally bounded away from zero with finite q-variation with $q < 1/(1-H)$ and the following condition (introduced by Barndorff-Nielsen and Shephard 2003) is satisfied (pathwise): for some $\gamma > 0$,

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{n} \big| |u|^\gamma(\eta_{n,j}) - |u|^\gamma(\chi_{n,j})\big| \to 0$$

as n tends to infinity, for any $\chi_{n,j}$ and $\eta_{n,j}$ such that

$$0 \leqslant \chi_{n,1} \leqslant \eta_{n,1} \leqslant \frac{1}{n} \leqslant \chi_{n,2} \leqslant \eta_{n,2} \leqslant \frac{2}{n} \ \ldots \leqslant \chi_{n,n} \leqslant \eta_{n,n} \leqslant T.$$

Then Theorem 4 holds for all $p > 0$. The proof would follow the same arguments as before.

Corollary 5. *Assume the same conditions as in Theorem 4. Consider a stochastic process $Y = \{Y_t, t \in [0, T]\}$ such that*

$$n^{-1/2+pH} V_p^n(Y)_T \xrightarrow{P} 0,$$

as n tends to infinity. Then

$$\left(B_t^H, n^{-1/2+pH} V_p^n(Y+Z)_t - c_p\sqrt{n}\int_0^t |u_s|^p\,\mathrm{d}s\right) \xrightarrow{\mathcal{L}} \left(B_t^H, v_1\int_0^t |u_s|^p\,\mathrm{d}W_s\right)$$

as n tends to infinity, where $W = \{W_t, t \geqslant 0\}$ is a Brownian motion independent of $\mathcal{F}_T^H$, and the convergence is in $\mathcal{D}([0, T])^2$.

The condition on Y is satisfied if it has b-Hölder continuous trajectories and $p(b-H) > \frac{1}{2}$. The condition is also satisfied for a jump semimartingale with Blumenthal–Getoor index β and $1/(2H) > p > \beta/(2(1-\beta H))$. For $1/(2H) > p > \beta$ the result is clear since the non-normed power variation is finite and the norming sequence tends to zero. For $p \leqslant \beta$ we split the process Y into two parts: one with only large jumps and Blumenthal–Getoor index 0, for which the result is clear; and another with infinitely many small jumps Y^ϵ. Using Hölder's inequality, we can deduce for $b > 1$,

$$m^{-1/2+pH} V_p^m(Y^\epsilon)_T \leqslant T^{1/a}\left(m^{-1+b/2+bpH}\sum_{j=1}^{[mT]} |Y_{j/m}^\epsilon - Y_{(j-1)/m}^\epsilon|^{pb}\right)^{1/b}.$$

As in Woerner (2003), using Hudson and Mason (1976) for the case $\beta < 1$, this term tends to zero as $m \to \infty$ and $\epsilon \to 0$, provided

$$1 - \frac{b}{2} - bpH + bp - \beta \; > 0,$$

$$pb \; > \beta.$$

This leads to the lower bound on p.

In the case $H = \frac{3}{4}$ the fluctuations of the power variation still converge to a Gaussian process, but with a different normalization.

Theorem 6. *Suppose that* $H = \frac{3}{4}$. *Then*

$$\left(B_t^H, (\log n)^{-1/2}\left(n^{-1/2+pH} V_p^n(B^H)_t - c_p t n^{1/2}\right)\right) \xrightarrow{\mathcal{L}} (B_t^H, \upsilon_2 W_t), \tag{9}$$

as n *tends to infinity, where* $W = \{W_t, t \in [0, T]\}$ *is a Brownian motion independent of the process* B^H *and* υ_2 *is given by*

$$\upsilon_2^2 = \lim_{n\to\infty} \frac{2}{\log n} \sum_{j=1}^{n} \frac{n-j}{n} \gamma_p(\rho_H(j)). \tag{10}$$

Proof. Notice that in this case we have

$$\sum_{j=1}^{n} \rho_H^2(j) \sim c \sum_{j=1}^{n} j^{-1} \sim c \log n.$$

As a consequence, we can apply the same arguments as in the proof of the Theorem 3. For instance, the convergence of the finite-dimensional distributions of the process $(\log n)^{-1/2} Z_t^{(n)}$ would follow from Breuer and Major (1983: Theorem 1′). □

We can also derive the following convergence in distribution for the fluctuations of the power variation of stochastic integrals, in the case $H = \frac{3}{4}$.

Theorem 7. *Suppose that* $H = \frac{3}{4}$ *and* $u = \{u_t, t \in [0, T]\}$ *is a stochastic process measurable with respect to* $\mathcal{F}_T^H$ *and and with Hölder continuous trajectories of order* $a > 1/(2(p \wedge 1))$. *Consider a stochastic process* $Y = \{Y_t, t \in [0, T]\}$ *such that*

$$n^{-1/2+pH} V_p^n(Y)_T \xrightarrow{P} 0,$$

as n *tends to infinity. Then we obtain*

$$(\log n)^{-1/2}\left(n^{-1/2+pH} V_p^n(Y+Z)_t - c_p\sqrt{n}\int_0^t |u_s|^p \, \mathrm{d}s\right) \xrightarrow{\mathcal{L}} \upsilon_2 \int_0^t |u_s|^p \, \mathrm{d}W_s,$$

as $n \to \infty$, *where* $W = \{W_t, t \in [0, T]\}$ *is a Brownian motion independent of* $\mathcal{F}_T^H$ *and* υ_2 *is given by* (10).

The condition on Y is satisfied for processes as before; we can only replace the greater than signs by greater than or equal to signs due to the faster rate of convergence.

If $H > \frac{3}{4}$, the fluctuations of the power variation converge to a process in the second chaos which is called the Rosenblatt process. In fact, we have the following result.

Theorem 8. *Fix $p > 0$ and assume that $\frac{3}{4} < H < 1$. Then*

$$n^{2-2H}(n^{-1+pH}V_p^n(B^H)_t - c_p t) \xrightarrow{\mathcal{L}} Z_t$$

where

$$Z_t = \frac{1}{\Gamma(2-2H)\cos((1-H)\pi)} d_p \times \int_0^\infty \int_0^{x_2} \frac{e^{i(x_1+x_2)t} - 1}{i(x_1+x_2)} |x_1|^{1/2-H} |x_2|^{1/2-H} \, dW_{x_1} \, dW_{x_2},$$

is the Rosenblatt process, $\{W_t, t \in [0, T]\}$ is a standard Brownian motion,

$$d_p = E(|B_1^H|^{2+p}) - E(|B_1^H|^p),$$

and the convergence is in $\mathcal{D}([0, T])$.

Proof. Since $\rho_H(n) = O(n^{2H-2})$, $\frac{3}{4} < H < 1$ and $(|B_j^H - B_{j-1}^H|^p - c_p)_{1 \leqslant j \leqslant n}$ is an L_2-functional, with Hermite range 2, of a stationary mean-zero Gaussian sequence, we can apply Taqqu (1979: Theorem 5.6) (see also Dobrushin and Major 1979). □

Appendix

For any real number $x \in (-1, 1)$ and for each $p > 0$, we set

$$\gamma_p(x) = (1-x^2)^{p+1/2} 2^p \sum_{k=0}^{\infty} \frac{(2x)^{2k}}{\pi (2k)!} \Gamma\left(\frac{p+1}{2} + k\right)^2. \tag{11}$$

Lemma 9. *Suppose that $(U, V) \sim N_2\left(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$, where $|\rho| < 1$. Then*

$$E(|U|^p |V|^p) = \gamma_p(\rho).$$

Proof. We have

$$
\begin{aligned}
&\mathrm{E}(|U|^p|V|^p)\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u|^p|v|^p}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(u^2+v^2-2\rho uv)\right\} \mathrm{d}u\,\mathrm{d}v\\
&= 2\int_{\mathbb{R}+}\int_{\mathbb{R}+} \frac{|u|^p|v|^p}{\pi\sqrt{1-\rho^2}} \cosh\left\{\frac{\rho uv}{1-\rho^2}\right\} \exp\left\{-\frac{1}{2(1-\rho^2)}(u^2+v^2)\right\} \mathrm{d}u\,\mathrm{d}v\\
&= 2(1-\rho^2)^{p+1/2}\int_{\mathbb{R}+}\int_{\mathbb{R}+} \frac{s^p t^p}{\pi} \exp\left\{-\frac{1}{2}(s^2+t^2)\right\} \cosh\{\rho st\}\,\mathrm{d}s\,\mathrm{d}t\\
&= 2(1-\rho^2)^{p+1/2}\sum_{k=0}^{\infty} \frac{\rho^{2k}}{\pi(2k)!}\left(\int_{\mathbb{R}+} s^{p+2k}\exp\left\{-\frac{1}{2}s^2\right\}\mathrm{d}s\right)^2\\
&= (1-\rho^2)^{p+1/2}2^p\sum_{k=0}^{\infty} \frac{(2\rho)^{2k}}{\pi(2k)!}\Gamma\left(\frac{p+1}{2}+k\right)^2\\
&= (1-\rho^2)^{p+1/2}\frac{2^p}{\pi}\Gamma\left(\frac{p+1}{2}\right)^2 {}_1\mathrm{F}_1\left(\frac{p+1}{2};\frac{1}{2};\rho^2\right),
\end{aligned}
$$

where ${}_1\mathrm{F}_1$ is the confluent hypergeometric function. □

The next proposition is the basic ingredient in the proof of Theorem 3.

Proposition 10. *Assume $H < \frac{3}{4}$ and let $B^{(n)}$ and $Y^{(n)}$ be the random vectors defined by* (7) *and* (8)*, respectively. Then*

$$(B^{(n)}, Y^{(n)}) \xrightarrow{\mathcal{L}} (B, V),$$

where B and V are independent centred Gaussian vectors, with $B_k = B^H_{b_k} - B^H_{a_k}$, and the components of V are independent with variances $v_1^2(b_k - a_k)$.

Proof. Denote by $\mathcal{H}_1$ the first Wiener chaos associated with the sequence $\{X_j\}$, that is, the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables X_j. For any $m \geqslant 2$, we denote by $\mathcal{H}_m$ the mth Wiener chaos, that is, the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $H_m(X)$, where $X \in \mathcal{H}_1$, $\mathrm{E}(X^2) = 1$, and H_m is the mth Hermite polynomial. We know that the mapping

$$I_m : \mathcal{H}_1^{\odot m} \to \mathcal{H}_m,$$

defined by $I_m(X^{\otimes m}) = H_m(X)$, is a linear isometry between the symmetric tensor product $\mathcal{H}_1^{\odot m}$, equipped with the norm $\sqrt{m!}\|\cdot\|_{\mathcal{H}_1^{\otimes m}}$ and the mth chaos. We will denote by J_m the projection operator on the mth Wiener chaos.

The function $H(x)$ can be expanded in the form

$$H(x) = \sum_{m=2}^{\infty} c_m H_m(x),$$

where $\sum_{m=2}^{\infty} c_m^2 m! = \mathrm{E}(H(Z)^2) < \infty$, Z being an $N(0, 1)$ random variable.

By the results of Nualart and Peccati (2005), Peccati and Tudor (2005) and Hu and Nualart (2005), in order to prove that the vector $(B^{(n)}, Y^{(n)})$ converges in distribution to a Gaussian vector (B, V), where B and Y are independent and Y has independent components, it suffices to show the following facts:

(i) For any $m \geqslant 2$ and $k = 1, \ldots, N$, the limit $\lim_{n\to\infty} \mathrm{E}(|J_m Y_k^{(n)}|^2) = \sigma_{m,k}^2$ exists and $\sum_{\mathrm{m}=2}^{\infty} \sup_n \mathrm{E}(|J_m Y_k^{(n)}|^2) < \infty$.

(ii) For any $m \geqslant 2$ and $k \neq h$, $\lim_{n\to\infty} \mathrm{E}(J_m Y_k^{(n)} J_m Y_h^{(n)}) = 0$.

(iii) For any $m \geqslant 2$, $k = 1, \ldots, N$ and $1 \leqslant p \leqslant m-1$,

$$\lim_{n\to\infty} I_m^{-1} J_m Y_k^{(n)} \otimes_p I_m^{-1} J_m Y_k^{(n)} = 0,$$

where $\otimes_p$ denotes the contraction of p indices.

We have

$$J_m Y_k^{(n)} = \frac{c_m}{\sqrt{n}} \sum_{[na_k] < j \leqslant [nb_k]} H_m(X_j).$$

Hence,

$$\begin{aligned}
\mathrm{E}\left(|J_m Y_k^{(n)}|^2\right) &= \frac{m! c_m^2}{n} \sum_{[na_k] < j, l \leqslant [nb_k]} \rho_H(j-l)^m \\
&= m! c_m^2 \left(\rho_H(0)^m \frac{[nb_k] - [na_k]}{n} . \right. \\
&\quad \left. + 2 \sum_{j=1}^{[nb_k]-[na_k]} \rho_H(j)^m \frac{[nb_k] - [na_k] - j}{n} \right).
\end{aligned}$$

This implies $\sum_{\mathrm{m}=2}^{\infty} \sup_n \mathrm{E}(|J_m Y_k^{(n)}|^2) < \infty$ and, furthermore,

$$\lim_{n\to\infty} \mathrm{E}(|J_m Y_k^{(n)}|^2) = m! c_m^2 |J_k| \left(\rho_H(0)^m + 2 \sum_{j=1}^{\infty} \rho_H(j)^m \right),$$

which implies (i).

Property (ii) follows from the estimates, for any $m \geqslant 2$ and $b_k < a_h$,

$$\mathrm{E}\Big(J_m Y_k^{(n)} J_m Y_h^{(n)}\Big) = \frac{m! c_m^2}{n} \sum_{j=[na_k]+1}^{[nb_k]} \sum_{i=[na_h]+1}^{[nb_h]} \rho_H(i-j)^m$$

$$\leqslant C \sum_{k=[na_h]-[nb_k]+1}^{[nb_h]-[na_k]-1} \rho_H(k)^m$$

which tend to zero as n tends to infinity.

Let us check (iii). Fix $m \geqslant 2$, $k = 1, \ldots, N$ and $1 \leqslant p \leqslant m-1$. We have

$$I_m^{-1} J_m Y_k^{(n)} \tilde{\otimes}_p I_m^{-1} J_m Y_k^{(n)} = \frac{c_m^2}{n} \sum_{[na_k]<j,l\leqslant[nb_k]} X_j^{\otimes m} \tilde{\otimes}_p X_l^{\otimes m}$$

$$= \frac{c_m^2}{n} \sum_{[na_k]<j,l\leqslant[nb_k]} \rho_H(j-l)^p \Big(X_j^{\otimes(m-p)} \tilde{\otimes} X_l^{\otimes(m-p)}\Big),$$

where the tilde denotes symmetrization. Thus, we have to show that the following quantity converges to zero as n tends to infinity:

$$n^{-2} \sum_{[na_k]<j,l,h,k\leqslant[nb_k]} \rho_H(j-l)^p \rho_H(h-k)^p$$
$$\times \langle X_j^{\otimes(m-p)} \tilde{\otimes} X_l^{\otimes(m-p)}, X_h^{\otimes(m-p)} \tilde{\otimes} X_k^{\otimes(m-p)} \rangle_{\mathcal{H}_1^{\otimes 2(m-p)}}.$$

It suffices to consider a term of the form

$$n^{-2} \sum_{[na_k]<j,l,h,k\leqslant[nb_k]} \rho_H(j-l)^p \rho_H(h-k)^p$$
$$\times\ \rho_H(j-h)^\alpha \rho_H(l-h)^{m-p-\alpha} \rho_H(j-k)^{m-p-\alpha} \rho_H(l-k)^\alpha,$$

where $0 \leqslant \alpha \leqslant m-p$. This is bounded by

$$Nn^{-1} \sum_{0\leqslant j,l,k\leqslant nN} \rho_H(j-l)^p \rho_H(k)^p \rho_H(j)^\alpha \rho_H(l)^{m-p-\alpha} \rho_H(j-k)^{m-p-\alpha} \rho_H(l-k)^\alpha,$$

for some natural number N. Without loss of generality, we can assume $p = m-p = 1$ and $\alpha = 0$ or $\alpha = 1$. For $\alpha = 0$ and any $0 < \varepsilon < 1$, we obtain

$$
\begin{aligned}
& n^{-1}\sum_{0\leqslant j\leqslant nN}\left(\sum_{0\leqslant l\leqslant nN}\rho_H(j-l)\rho_H(l)\right)^2 \\
& \leqslant n^{-1}\sum_{0\leqslant j\leqslant [nN\varepsilon]}\left(\sum_{0\leqslant l\leqslant nN}\rho_H(j-l)\rho_H(l)\right)^2 \\
& \quad + 2n^{-1}\sum_{[nN\varepsilon]<j\leqslant nN}\left(\sum_{0\leqslant l\leqslant [nN\varepsilon/2]}\rho_H(j-l)\rho_H(l)\right)^2 \\
& \quad + 2n^{-1}\sum_{[nN\varepsilon]<j\leqslant nN}\left(\sum_{[nN\varepsilon/2]<l\leqslant nN}\rho_H(j-l)\rho_H(l)\right)^2 \\
& \leqslant 2N\varepsilon\left(\sum_{0\leqslant l<\infty}\rho_H(l)^2\right)^2 + 6N\sum_{0\leqslant l<\infty}\rho_H(l)^2\sum_{[nN\varepsilon/2]<l<\infty}\rho_H(l)^2,
\end{aligned}
$$

which converges to $2N\varepsilon(\sum_{0\leqslant l<\infty}\rho_H(l)^2)^2$ as n tends to infinity and the result follows by letting ε tend to zero.

Let us compute the variance of the limit. We have

$$
\begin{aligned}
\mathrm{E}(|Y_k^{(n)}|^2) &= \frac{[nb_k]-[na_k]}{n}\mathrm{var}(|B_1^H|^p) \\
&\quad + 2\sum_{j=1}^{[nb_k]-[na_k]}\frac{[nb_k]-[na_k]-j}{n}\mathrm{cov}\big(H(X_1),\ H(X_{1+j})\big),
\end{aligned}
$$

where $\mathrm{var}(|B_1^H|^p)=\delta_p$. By Lemma 9 we obtain

$$
\mathrm{cov}(H(X_1),\ H(X_{1+j})) = \gamma_p(\rho_H(j)) - \gamma_p(0),
$$

and hence we get the desired limit variance. □

Acknowledgements

The work of J.M. Corcuera and D. Nualart is supported by MCyT Grant no. BFM2003-04294. J. Woerner would like to thank J.M. Corcuera and D. Nualart for their hospitality during her stay at the University of Barcelona where part of this research was carried out.

References

Aldous, D.J. and Eagleson, G.K. (1978) On mixing and stability of limit theorems. *Ann. Probab.*, **6**(2), 325–331.

Barndorff-Nielsen, O.E. and Shephard, N. (2002) Econometric analysis of realized volatility and its use in estimating stochastic volatility models. *J. Roy. Statist. Soc., Ser. B*, **64**, 253–280.

Barndorff-Nielsen, O.E. and Shephard, N. (2003) Realized power variation and stochastic volatility models. *Bernoulli*, **9**, 243–265.

Barndorff-Nielsen, O.E. and Shephard, N. (2004a) Power and bipower variation with stochastic volatility and jumps (with discussion). *J. Financial Econometrics*, **2**, 1–48.

Barndorff-Nielsen, O.E. and Shephard, N. (2004b) Econometric analysis of realised covariation: high frequency covariance, regression and correlation in financial economics. *Econometrica*, **72**, 885–925.

Barndorff-Nielsen, O.E., Graversen S. E., Jacod, J., Podolskij, M. and Shephard, N. (2006) A central limit theorem for realised power and bipower variations of continuous semimartingales. In Y. Kabanov, R. Liptser and J. Stoyanov (eds), *From Stochastic Analysis to Mathematical Finance: The Shiryaev Festschrift*. Berlin: Springer-Verlag.

Billingsley, P. (1968). *Convergence of Probability Measures*. New York: Wiley.

Breuer, P. and Major, P. (1983) Central limit theorems for non-linear functionals of Gaussian fields. *J. Multivariate Anal.*, **13**, 425–441.

Dobrushin, R.L. and Major, P. (1979) Non-central limit theorems for non-linear functionals of Gaussian fields, *Z. Wahrscheinlichkeitstheorie Verw. Geb.,* **50**, 27–52.

Fernique, X. (1975) Regularité des trajectoires des fonctions aléatoires gaussiennes. In P.L. Hennequin, *École d'Été de Probabilités de Saint-Flour IV – 1974*, Lecture Notes in Math. 480. Berlin: Springer-Verlag.

Giraitis, L. and Surgailis, D. (1985) CTL and other limit theorems for functionals of Gaussian processes. *Z. Wahrscheinlichkeitstheorie Verw. Geb.,* **70**, 191–212.

Hu, Y. and Nualart, D. (2005) Renormalized self-intersection local time for fractional Brownian motion. *Ann. Probab.*, **33**, 948–983.

Hudson, W.N. and Mason J.D. (1976) Variational sums for additive processes. *Proc. Amer. Math. Soc.*, **55**, 395–399.

León, J.R. and Ludeña, C. (2004) Stable convergence of certain functionals of diffusions driven by fBm. *Stochastic Anal. Appl.*, **22**, 289–314.

Nualart, D. and Peccati, G. (2005) Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.*, **33**, 177–193.

Peccati, G. and Tudor, C.A. (2005) Gaussian limits for vector-valued multiple stochastic integrals. In M. Émery, M. Ledoux and M. Yor (eds), *Séminaire de Probabilités XXXVIII*, Lecture Notes in Math. 1857, pp. 247–262. Berlin: Springer-Verlag.

Rosenblatt, M. (1961) Independence and dependence. In J. Neyman (ed.), *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 2, pp. 411–443. Berkeley: University of California Press.

Shiryaev, A.N. (1999) *Essentials of Stochastic Finance.* Singapore: World Scientific.

Sun, T.C. and Ho, H.C. (1986) On central and non-central limit theorems for non-linear functions of stationary Gaussian process. In E. Eberlein and M.S. Taqqu (eds.). *Dependence in Probability and Statistics: A Survey of recent results*, Progr. Probab. 11. Boston: Birkhäuser.

Taqqu, M.S. (1975) Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrscheinlichkeitstheorie Verw. Geb.,* **31**, 287–302.

Taqqu, M.S. (1977) Law of the iterated logarithm for sums of non-linear functions of Gaussian variables that exhibit a long range dependence. *Z. Wahrscheinlichkeitstheorie Verw. Geb.,* **40**, 203–238.

Taqqu, M.S. (1979) Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **50**, 53–83.

Woerner, J.H.C. (2003) Variational sums and power variation: a unifying approach to model selection and estimation in semimartingale models. *Statist. Decisions*, **21**, 47–68.

Woerner, J.H.C. (2005) Estimation of Integrated Volatility in Stochastic Volatility Models. *Appl. Stochastic Models Business Industry*, **21**, 27–44.

Young, L.C. (1936) An inequality of the Hölder type connected with Stieltjes integration, *Acta Math.*, **67**, 251–282.

Received April 2005 and revised January 2006