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Fractional integral equations and state space transforms

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We introduce a class of stochastic differential equations driven by fractional Brownian motion which allow for a constructive method in order to obtain stationary solutions. This leads to a substantial extension of the fractional Ornstein–Uhlenbeck processes. Structural properties of this class of new models are investigated, and their stationary densities are explicitly given.

Keywords: fractional Brownian motion; fractional integral; fractional Ornstein–Uhlenbeck process; fractional Vasicek model; Langevin equation; long-range dependence; Riemann–Stieltjes integrals; state space transform; stochastic calculus; solution of stochastic differential equations

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space carrying a two-sided fractional Brownian motion $(B_t^H)_{t\in\mathbb{R}}$ (FBM) with Hurst index $H \in (0, 1)$, i.e., a centred Gaussian process $(t, \omega) \mapsto B_t^H(\omega)$, $t \in \mathbb{R}$ and $\omega \in \Omega$ with locally Hölder continuous sample paths up to every order $\alpha < H$ and covariance function

$$\mathrm{E}B_t^H B_s^H = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \qquad s,\ t \in \mathbb{R}. \tag{1.1}$$

The process has stationary increments and is self-similar, i.e., for all $c \in \mathbb{R}$,

$$(B_{ct}^H) \overset{\mathrm{d}}{=} |c|^H (B_t^H), \qquad t \in \mathbb{R};$$

in particular, $B_0^H = 0$ with probability one.

Moreover, sample paths of B^H are nowhere differentiable and its variation is always infinite. A Hurst index of $H = 1/2$ corresponds to a standard Brownian motion. As the quadratic variation of B^H is 0 for $H > 1/2$ and infinite for $H < 1/2$, an FBM is not a semimartingale for $H \neq 1/2$. The Itô integral with respect to an FBM is therefore not defined for any $H \neq 1/2$. Moreover, an FBM exhibits short-range dependence for $H < 1/2$ and long-range dependence for $H > 1/2$. Further properties can be found in Samorodnitsky and Taqqu (1994).

We wish to use an FBM as the driving process of a stochastic differential equation (SDE)

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and start with an Ornstein–Uhlenbeck model. For $\gamma > 0$, consider the stationary fractional Ornstein–Uhlenbeck process (FOUP), i.e.,

$$O_t^{H,\gamma} = \int_{-\infty}^t \mathrm{e}^{-\gamma(t-s)}\,\mathrm{d}B_s^H, \qquad t \in \mathbb{R}. \tag{1.2}$$

This stochastic integral converges as a pathwise improper Riemann–Stieltjes (RS) integral. In this RS sense $O^{H,\gamma} = (O_t^{H,\gamma})_{t\in\mathbb{R}}$ solves pathwise the SDE

$$O_0 = \int_{-\infty}^0 \mathrm{e}^{\gamma s}\,\mathrm{d}B_s^H, \qquad \mathrm{d}O_t = -\gamma O_t\,\mathrm{d}t + \mathrm{d}B_t^H, \quad t > 0. \tag{1.3}$$

This SDE is called the Langevin equation, and it has a long and successful history, particularly in physics; for details, see Mikosch and Norvaiša (2000).

In our paper we consider fractional integral equations of the type

$$X_t - X_s = \int_s^t \mu(X_u)\mathrm{d}u + \int_s^t \sigma(X_u)\mathrm{d}B_u^H, \qquad s \leqslant t, \tag{1.4}$$

where all integrals are interpreted in the pathwise RS sense. We ask what functions μ and σ allow for a stationary solution $X = (f(O_t))_{t\in\mathbb{R}}$ where f is a monotone transformation. We summarize relations on μ and σ in the concept of H-*proper triples* in Section 3.

The paper is organized as follows. In Section 2 we introduce the notation and summarize the material regarding RS integrals. Furthermore, we discuss and refine (without proof) some results on the Langevin equation and the FOUP as given in Cheridito *et al.* (2003). Section 3 includes our main theorems and relates the concept of H-proper triples to the transformation f. Various authors have studied the existence and uniqueness of the SDE (1.4). Our approach is mainly based on Zähle (1998) and Klingenhöfer and Zähle (1999). In Section 4 the structural properties of H-proper triples are analysed in detail. Those results are applied in Section 5, where we discuss some examples of μ and σ.

2. Preliminaries

Throughout this paper we use the convention $\int_\beta^\alpha g(x)\mathrm{d}x = -\int_\alpha^\beta g(x)\mathrm{d}x$ for $\alpha < \beta$. We start with some smoothness and integrability conditions.

Definition 2.1. *For $M \subseteq \mathbb{R}$ we define the following spaces of functions $f : M \to \mathbb{R}$.*

(i) *$\mathcal{C}(M)$ denotes the space of continuous functions.*
(ii) *If $M \subseteq \mathbb{R}$ is open and $k \in \mathbb{N} \cup \infty\}$, then $\mathcal{C}^k(M)$ denotes the space of k-times continuously differentiable functions.*
(iii) *For $\beta < 0$, $\mathcal{C}^{\beta-}(M)$ ($\mathcal{C}^{\beta+}(M)$) denotes the space of functions such that, for all compact intervals $K \subseteq M$, the restriction $f : K \to \mathbb{R}$ is Hölder continuous of all orders $\alpha < \beta$ (of at least some order $\alpha = \alpha(K) > \beta$)*
(iv) *For any Borel set $M \subseteq \mathbb{R}$, $\mathcal{L}_C(M)$ denotes the set of measurable functions*

$g : M \to \overline{\mathbb{R}}$ which are locally integrable, i.e., the Lebesgue integral $\int_K |g(t)|\,\mathrm{d}t < \infty$ for all compact $K \subseteq M$.

(v) *$\mathcal{AC}(M)$ denotes the set of locally absolutely continuous functions $f : M \to \mathbb{R}$, i.e., there exists $g \in \mathcal{L}_C(M)$ such that $f(y) = f(x) + \int_x^y g(z)\mathrm{d}z$ for all $[x, y] \subseteq M$.*

We summarize some well-known facts in the following remark.

Remark 2.1. For $M, N \subseteq \mathbb{R}$, let $f : N \to \mathbb{R}$ and $g : M \to N$. For $\beta > 0$ and $H \in (0, 1)$, the following assertions hold.

(i) If $f \in \mathcal{C}^{((1-H)/H)+}(N)$ and $g \in \mathcal{C}^{H-}(M)$, then $f \circ g \in \mathcal{C}^{(1-H)+}(M)$.
(ii) If M is open and $g \in \mathcal{C}^1(M)$ and $f \in \mathcal{C}^{\beta+}(N)$, then $f \circ g \in \mathcal{C}^{\beta+}(M)$.
(iii) If N is open and $g \in \mathcal{C}^{\beta-}(M)$ and $f \in \mathcal{C}^1(N)$, then $f \circ g \in \mathcal{C}^{\beta-}(M)$.
(iv) If $M = \mathbb{R}$ and $g \in \mathcal{C}^1(\mathbb{R})$ with $g' \in \mathcal{C}^{\beta+}(\mathbb{R})$, where $g : \mathbb{R} \to g(\mathbb{R}) = N$ is strictly increasing, then $g^{-1} \in \mathcal{C}^1(N \backslash g(Z(g')))$ with $(g^{-1})' \in \mathcal{C}^{\beta+}(N \backslash g(Z(g')))$.
(v) If $\beta \geqslant 1$ and $g \in \mathcal{C}^{\beta+}(\mathbb{R})$, then g is constant.

The following properties will be used throughout.

Remark 2.2. (i) Almost all sample paths of B^H are elements of $\mathcal{C}^{H-}(\mathbb{R}) \cap \mathcal{C}^{H'+}(\mathbb{R})$ for all $H' \in (0, H)$ and all $H \in (0, 1)$; but recall that B^H has Hölder continuous paths of order α only strictly less than H (Decreusefond and Üstünel 1999, Theorem 3.1).

(ii) For $H > 0$ we define a Banach space of continuous functions by

$$V_H := \left\{ g \in \mathcal{C}(\mathbb{R}) : \|g\|_H := \sup_{t \in \mathbb{R}} \frac{|g(t)|}{1 + |t|^H L(|t|)} < \infty \right\}, \tag{2.1}$$

where $L(x) = \sqrt{\log \log x}$ for $x \geqslant \mathrm{e}$, and $L(x) = 0$ otherwise. If $H \in (0, 1)$, then almost all sample paths of B^H belong to V_H by the law of the iterated logarithm (Arcones 1995, Corollary 3.1).

(iii) Define $Z(h) := \{x \in M : h(x) = 0\}$, the set of zeros of h in M of a function $h : M \to \mathbb{R}$. If $M \subseteq \mathbb{R}$ is an open interval, $g \in \mathcal{L}_C(M)$ and $f \in \mathcal{AC}(M)$ is strictly increasing with $f(t) = f(s) + \int_s^t g(z)\mathrm{d}z$, then $f^{-1} \in \mathcal{AC}(f(M))$ if and only if $Z(g)$ has Lebesgue measure zero. In this case, if $h \in \mathcal{L}_C(f(M))$ with $f^{-1}(y) = f^{-1}(x) + \int_x^y h(z)\mathrm{d}z$ for all $x, y \in f(M)$, then almost everywhere

$$h = 1/(g \circ f^{-1}).$$

The next proposition rephrases results on RS integrals as given in Zähle (1998, Theorems 4.2.1, 4.3.1 and 4.4.2) in terms of the Hölder spaces $\mathcal{C}^{H-}(M)$ and $\mathcal{C}^{H+}(M)$. Assertion (i) was shown by Young (1936, Sections 8 and 10).

Proposition 2.1. *Let $H \in (0, 1)$ and $M, N \subseteq \mathbb{R}$. Suppose that $a \leqslant b$ exist such that $[a, b] \subseteq M$.*

(i) *If $f \in \mathcal{C}^{(1-H)+}(M)$ and $g \in \mathcal{C}^{H-}(M)$, then $\int_a^b f(x)\mathrm{d}g(x)$ exists.*
(ii) *(Chain rule) Suppose $g \in \mathcal{C}^{H-}(M)$ and let $f \in \mathcal{C}^1(N)$, where $g(M) \subseteq N$.*

If $f' \circ g \in \mathcal{C}^{(1-H)+}(M)$, *then*

$$f(g(y)) - f(g(a)) = \int_a^y f'(g(x))\mathrm{d}g(x), \qquad a \leqslant y \leqslant b.$$

(iii) *(Density formula) Let* $f,\, h \in \mathcal{C}^{(1-H)+}(M)$ *and* $g \in \mathcal{C}^{H-}(M)$. *Then*

$$\phi(y) := \int_a^y h(x)\mathrm{d}g(x), \qquad a \leqslant y \leqslant b,$$

exists. Furthermore, $\phi \in \mathcal{C}^{H-}([a, b])$ *and*

$$\int_a^b f(x)h(x)\mathrm{d}g(x) = \int_a^b f(x)\mathrm{d}\phi(x).$$

Throughout this paper all integrals $\int_s^t \sigma(X_u)\mathrm{d}B_t^H$ will be interpreted in the pathwise RS sense. In our analysis we replace B^H by sample paths g from a suitable subspace of $\mathcal{C}(\mathbb{R})$. By Remark 2.2(ii), $B^H \in V_H \cap \mathcal{C}^{H-}(\mathbb{R})$ with probability one; it is therefore convenient to work with functions $g \in V_H \cap \mathcal{C}^{H-}(\mathbb{R})$. We need the following definition to make our approach precise.

Definition 2.2. *Let* $H \in (0, 1)$ *and* $g \in V_H \cap \mathcal{C}^{H-}(\mathbb{R})$. *Suppose that* $I \subseteq \mathbb{R}$ *is non-empty and* $\mu, \sigma \in \mathcal{C}(I)$. *We refer to* x *as a solution of*

$$\mathrm{d}x(t) = \mu(x(t))\mathrm{d}t + \sigma(x(t))\mathrm{d}g(t) \tag{2.2}$$

if $x \in \mathcal{C}^{H-}(\mathbb{R})$ *and* x *takes values in* I *such that for* $s \leqslant t$,

(S1) $\sigma \circ x$ *is RS integrable with respect to* g *on* $[s, t]$;
(S2) *the integral equation*

$$x(t) - x(s) = \int_s^t \mu(x(u))\mathrm{d}u + \int_s^t \sigma(x(u))\mathrm{d}g(u).$$

holds.

The space of all solutions x *of* (2.2) *is denoted by* $\mathcal{S}^H(I, \mu, \sigma, g)$.

Remark 2.3. Under our assumptions on μ and g, $\int_s^t \mu(x(u))\mathrm{d}u$ in (S2) always exists as a Riemann integral. If, additionally, $\sigma \in \mathcal{C}^{((1-H)/H)+}(I)$, then (S1) is satisfied by Remark 2.1(i) and Proposition 2.1(i). However, the RS integral may also exist under weaker assumptions. Thus, we will not explicitly state such a condition on σ; see also Remark 5.1 below.

For $g \in V_H$ and $\gamma > 0$, we define the (Ornstein–Uhlenbeck) operator O^γ by

$$O_t^\gamma(g) := \int_{-\infty}^t \mathrm{e}^{\gamma(t-s)}\,\mathrm{d}g(s) := g(t) - \gamma\int_{-\infty}^t \mathrm{e}^{\gamma(t-s)}g(u)\mathrm{d}u, \qquad t \in \mathbb{R}. \tag{2.3}$$

For all $g \in V_H$, the right-hand integral exists as an improper Riemann integral or Lebesgue integral. Partial integration shows that the integral on the left-hand side converges as an improper RS integral (see Lang 1993, Proposition X, Chapter 1, Section 1.4.). In particular,

O^γ is well defined as an operator on V_H. Furthermore, Remarks 2.2(i)–(ii) imply that the pathwise identity $O^{H,\gamma} = O^\gamma(B^H)$ holds with probability one.

The next proposition summarizes properties of the operator O^γ.

Proposition 2.2 *Let* $H \in (0, 1)$ *and* $\gamma > 0$. *The following assertions hold.*

(i) $O^\gamma : V_H \to V_H$ *defines a continuous linear operator.*
(ii) *Let* $g \in V_H$ *and* $H' \in (0, 1)$. *Then*

$$g \in \mathcal{C}^{H'-}(\mathbb{R}) \iff O^\gamma(g) \in \mathcal{C}^{H'-}(\mathbb{R}),$$

$$g \in \mathcal{C}^{H'+}(\mathbb{R}) \iff O^\gamma(g) \in \mathcal{C}^{H'+}(\mathbb{R}).$$

Proof. (i) and (ii) are direct consequences of the second equality in (2.3). □

Next we define, for $g \in V_H$, $\tau, y \in \mathbb{R}$ and $\gamma > 0$,

$$O_t^\gamma(g, \tau, y) = O_t^\gamma(g) - \mathrm{e}^{-\gamma(t-\tau)} O_\tau^\gamma(g) + \mathrm{e}^{-\gamma(t-\tau)} y, \qquad t \in \mathbb{R}. \tag{2.4}$$

Observe that $O_\tau^\gamma(g, \tau, y) = y$ for all $\tau, y \in \mathbb{R}$; moreover,

$$O^\gamma(g, \tau, O_\tau^\gamma(g)) = O^\gamma(g). \tag{2.5}$$

The following theorem connects operators to solutions of (1.3). It is a refinement of Proposition A.1 c) of Cheridito *et al.* (2003). The proof is omitted.

Theorem 2.3. *Let* $H \in (0, 1)$, $\gamma > 0$ *and* $g \in V_H$. *Then the following assertions hold.*

(i) *If* $o \in \mathcal{C}(\mathbb{R})$, *then* $o = O^\gamma(g, \tau, o(\tau))$ *for all* $\tau \in \mathbb{R}$ *if and only if*

$$o(t) - o(s) = -\gamma \int_s^t o(u)\mathrm{d}u + g(t) - g(s), \qquad s \leqslant t. \tag{2.6}$$

(ii) *If* $o \in V_H$ *is a solution of* (2.6), *then* $o = O^\gamma(g)$.
(iii) $O^{H,\gamma} = O^\gamma(B^H)$ *is the unique strictly stationary pathwise solution of*

$$O_t^{H,\gamma} - O_s^{H,\gamma} = -\gamma \int_s^t O_u^{H,\gamma}\, \mathrm{d}u + B_t^H - B_s^H, \qquad s \leqslant t.$$

3. Stationary solutions via state space transforms

We start with a preliminary definition which will later be made more specific.

Definition 3.1. *A function* $f : \mathbb{R} \to \mathbb{R}$ *is called a* state space transform (SST) *if it is continuous and strictly increasing; the open interval* $I = f(\mathbb{R})$ *is called the* state space.

As the FOUP process $O^{H,\gamma}$ as defined in (1.2) takes values in $\mathbb{R}$ and is stationary, any process of the form $f(O^{H,\gamma})$ has state space $f(\mathbb{R})$ and strict stationarity is preserved under

f. On the other hand, for any given interval I, an SST f with state space I exists and we can construct a new stochastic process $(X_t)_{t\in\mathbb{R}}$ with state space I by setting $X_t := f(O_t^{H,\gamma})$. The concept of state space transforms is borrowed from the theory of regular diffusions (e.g. Karlin and Taylor 1981, Theorem 2.1). A standard tool in the theory of diffusions is the concept of a scale function (Itô and McKean 1974; Revuz and Yor 1998), which turns one-dimensional diffusions into continuous local martingales. As we are dealing with neither martingales nor Markov processes, these concepts are only loosely connected to our work. We aim at stationary solutions of integral equations (1.4) driven by FBM. More precisely, we are interested in the existence of solutions $X = f(O^{H,\gamma})$, where f is an SST and $\gamma > 0$. We start with a simple example.

Example 3.1 *Fractional Vasicek model*. Define the SST $f(x) = \sigma x - \alpha/\beta,\ x \in \mathbb{R}$, for $\beta < 0$, $\sigma > 0$ and $\alpha \in \mathbb{R}$. Set $\gamma = -\beta$. Then the process $V_t = f(O_t^\gamma) = \sigma\, O_t^{H,\gamma} - \alpha/\beta$, $t \in \mathbb{R}$, inherits its stationarity from the FOUP. Moreover, the following equality holds for all $s \leqslant t$:

$$V_t - V_s = -\gamma \int_s^t \sigma\, O_u^{H,\gamma}\,\mathrm{d}u + \sigma(B_t^H - B_s^H) = \int_s^t (\alpha + \beta\, V_u)\mathrm{d}u + \sigma(B_t^H - B_s^H).$$

Hence, V is the solution of the SDE $\mathrm{d}V_t = \mu(V_t)\mathrm{d}t + \sigma\mathrm{d}B_t^H$, where $\mu(x) = \alpha + \beta x$. Obviously, V serves as a natural extension of the usual Vasicek model driven by the Wiener process to the fractional world.

In order to generalize this approach, we introduce the following concept. Recall from Remark 2.2(iii) the definition of $Z(h)$.

Definition 3.2. *A triple* (I, μ, σ) *is called* proper *if the following properties are satisfied:*

(P1) $I = (l, r) \subseteq \mathbb{R}$ *is an open non-empty interval and* $\mu, \sigma \in \mathcal{C}(I)$.
(P2) *There exists* $\psi \in \mathcal{AC}(I)$ *strictly decreasing such that* $\psi = \mu/\sigma$ *on* $I\backslash Z(\sigma)$ *and*

$$\lim_{x\uparrow r} \psi(x) = -\lim_{x\downarrow l} \psi(x) = -\infty. \tag{3.1}$$

(P3) *There exists* $\gamma > 0$ *such that* $\sigma\psi' \equiv -\gamma$ *Lebesgue-a.e. on* I.

Let $H \in (0, 1)$. *A triple* (I, μ, σ) *is called* H-proper *if* (I, μ, σ) *is proper and, in addition, the following property holds:*

(P4) *The inverse function* $\psi^{-1} : \mathbb{R} \to \psi^{-1}(\mathbb{R}) = I$ *is differentiable with* $(\psi^{-1})' \in \mathcal{C}^{((1-H)/H)+}(\mathbb{R})$.

Remark 3.1. (i) By (P2), $\psi : I \to \psi(I) = \mathbb{R}$ is strictly decreasing and absolutely continuous. In particular, the additional property (P4) makes sense. Furthermore, ψ is a.e. differentiable on I with $\psi' \leqslant 0$; (P3) implies that both sets $Z(\sigma)$ and $Z(\psi')$ have Lebesgue measure zero. Furthermore, σ is non-negative and $1/\sigma \in \mathcal{L}_C(I)$. Additionally, $I\backslash Z(\sigma)$ is dense in I and open by (P1). By continuity the equality $\mu = \sigma\psi$ extends to I. Consequently, ψ and, therefore, γ are uniquely determined by μ and σ.

(ii) Let $H \in (0, 1/2]$ and (I, μ, σ) be H-proper. By Remark 2.1, $\psi' \in \mathcal{C}^{((1-H)/H)+}(\mathbb{R})$ implies that ψ' is constant. Thus, for some $\alpha, \beta \in \mathbb{R}$, $\psi(x) = \alpha x + \beta$, $x \in I$. (P2) implies that $I = \mathbb{R}$ and $\alpha < 0$. By (P3) the function σ reduces to a non-negative constant; furthermore, $\mu = \sigma\psi$ is affine. Thus, (I, μ, σ) is a Vasicek model as considered in Example 3.1.

We summarize some notation in the following definition.

Definition 3.3. *Let* (I, μ, σ) *be proper.*

(i) *The interval* I *is called the* state space.
(ii) *The (unique)* $\gamma > 0$ *in* (P3) *is called the* friction coefficient (FC) (for (I, μ, σ)).
(iii) *The (unique) SST* $f : \mathbb{R} \to I = f(\mathbb{R})$, $f(x) := \psi^{-1}(-\gamma x)$, *is called the* SST (for (I, μ, σ)).

We introduce the concept of a centre.

Definition 3.4. *Let* $(l, r) \subseteq \mathbb{R}$ *be an open interval and* $h \in \mathcal{C}((l, r))$. *The (unique) number* $\xi \in (l, r)$ *is called a centre for* h *if* $h(x)$ *is non-negative for* $x \in (l, \xi)$ *and non-positive for* $x \in (\xi, r)$, *and* $Z(h)$ *has Lebesgue measure zero.*

Every proper triple (I, μ, σ) has a centre for μ.

Lemma 3.1. *If* (I, μ, σ) *is proper then there exists a centre* ξ *for* μ *with the following properties:*

$$Z(\psi) = \{\xi\}, \qquad f(0) = \xi, \qquad Z(\mu) = Z(\sigma) \cup \{\xi\}.$$

Proof. Note that there exists a unique ξ such that $Z(\psi) = \{\xi\}$ as $\psi : I \to \psi(I) = \mathbb{R}$ is strictly decreasing and continuous by (P2). By Definition 3.3(iii), $f(0) = \xi$ is immediate by construction. As $\psi : I \to \mathbb{R}$ is strictly decreasing we obtain $\psi(x) > 0$ for all $l < x < \xi$, while $\psi(x) < 0$ for all $\xi < x < r$. By Remark 3.1(i), the equality $\mu = \sigma\psi$ holds on the whole of I and therefore $Z(\sigma) \cup \{\xi\} = Z(\mu)$ is immediate. Since σ is non-negative on I, $\mu(x)$ is non-negative for $x \leqslant \xi$ and non-positive for $x \geqslant \xi$. Finally, as $Z(\sigma)$ has Lebesgue measure zero by Remark 3.1(i), the same holds for $Z(\mu)$; thus, ξ is a centre for μ. $\square$

In the next two lemmas we present a differential equation for the corresponding SST f which shows that f is determined by σ and the centre ξ only. Furthermore, we give a sufficient condition such that a proper triple is H-proper.

Lemma 3.2. *Suppose that* (I, μ, σ) *is proper with the corresponding SST* f. *Let* ξ *be the centre for* μ. *Then the following assertions hold:*

(i) $f \in \mathcal{C}^1(\mathbb{R})$ *with* $f' = \sigma \circ f$ *and* $f(0) = \xi$. *Furthermore,* $f^{-1} \in \mathcal{C}^1(I \backslash Z(\sigma))$ *with* $(f^{-1})'(x) = 1/\sigma(x)$ *for all* $x \in I \backslash Z(\sigma)$.
(ii) *If* $g \in \mathcal{C}^1(\mathbb{R})$ *is an SST with state space* I *such that* $g' = \sigma \circ g$ *and* $g(0) = \xi$, *then*

$$f(x) \leqslant g(x) \ \textit{ for all } x \leqslant 0 \quad \textit{and} \quad f(x) \geqslant g(x) \textit{ for all } x \geqslant 0.$$

Furthermore, $f = g$ *if and only if* $g^{-1} \in \mathcal{AC}(I)$.

Proof. (i) By Definition 3.3, $f(0) = \xi$. Construction of f and Remark 3.1(i) guarantee that $1/\sigma \in \mathcal{L}_C(I)$ and $f^{-1} \in \mathcal{AC}(I)$ with $(f^{-1})' = 1/\sigma$ a.e. In particular, the set $Z((f^{-1})')$ has Lebesgue measure zero. Thus, $f \in \mathcal{AC}(\mathbb{R})$ and, a.e.,

$$f' = \frac{1}{(f^{-1})' \circ f} = \sigma \circ f. \tag{3.2}$$

As the right-hand side is continuous we even obtain $f \in C^1(\mathbb{R})$. In particular, (3.2) extends to the whole of $\mathbb{R}$. Furthermore, (3.2) implies $f' \circ f^{-1} = \sigma$ on I; consequently, $(f^{-1})' = 1/\sigma$ on $I \backslash Z(\sigma)$. The set $I \backslash Z(\sigma)$ is open; thus, $f^{-1} \in \mathcal{C}^1(I \backslash Z(\sigma))$.

(ii) Let $g \in \mathcal{C}^1(\mathbb{R})$ be an SST with state space I, $g(0) = \xi$ and $g' = \sigma \circ g$. Lebesgue's decomposition theorem states the existence of non-decreasing and continuous functions $h_1, h_2 \in \mathcal{C}(I)$ such that $g^{-1} = h_1 + h_2$, where $h_1 \in \mathcal{AC}(I)$ and h_2 is the distribution function of a positive σ-finite measure ρ which is singular to the Lebesgue measure. Without loss of generality, suppose that $h_1(\xi) = 0$ and $h_2(0) = 0$. As $g \in \mathcal{C}^1(\mathbb{R})$ and $g' = \sigma \circ g$ we find g^{-1} differentiable on $I \backslash Z(\sigma)$ with $(g^{-1})'(x) = 1/\sigma(x)$ for all $x \in I \backslash Z(\sigma)$. As h_1 is differentiable a.e. on I and $Z(\sigma)$ has Lebesgue measure zero by Remark 3.1(i), $h_2 = g^{-1} - h_1$ is differentiable a.e. on I with $h_2' = 0$ since ρ is singular to the Lebesgue measure. Thus, $(f^{-1})' = (g^{-1})' = h_1'$ a.e. on I. Since $h_1(\xi) = 0$ we obtain $h_1 = f^{-1}$ and therefore $g^{-1} = f^{-1} + h_2$ on I.

The rest of the assertion is immediate. □

Lemma 3.3. *Let* $H \in (0, 1)$ *and* (I, μ, σ) *be proper with SST* f.

(i) (I, μ, σ) *is* H*-proper if and only if* $f \in \mathcal{C}^1(\mathbb{R})$ *and* $f' \in \mathcal{C}^{((1-H)/H)+}(\mathbb{R})$.
(ii) *If* $\sigma \in \mathcal{C}^{((1-H)/H)+}(I)$*, then* $f' \in \mathcal{C}^{((1-H)/H)+}(\mathbb{R})$ *and* (I, μ, σ) *is* H*-proper.*
(iii) *If* (I, μ, σ) *is* H*-proper, then both* $(f^{-1})'$*,* $\sigma \in \mathcal{C}^{((1-H)/H)+}(I \backslash Z(\sigma))$.
(iv) *If* $Z(\sigma) = \varnothing$*, then* (I, μ, σ) *is* H*-proper if and only if* $\sigma \in \mathcal{C}^{((1-H)/H)+}(\mathbb{R})$.

Proof. (i) is immediate by definition. As $f' = \sigma \circ f$, (ii) follows from Remark 2.1(ii) and $f \in \mathcal{C}^1(\mathbb{R})$. Furthermore, $f' = \sigma \circ f$ implies $f(Z(f')) = Z(\sigma)$. As $\sigma = f' \circ f^{-1}$ holds on I (iii) is implied by (i) and Remarks 2.1(ii) and 2.1(iv); (iv) follows from (ii) and (iii). □

If (I, μ, σ) is H-proper, then there exists a simple method to construct solutions of (2.2) explicitly. For $g \in V_H$, an SST f with state space I, FC $\gamma > 0$, $\tau \in \mathbb{R}$ and $z \in I$, we define $X^{f,\gamma}(g, \tau, z) : V_H \times \mathbb{R} \times I \to \mathcal{C}(\mathbb{R})$ by

$$X_t^{f,\gamma}(g, \tau, z) := f(O_t^\gamma(g, \tau, f^{-1}(z))), \qquad t \in \mathbb{R},$$

with $O^\gamma(g, \tau, y)$ as in (2.4). By definition, for all $\tau, z \in \mathbb{R}$ (cf. (2.5)), we have $X_\tau^{f,\gamma}(g, \tau, z) = z$; moreover,

$$X^{f,\gamma}(g, \tau, f(O_\tau^\gamma)) = f(O^\gamma(g)). \tag{3.3}$$

Theorem 3.4. *Let $H \in (0, 1)$. Then the following assertions hold:*

(i) *If (I, μ, σ) is H-proper with SST f and FC $\gamma > 0$, then, for all $g \in V_H \cap \mathcal{C}^{H-}(\mathbb{R})$,*

$$\{X^{f,\gamma}(g, \tau, z) : \tau \in \mathbb{R}, z \in I\} \subseteq \mathcal{S}^H(I, \mu, \sigma, g). \tag{3.4}$$

(ii) *Let $I \subseteq \mathbb{R}$ be an open interval. Let $\gamma > 0$ and $\mu, \sigma \in \mathcal{C}(I)$. Let $f \in \mathcal{C}^1(\mathbb{R})$ be an SST with state space I, where $Z(f')$ has Lebesgue measure zero. If* (3.4) *holds for all $g \in V_H \cap \mathcal{C}^{H-}(\mathbb{R})$, then (I, μ, σ) is proper.*

Proof. (i) Let $\gamma > 0$ be the FC and f the SST for (I, μ, σ). Fix $g \in V_H \cap \mathcal{C}^{H-}(\mathbb{R})$ and let $\tau \in \mathbb{R}$ and $z \in I$. Set

$$o(t) := O_t^{\gamma}(g, \tau, f^{-1}(z)), \quad x(t) := X_t^{f,\gamma}(g, \tau, z), \qquad t \in \mathbb{R}.$$

We shall show that $x \in \mathcal{S}^H(I, \mu, \sigma, g)$.

By construction, x takes values in I. Proposition 2.2(ii) implies $o \in \mathcal{C}^{H-}(\mathbb{R})$. Furthermore, $f \in \mathcal{C}^1(\mathbb{R})$ and $f' \in \mathcal{C}^{((1-H)/H)+}(\mathbb{R})$ by Lemma 3.3(i). Thus, $f' \circ o \in \mathcal{C}^{(1-H)+}(\mathbb{R})$ and $x = f \circ o \in \mathcal{C}^{H-}(\mathbb{R})$ by Remarks 2.1(i) and 2(iii), respectively. Proposition 2.1(ii) applies to $f \circ o$, i.e.,

$$x(t) - x(s) = f(o(t)) - f(o(s)) = \int_s^t f'(o(u))\mathrm{d}o(u), \qquad s \leqslant t, \tag{3.5}$$

where the right-hand side exists as an RS integral. As o is the solution of (2.6) we can rewrite (3.5) as

$$o(u) = o(s) - \gamma \int_s^u o(v)\mathrm{d}v + g(u) - g(s), \qquad s \leqslant u. \tag{3.6}$$

Recall that the RS integral is additive with respect to a sum of integrators if the RS integrals exist separately for each of the integrators. Clearly, $f' \circ o$ is RS integrable with respect to the constant functions $u \mapsto o(s)$ and $u \mapsto g(s)$. Furthermore, $u \mapsto -\gamma \int_s^u o(v)\mathrm{d}v$ is of bounded variation and $f' \circ o$ is continuous; thus, $f' \circ o$ is RS integrable with respect to $u \mapsto -\gamma \int_s^u o(v)\mathrm{d}v$. As $g \in \mathcal{C}^{H-}(\mathbb{R})$ and $f' \circ o \in \mathcal{C}^{(1-H)+}(\mathbb{R})$ the existence of the RS integral of $f' \circ o$ with respect to g is ensured by Proposition 2.1(i). Thus, (3.5) and (3.6) imply

$$x(t) - x(s) = -\gamma \int_s^t f'(o(u))d\int_s^u o(v)\mathrm{d}v + \int_s^t f'(o(u))\mathrm{d}g(u), \qquad s \leqslant t.$$

As $u \mapsto \int_s^u o(v)\mathrm{d}v$ is continuously differentiable and $u \mapsto f'(o(u))o(u)$ is continuous, we obtain by the density formula for Riemann integrals

$$x(t) - x(s) = -\gamma \int_s^t f'(o(u))o(u)\mathrm{d}u + \int_s^t f'(o(u))\mathrm{d}g(u), \qquad s \leqslant t.$$

Lemma 3.2(i) states $f' = \sigma \circ f$; hence, $\sigma \circ x = f' \circ o \in \mathcal{C}^{(1-H)+}(\mathbb{R})$. By Proposition 2.1(i), $\sigma \circ x$ is RS integrable with respect to $g \in \mathcal{C}^{H-}(\mathbb{R})$. Thus, x satisfies (S1) in Definition 2.2. Invoking Remark 3.1(i) and Definition 3.3(iii), we observe that $\sigma f^{-1} = -\sigma\psi/\gamma = -\mu/\gamma$. Finally, (S2) is satisfied as

$$x(t) - x(s) = -\gamma \int_s^t \sigma(f(o(u)))o(u)\mathrm{d}u + \int_s^t \sigma(f(o(u)))\mathrm{d}g(u)$$
$$= \int_s^t \mu(x(u))\mathrm{d}u + \int_s^t \sigma(x(u))\mathrm{d}g(u).$$

Thus, $x = X^{f,\gamma}(g, \tau, z) \in \mathcal{S}^H(I, \mu, \sigma, g)$.

(ii) For $0 < \alpha < 1$, set

$$g_\alpha(t) := \exp\Big[\gamma \frac{\alpha}{1-\alpha}(t \wedge 1)\Big], \qquad t \in \mathbb{R}.$$

We consider the family

$$T := \{\delta\, g_\alpha : \delta \in \mathbb{R}, \alpha \in (0, 1)\} \subseteq V_H \cap \mathcal{C}^{H-}(\mathbb{R}).$$

For $0 < \alpha < 1$, we obtain

$$g'_\alpha(t) = \frac{\alpha\gamma}{1-\alpha} g_\alpha(t), \quad O^\gamma_t(g_\alpha) = \alpha\, g_\alpha(t), \quad [O^\gamma_t(g_\alpha)]' = \frac{\alpha^2\gamma}{1-\alpha} g_\alpha(t), \qquad t < 1.$$

For $g \in T$ arbitrary, as $\sigma \in \mathcal{C}(I)$ and, therefore, $\sigma(f(O^\gamma(g))g' \in \mathcal{C}((-\infty, 1))$, the density formula for Riemann integrals applies, i.e.,

$$\int_s^t \sigma(O^\gamma_u(g))\mathrm{d}g(u) = \int_s^t \sigma(O^\gamma_u(g))g'(u)\mathrm{d}u, \qquad s \leqslant t < 1,\ g \in T.$$

In particular, for all $g \in T$, we know from (3.3) that $X^{f,\gamma}(g, 0, O^\gamma_0(g)) = f(O^\gamma(g)) \in \mathcal{S}^H(I, \mu, \sigma, g)$; hence, for $g \in T$,

$$f(O^\gamma_t(g)) - f(O^\gamma_s(g)) = \int_s^t \mu(f(O^\gamma_u(g)))\mathrm{d}u + \int_s^t \sigma(f(O^\gamma_u(g)))g'(u)\mathrm{d}u, \qquad s \leqslant t < 1.$$

As the integrands are continuous, we may differentiate both sides and obtain

$$f'(O^\gamma_t(g))[O^\gamma_t(g)]' = \mu(f(O^\gamma_t(g))) + \sigma(f(O^\gamma_t(g)))g'(t), \qquad s \leqslant t < 1,\ g \in T.$$

Specifying $g = \delta\, g_\alpha$ for $\delta \in \mathbb{R}$ and $\alpha \in (0, 1)$, this is equivalent to

$$\frac{\alpha^2\delta\gamma}{1-\alpha} g_\alpha(t) f'(\alpha\,\delta\, g_\alpha(t)) = \mu(f(\alpha\,\delta\, g_\alpha(t))) + \frac{\alpha\delta\gamma}{1-\alpha} g_\alpha(t)\sigma(f(\alpha\,\delta\, g_\alpha(t))), \qquad s \leqslant t < 1.$$

For $\alpha \in (0, 1)$ and $x \in \mathbb{R}$ fixed, choose $t < 1$ and $\delta \in \mathbb{R}$ such that $x = \alpha\,\delta g_\alpha(t)$; this then implies

$$\gamma\, x\, f'(x) = \frac{1-\alpha}{\alpha}\mu(f(x)) + \frac{\gamma}{\alpha}\, x\, \sigma(f(x)), \qquad x \in \mathbb{R}, 0 < \alpha < 1.$$

Specifying $\alpha_i = 1/i$, $i = 2, 3$, this generates a system of linear equations with non-vanishing determinant. The unique solution is

$$\mu(f(x)) = -\gamma\, x\, f'(x), \quad \gamma\, x\sigma(f(x)) = \gamma\, x\, f'(x), \qquad x \in \mathbb{R}.$$

By continuity, for all $y \in I$, we obtain the equivalent formulation

$$\mu(y) = -\gamma f^{-1}(y) f' \circ f^{-1}(y), \quad \sigma(y) = f' \circ f^{-1}(y), \qquad y \in I.$$

It remains to show that (I, μ, σ) is proper. Clearly (P1) holds. As $f \in \mathcal{C}^1(\mathbb{R})$ and $Z(f')$ has Lebesgue measure zero, we have $f^{-1} \in \mathcal{AC}(I)$. Thus, $\psi(x) := -\gamma f^{-1}(x)$ is an absolutely continuous strictly decreasing extension of μ/σ to I where $\lim_{x\uparrow r}\psi(x) = -\lim_{x\downarrow l}\psi(x) = -\infty$. By Remark 2.2(iii), $\sigma\psi' = -\gamma\sigma/[f' \circ f^{-1}] = -\gamma$ a.e. on I; thus, (P3) is satisfied. □

If (I, μ, σ) is H-proper, then the solution of (2.2) is unique in the following sense.

Theorem 3.5. *Let $H \in (0, 1)$ and (I, μ, σ) be H-proper with SST f and FC $\gamma > 0$. Suppose that $g \in V_H \cap \mathcal{C}^{H-}(\mathbb{R})$. Then the following assertions hold.*

(i) *If $Z(\sigma) = \varnothing$, then*

$$\mathcal{S}^H(I, \mu, \sigma, g) = \{X^{f,\gamma}(g, \tau, z) : \tau \in \mathbb{R}, z \in I\}. \tag{3.7}$$

(ii) *If* (3.7) *holds and g is not a constant function, then $Z(\sigma) = \varnothing$.*

Proof. (i) Lemmas 3.2 and 3.3 state that $f^{-1} \in \mathcal{C}^1(I)$ and $(f^{-1})' \in \mathcal{C}^{(1-H)/H+}(I)$, where $(f^{-1})'(z) = 1/\sigma(z)$ for all $z \in I$.

Let $s \leqslant t$ and $x \in \mathcal{S}^H(I, \mu, \sigma)$. By Definition 2.2, we have $x \in \mathcal{C}^{H-}(\mathbb{R})$. Both Remark 2.1(iv) and Lemma 3.2(i) apply, i.e.,

$$(1/\sigma) \circ x = (f^{-1})' \circ x \in \mathcal{C}^{(1-H)+}(\mathbb{R}). \tag{3.8}$$

Therefore, Proposition 2.1(ii) applies to $f^{-1} \circ x$ and we obtain

$$f^{-1}(x(t)) - f^{-1}(x(s)) = \int_s^t (f^{-1})'(x(u))\mathrm{d}x(u) = \int_s^t \frac{1}{\sigma(x(u))}\,\mathrm{d}x(u), \qquad s \times 6t. \tag{3.9}$$

On the other hand, $x \in \mathcal{S}^H(I, \mu, \sigma, g)$ satisfies (S1) and (S2) of Definition 2.2, i.e.,

$$x(u) = x(s) + \int_s^u \mu(x(v))\mathrm{d}v + \int_s^u \sigma(x(v))\mathrm{d}g(v), \qquad s \leqslant u. \tag{3.10}$$

We shall ensure that the RS integral on the right-hand side in (3.9) is additive with respect to the integrators in (3.10). This is guaranteed if the RS integrals exist separately. As $(1/\sigma) \circ x$ is continuous, this obviously holds for the first two addends in (3.10). Lemma 3.3(iv) states that $\sigma \in \mathcal{C}^{((1-H)/H)+}(I)$; thus, $\sigma \circ x \in \mathcal{C}^{(1-H)+}(\mathbb{R})$ by Remark 2.1(i). As $g \in \mathcal{C}^{H-}(\mathbb{R})$, Proposition 2.1(iii) applies to $\psi(u) = \int_s^u \sigma(x(v))\mathrm{d}g(v)$, i.e., $\psi \in \mathcal{C}^{H-}([s, t])$. By (3.8), we have $(1/\sigma) \circ x \in \mathcal{C}^{(1-H)+}(\mathbb{R})$. Thus, $(1/\sigma) \circ x$ is RS integrable with respect to ψ. Thus, the RS integral in (3.9) is additive with respect to the integrators in (3.10). Furthermore, Proposition 2.1(iii) provides a density formula. The following chain of equalities summarizes our reasoning:

$$\begin{aligned} & f^{-1}(x(t)) - f^{-1}(x(s)) \\ &= \int_s^t \frac{1}{\sigma(x(u))} \,\mathrm{d}\left[\int_s^u \mu(x(v))\mathrm{d}v\right] + \int_s^t \frac{1}{\sigma(x(u))} \,\mathrm{d}\left[\int_s^u \sigma(x(v))\mathrm{d}g(v)\right] \\ &= \int_s^t \frac{\mu(x(u))}{\sigma(x(u))} \mathrm{d}u + g(t) - g(s), \qquad s \leqslant t. \end{aligned}$$

As (I, μ, σ) is proper, (P2) states $\psi(z) = \mu(z)/\sigma(z)$ for all $z \in I$; additionally, $\psi = -\gamma f^{-1}$ by Definition 3.3. Hence $\mu(z)/\sigma(z) = -\gamma f^{-1}(z)$ holds for all $z \in I$; therefore

$$f^{-1}(x(t)) - f^{-1}(x(s)) = -\gamma \int_s^t f^{-1}(x(u))\mathrm{d}u + g(t) - g(s), \qquad s \leqslant t.$$

Thus, $f^{-1} \circ x : \mathbb{R} \to \mathbb{R}$ is a solution of (2.6). Fix some $\tau \in \mathbb{R}$. For all $t \in \mathbb{R}$, Theorem 2.3 states that $f^{-1}(x(t)) = O_t^\gamma(g, \tau, f^{-1}(x(\tau)))$; thus, $x = X^{f,\gamma}(g, \tau, x(\tau))$.

(ii) Assume contrary to the hypothesis that, at the same time, $Z(\sigma) \neq \varnothing$ and (3.7) holds. In particular, there exists $z_0 \in I$ such that $\sigma(z_0) = 0$ and, therefore, $\mu(z_0) = 0$ by Lemma 3.1. Observe that $y_{z_0} \in \mathcal{S}^H(I, \mu, \sigma, g)$ where $y_{z_0} : \mathbb{R} \to I$, $y_{z_0}(t) = z_0$.

By assumption, y_{z_0} has the form $X^{f,\gamma}(g, \tau, z)$ for some $\tau \in \mathbb{R}$ and $z \in I$. Thus,

$$f^{-1}(z_0) = \mathrm{e}^{-\gamma(t-\tau)} f^{-1}(z) - \mathrm{e}^{-\gamma(t-\tau)} O_\tau^\gamma(g) + O_t^\gamma(g), \qquad t \in \mathbb{R}.$$

Set $G := \mathrm{e}^{\gamma\tau} f^{-1}(z) - \mathrm{e}^{\gamma\tau} O_\tau^\gamma(g)$. By partial integration, we obtain

$$g(t) = \gamma \int_{-\infty}^t \mathrm{e}^{-\gamma(t-s)} g(s)\mathrm{d}s - G\mathrm{e}^{-\gamma t} + f^{-1}(z_0), \qquad t \in \mathbb{R}.$$

Thus, $g \in \mathcal{C}^\infty(\mathbb{R})$. Multiplying both sides by $\mathrm{e}^{\gamma t}$ and differentiating yields a linear differential equation for $h(t) = \mathrm{e}^{\gamma t} g(t)$, namely, $h'(t) = \gamma h(t) + \gamma \mathrm{e}^{\gamma t} f^{-1}(z_0)$. Both $h(t) = \mathrm{e}^{\gamma t}[h_0 + \gamma f^{-1}(z_0)t]$ and $g(t) = \mathrm{e}^{-\gamma t} h(t) = h_0 + \gamma f^{-1}(z_0)t$ are uniquely determined up to the choice of $h_0 \in \mathbb{R}$. As g is not a constant function, we obtain $f^{-1}(z_0) \neq 0$. However, an affine non-constant function is not an element of V_H, contradicting our assumption. □

We return to the probabilistic setting.

Theorem 3.6. *Let $H \in (0, 1)$ and (I, μ, σ) be H-proper with FC $\gamma > 0$ and SST f. Let $X = f(O^{H,\gamma})$. Let $\Gamma(\cdot)$ denote Euler's gamma function. Then the following assertions hold.*

(i) *X is a strictly stationary pathwise solution of the stochastic integral equation*

$$X_t - X_s = \int_s^t \mu(X_u)\mathrm{d}u + \int_s^t \sigma(X_u)\mathrm{d}B_u^H, \qquad s \leqslant t. \tag{3.11}$$

The distribution of X_t has a Lebesgue density p, where

$$p(\cdot) = \frac{\gamma^H}{\sqrt{\pi\Gamma(2H+1)\sigma(\cdot)^2}} \exp\left[-\frac{\gamma^{2H-2}}{\Gamma(2H+1)}\left(\frac{\mu(\cdot)}{\sigma(\cdot)}\right)^2\right] \qquad \textit{a.e. on } I. \tag{3.12}$$

(ii) *If* $Z(\sigma) = \varnothing$ *then* X *is the unique stationary pathwise solution of* (3.11).

Proof. (i) Let f be the SST of the H-proper triple (I, μ, σ). Theorem 3.4 states that $X = X^{f,\gamma}(B^H, 0, f(O_0^\gamma(B^H)) = f(O^{H,\gamma})$ is a pathwise solution of (3.11). Theorem 2.3(iii) states that $O^{H,\gamma}$ is strictly stationary and so is $X = f(O^{H,\gamma})$. The FOUP is a stationary mean-zero Gaussian process with variance $\mathrm{E}(O_t^{H,\gamma})^2 = \Gamma(2H+1)/(2\gamma^{2H})$ (cf. Buchmann and Klüppelberg 2005, Lemma 2.1). By definition of f, the relation $f^{-1} = -\psi/\gamma$ holds; thus

$$P(X_t \leqslant x) = P(f(O^{H,\gamma}(t)) \leqslant x) = \Phi\left[-\sqrt{\frac{2}{\Gamma(2H+1)}}\gamma^{H-1}\psi(x)\right], \qquad x \in I,$$

where $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^x \mathrm{e}^{-y^2/2}\,\mathrm{d}y$, $x \in \mathbb{R}$.

Therefore the distribution of X_t is absolutely continuous with respect to Lebesgue measure. Formula (3.12) is verified by differentiating and (P3).

(ii) If Y is a pathwise solution, then Theorem 3.5 guarantees the existence of a random variable G such that $Y_t = f(\mathrm{e}^{-\gamma t}G + O_t^\gamma(B^H))$ almost surely. Additionally, if Y is strictly stationary then $G = 0$ with probability one. Thus, $Y = X$. □

We conclude this section with some remarks on the existing literature.

Remark 3.2. Nualart and Rășcanu (2002) also consider the case $H > 1/2$. They prove existence and uniqueness for time-dependent multivariate integral equations defined on compact time intervals. They apply Banach's fixpoint theorem which requires strong regularity conditions on μ and σ. Hairer (2005) shows by a different method the existence of stationary solution of (3.11) for constant $\sigma > 0$. Mikosch and Norvaiša (2000) discuss linear equations driven by an FBM with Hurst index $H \in (1/2, 1)$. They consider the coefficients $\mu(t)$ and $\sigma(t)$, $t \in \mathbb{R}^+$, of bounded α-variation for $\alpha > 1/H$. Our work has mostly profited from Zähle (1998); in fact it is related to Klingenhöfer and Zähle (1999), where drift and volatility may depend on time and the level of the process. They take μ to be continuous and $\sigma \in \mathcal{C}^1$. Our approach gives a constructive method to find stationary solutions of integral equations under weakest possible conditions – only Hölder continuity – for drift and volatility.

4. Structural properties

4.1. Construction of proper triples when σ is given

Suppose that $I \subseteq \mathbb{R}$ is an open interval and $\sigma \in \mathcal{C}(I)$ is non-negative. When does there exist a $\mu \in \mathcal{C}(I)$ such that (I, μ, σ) is proper? Define

$$\mathcal{K}_{I,\sigma} = \{(\gamma, \mu) \in \mathbb{R}^+ \times \mathcal{C}(I) : (I, \mu, \sigma) \text{ is proper with FC } \gamma\}. \tag{4.1}$$

Recall that a subset $M \subseteq \mathcal{C}(I)$ is a cone if $\alpha M \subseteq M$ for all $\alpha > 0$. The next proposition lists some important properties of $\mathcal{K}_{I,\sigma}$.

Proposition 4.1. *Let $I = (l, r) \subseteq \mathbb{R}$ and $\sigma \in \mathcal{C}(I)$ be non-negative.*

(i) *The following assertions are equivalent:*
   (a) $\mathcal{K}_{I,\sigma} \neq \emptyset$.
   (b) $1/\sigma \in \mathcal{L}_C(I)$ *and, for all* $x \in I$,

$$\int_l^x \frac{\mathrm{d}z}{\sigma(z)} = \int_x^r \frac{\mathrm{d}z}{\sigma(z)} = \infty. \tag{4.2}$$

(ii) $\mathcal{K}_{I,\sigma}$ *is a cone. If* (a) *or* (b) *of* (i) *holds, then*

$$\mathcal{K}_{I,\sigma} = \left\{ (\gamma, \mu) \in \mathbb{R}^+ \times \mathcal{C}(I) : \mu(x) = -\gamma\sigma(x) \int_\xi^x \frac{\mathrm{d}z}{\sigma(z)},\ \xi \in I \right\},$$

where γ and ξ are uniquely determined by σ and μ.

Proof. (i) We first show that (a) $\Rightarrow$ (b). Suppose $(\gamma, \mu) \in \mathcal{K}_{I,\sigma}$. Let $\psi \in \mathcal{AC}(I)$ be the function as in (P2) and (P3). By Remark 3.1(i), $1/\sigma \in \mathcal{L}_C(I)$. As by Lemma 3.1 (I, μ, σ) defines a centre $\xi \in I$ for μ with $\psi(\xi) = 0$, (P3) implies $\psi(x) = -\gamma \int_\xi^x 1/\sigma(z)\mathrm{d}z$ for all $x \in I$. By property (P2),

$$\lim_{x\uparrow r} \gamma \int_\xi^x \frac{\mathrm{d}z}{\sigma(z)} = -\lim_{x\uparrow r} \psi(x) = \infty \quad \text{and} \quad \lim_{x\downarrow l} \gamma \int_x^\xi \frac{\mathrm{d}z}{\sigma(z)} = \lim_{x\downarrow l} \psi(x) = \infty.$$

We now show that (b) $\Rightarrow$ (a). Let $\gamma > 0$, $\xi \in I$. Define $\mu(x) = -\gamma\,\sigma(x) \int_\xi^x 1/\sigma(z)\mathrm{d}z$, $x \in I$. We have to show that (I, μ, σ) is a proper triple with FC $\gamma > 0$. Property (P1) holds by assumption. Define $\psi(x) = -\gamma \int_\xi^x 1/\sigma(z)\mathrm{d}z$, $x \in I$. Then $\psi = \mu/\sigma$ on $I\backslash Z(\sigma)$ and $\psi \in \mathcal{AC}(I)$ is strictly decreasing. The limits in (3.1) are implied by (4.2). Therefore, (P2) holds. Moreover, $\psi' = -\gamma/\sigma$ a.e. on I. Hence (P3) is true. Thus, $(\gamma, \mu) \in \mathcal{K}_{I,\sigma}$.

(ii) Without loss of generality, assume $\mathcal{K}_{I,\sigma} \neq \emptyset$. Let $(\gamma, \mu) \in \mathcal{K}_{I,\sigma}$ and $\alpha > 0$. We shall show that $(I, \tilde{\mu}, \sigma)$ is proper with FC $\tilde{\gamma}$, where $\tilde{\mu} = \alpha\mu$ and $\tilde{\gamma} = \alpha\gamma$. (P1) is immediate. Let $\psi \in \mathcal{AC}(I)$ be the extension of μ/σ to I. Obviously, $\tilde{\psi} = \alpha\psi \in \mathcal{AC}(I)$ is a strictly decreasing extension of $\tilde{\mu}/\sigma$ to I satisfying (P2). Property (P3) follows from $\tilde{\psi}' = \alpha\psi' = -\alpha\gamma/\sigma = -\tilde{\gamma}/\sigma$. Suppose that (a) or (b) of (i) holds. Let $(\gamma, \mu) \in \mathcal{K}_{I,\sigma}$. Then (I, μ, σ) is a proper triple with FC $\gamma > 0$ and centre $\xi \in I$ for μ. If $\psi \in \mathcal{AC}(I)$ is the extension of μ/σ from $I\backslash Z(\sigma)$ to I, then as in the proof of (i) we obtain the representation

$$\psi(x) = \int_\xi^x \psi'(z)\mathrm{d}z = -\gamma \int_\xi^x \frac{\mathrm{d}z}{\sigma(z)}, \qquad x \in I;$$

thus, $\mu(x) = -\gamma\sigma(x) \int_\xi^x 1/\sigma(z)\mathrm{d}z$, $x \in I$. Uniqueness of γ and ξ follows from the construction. □

4.2. Construction of proper triples when μ is given

Suppose that $I \subseteq \mathbb{R}$ is an open interval and that $\mu \in \mathcal{C}(I)$. When does there exist a continuous non-negative function $\sigma : I \to \mathbb{R}$ such that (I, μ, σ) is a proper triple? Combining (P2) and (P3) yields a differential equation for ψ in terms of μ, namely, a.e.,

$$\psi' = -\gamma \frac{1}{\sigma} = -\gamma \frac{\psi}{\mu}. \tag{4.3}$$

Every solution ψ of (4.3) defines a candidate σ by (P2), i.e., set $\sigma = \mu/\psi$.

However, every proper triple has a centre $\xi \in I$ for μ; in particular, $\psi(\xi) = 0$. It is not obvious that (4.3) leads to a continuous σ. We split the state space I into (l, ξ) and (ξ, r) and (4.3) is solved on (l, ξ) and (ξ, r) separately. This yields two branches of a solution. Their behaviour close to ξ has to be investigated more carefully. Before we do that we state the following lemma.

Lemma 4.2. *Let $I = (l, r) \subseteq \mathbb{R}$ and $\mu \in \mathcal{C}(I)$. Suppose that there exists a centre $\xi \in I$ for μ and that $1/\mu \in \mathcal{L}_C(I\backslash\{\xi\})$. Let $\gamma > 0$, $x \in (l, \xi)$ and $y \in (\xi, r)$. Then $\psi_{x,y,\gamma} : I\backslash\{\xi\} \to \mathbb{R}$ is well defined, where*

$$\psi_{x,y,\gamma}(w) := \begin{cases} \exp\left[-\gamma \int_x^w \frac{\mathrm{d}z}{\mu(z)}\right] & \textit{for } w \in (\mathrm{l}, \xi), \\ -\exp\left[-\gamma \int_y^w \frac{\mathrm{d}z}{\mu(z)}\right] & \textit{for } w \in (\xi, r). \end{cases} \tag{4.4}$$

Moreover, $\psi_{x,y,\gamma}$ is the unique absolutely continuous solution of (4.3) *on $I\backslash\{\xi\}$, with*

$$\psi_{x,y,\gamma}(x) = -\psi_{x,y,\gamma}(y) = 1.$$

Additionally, if $\psi_{x,y,\gamma}$ extends continuously to I, then $\psi_{x,y,\gamma} \in \mathcal{AC}(I)$ and $\psi_{x,y,\gamma}$ is strictly increasing on I.

Proof. Clearly, $\psi = \psi_{x,y,\gamma}$ is well defined and $\psi \in \mathcal{AC}(I\backslash\{\xi\})$. Furthermore, it is a solution of (4.3) with $\psi(x) = -\psi(y) = 1$. Suppose that $\overline{\psi} \in \mathcal{AC}(I\backslash\{\xi\})$ is another solution of (4.3) on $I\backslash\{\xi\}$ with $\overline{\psi}(x) = -\overline{\psi}(y) = 1$. As $\overline{\psi}$ is continuous there exists an open interval U containing x such that $\overline{\psi}(z) > 0$ for all $z \in U$; thus, $\log \overline{\psi} \in \mathcal{AC}(\mathrm{U})$. Analogously, $\psi > 0$ on (l, ξ); hence, $\log \psi \in \mathcal{AC}((l, \xi))$. Thus, a.e. on $U \cap (l, \xi)$,

$$(\log \overline{\psi})' = \frac{\overline{\psi}'}{\overline{\psi}} = -\gamma \frac{1}{\mu} = \frac{\psi'}{\psi} = (\log \psi)'.$$

Integrating both sides shows $\overline{\psi} = \psi$ on $U \cap (l, \xi)$. As (l, ξ) is connected and $\psi > 0$ on (l, ξ) we may proceed and obtain $\overline{\psi} = \psi$ on (l, ξ). Analogous reasoning holds for (ξ, r).

Clearly, $\psi' < 0$ a.e. on I by (4.3) and (4.4). Suppose that ψ extends continuously to I and set $\psi'(\xi) = 0$. For all $s \in (l, \xi)$, the monotone convergence theorem and continuity of ψ imply

$$\psi(\xi) - \psi(s) = \lim_{z\uparrow\xi} \psi(z) - \psi(s) = \lim_{z\uparrow\xi} \int_s^z \psi'(w)\mathrm{d}w = \int_s^\xi \psi'(w)\mathrm{d}w.$$

This shows $\psi \in \mathcal{AC}((l, \xi])$. Analogous reasoning holds for $[\xi, r)$. The extension ψ is strictly decreasing as $\psi' < 0$ a.e. □

In the previous subsection, for σ given, we were able to choose an FC $\gamma > 0$ freely. In general this is not possible when μ is given. Therefore, we shall treat γ and σ separately. Firstly, we set

$$\Gamma_{I,\mu} := \{\gamma \in \mathbb{R}^+ : \exists \sigma \in \mathcal{C}(I) \text{ such that } (I, \mu, \sigma) \text{ is proper with FC } \gamma\}.$$

For $\gamma \in \Gamma_{I,\mu}$, we require the set of possible candidates for σ, i.e., we set

$$\mathcal{H}_{I,\mu,\gamma} := \{\sigma \in \mathcal{C}(I) : (I, \mu, \sigma) \text{ is proper with the FC } \gamma\}.$$

Although the set of all pairs (γ, σ), where (I, μ, σ) is proper with FC γ, is no longer a cone, an analogous property still holds for $\mathcal{H}_{I,\mu,\gamma}$.

Proposition 4.3. *Let $I \subseteq \mathbb{R}$ be an open interval, $\mu \in \mathcal{C}(I)$ and $\gamma \in \Gamma_{I,\mu}$. Then $\mathcal{H}_{I,\mu,\gamma}$ is a cone.*

Proof. Let $\sigma \in \mathcal{H}_{I,\mu,\gamma}$ and $\alpha > 0$. We have to show that $\tilde{\sigma} = \alpha\sigma \in \mathcal{H}_{I,\mu,\gamma}$. (P1) is obvious. If $\psi \in \mathcal{AC}(I)$ is the strictly decreasing extension of μ/σ to I such that (P2) and (P3) are satisfied, then $\tilde{\psi} := \psi/\alpha \in \mathcal{AC}(I)$ is a strictly decreasing extension of $\mu/\tilde{\sigma}$ to I such that both (P2) and (P3) hold for I, μ, $\tilde{\sigma}$ and γ. □

We are now ready to investigate $\Gamma_{I,\mu}$.

Proposition 4.4. *Let $I = (l, r) \subseteq \mathbb{R}$ be an open interval. Let $\mu \in \mathcal{C}(I)$. The following assertions are equivalent:*

(i) $\Gamma_{I,\mu} \neq \varnothing$.
(ii) *There exists a centre $\xi \in I$ for μ such that:*
   (a) $1/\mu \in \mathcal{L}_C(I\backslash\{\xi\})$;
   (b) *for all $l < x < \xi < y < r$,*

$$\int_l^x \frac{\mathrm{d}z}{\mu(z)} = \int_y^r \frac{\mathrm{d}z}{|\mu(z)|} = \int_x^\xi \frac{\mathrm{d}z}{\mu(z)} = \int_\xi^y \frac{\mathrm{d}z}{|\mu(z)|} = \infty; \tag{4.5}$$

   (c) *the set $\Theta_{I,\mu}$ is non-empty, where*

$$\Theta_{I,\mu} := \left\{\gamma \in \mathbb{R}^+ : \exists\, x \in (l, \xi) \exists\, y \in (\xi, r) \lim_{w\uparrow\xi} \frac{\mu(w)}{\psi_{x,y,\gamma}(w)} = \lim_{w\downarrow\xi} \frac{\mu(w)}{\psi_{x,y,\gamma}(w)} = 0\right\}.$$

Proof. (i) $\Rightarrow$ (ii) Suppose $\Gamma_{I,\mu} \neq \varnothing$. By Lemma 3.1 there exists a centre $\xi \in I$ for μ. Let $\gamma \in \Gamma_{I,\mu}$. Then there exists $\sigma \in \mathcal{C}(I)$ such that (I, μ, σ) is proper with FC $\gamma > 0$. Let

$\psi \in \mathcal{AC}(I)$ be the extension of μ/σ from $I\backslash Z(\sigma)$ to I. Lemma 3.1 states that $Z(\psi) = \{\xi\}$. As $Z(\mu) = \{\xi\} \cup Z(\sigma)$ has Lebesgue measure zero, (P3) yields $\psi' = -\gamma/\sigma = -\gamma\psi/\mu$ a.e. on I.

As $\psi \in \mathcal{C}(I)$ is continuous and $Z(\psi) = \{\xi\}$ holds, ψ is bounded away from zero on compact subsets of $I\backslash\{\xi\}$. By Remark 3.1(i), $1/\sigma \in \mathcal{L}_C(I)$ and thus $1/\mu = 1/(\psi\sigma) \in \mathcal{L}_C(I\backslash\{\xi\})$.

Next we shall prove (4.5). Recall that $\psi : I \to \mathbb{R}$ is strictly decreasing and continuous. As $\psi(\xi) = 0$ there exist $x \in (l, \xi)$ and $y \in (\xi, r)$ such that $\psi(x) = -\psi(y) = 1$. Lemma 4.2 applies to μ; thus, $\psi(w) = \psi_{x,y,\gamma}(w)$ for all $w \in I\backslash\{\xi\}$.

Formula (4.4) and (P2) yield

$$\infty = \lim_{w\downarrow\infty} \psi(w) = \lim_{w\downarrow\infty} \psi_{x,y,\gamma}(w) = \lim_{x\downarrow l} \exp\left[\gamma\int_x^w \frac{\mathrm{d}z}{\mu(z)}\right].$$

Analogously, continuity of ψ in ξ implies that

$$0 = \lim_{w\uparrow\xi} \psi(w) = \lim_{w\uparrow\xi} \psi_{x,y,\gamma}(w) = \lim_{w\uparrow\xi} \exp\left[-\gamma\int_x^w \frac{\mathrm{d}z}{\mu(z)}\right].$$

Finally, for any $\tilde{\gamma} \in (0, \gamma)$,

$$\begin{aligned}\lim_{w\uparrow\xi} \frac{\mu(w)}{\psi_{x,y,\tilde{\gamma}}(w)} &= \lim_{w\uparrow\xi} \frac{\mu(w)}{\psi(w)} \exp\left[(\tilde{\gamma} - \gamma)\int_x^w \frac{\mathrm{d}z}{\mu(z)}\right] \\ &= \lim_{w\uparrow\xi} \sigma(w)\exp\left[(\tilde{\gamma} - \gamma)\int_x^w \frac{\mathrm{d}z}{\mu(z)}\right] = 0. \end{aligned}\tag{4.6}$$

Analogous reasoning for (ξ, r) shows that $\Theta_{I,\mu} \neq \emptyset$.

(ii) $\Rightarrow$ (i) Let $\gamma \in \Theta_{I,\mu,\sigma}$. As $1/\mu \in \mathcal{L}_C(I\backslash\{\xi\})$, Proposition 4.2 applies to μ and γ. Thus, there exist $x \in (l, \xi)$ and $y \in (\xi, r)$ such that

$$\lim_{w\uparrow\xi} \frac{\mu(w)}{\psi_{x,y,\gamma}(w)} = \lim_{w\downarrow\xi} \frac{\mu(w)}{\psi_{x,y,\gamma}(w)} = 0; \tag{4.7}$$

additionally, $\psi_{x,y,\gamma}$ is the unique solution of (4.3) on $I\backslash\{\xi\}$ with $\psi_{x,y,\gamma}(x) = -\psi_{x,y,\gamma}(y) = 1$. By (4.4) and (4.5), $\psi_{x,y,\gamma}$ extends continuously on I by setting $\psi_{x,y,\gamma}(\xi) := 0$. Furthermore,

$$\lim_{w\downarrow l} \psi_{x,y,\gamma}(w) = -\lim_{w\uparrow r} \psi_{x,y,\gamma}(w) = \infty.$$

Lemma 4.2 states that $\psi_{x,y,\gamma} \in \mathcal{AC}(I)$ and that $\psi_{x,y,\gamma}$ is strictly decreasing.

In particular, $Z(\psi_{x,y,\gamma}) = \{\xi\}$. Set $\sigma(\xi) := 0$ and $\sigma(w) := \mu(w)/\psi_{x,y,\gamma}(w)$ for $w \in I\backslash\{\xi\}$. We shall show that (I, μ, σ) is proper. $\sigma \in \mathcal{C}(I\backslash\{\xi\})$ is immediate; continuity of σ in ξ is a consequence of $\gamma \in \Theta_{I,\mu}$. Consequently, (P1) holds. By definition, $\psi = \psi_{x,y,\gamma}$ is a function satisfying (P2). By (4.3), a.e. $\psi' = -\gamma\psi/\mu = -\gamma/\sigma$ implying (P3). Thus, (I, μ, σ) is proper and hence $\Gamma_{I,\mu}$ is non-empty. □

The next proposition investigates $\Theta_{I,\mu}$ in more detail. We denote

$$\gamma_{I,\mu} := \sup \Gamma_{I,\mu}.$$

Proposition 4.5. *Suppose that $I \subseteq \mathbb{R}$ is an open interval and $\mu \in \mathcal{C}(I)$. Let $\Gamma_{I,\mu} \neq \emptyset$ and let $\xi \in I$ be the centre of μ. Then $(0, \gamma_{I,\mu}) \subseteq \Theta_{I,\mu} \subseteq \Gamma_{I,\mu}$. Furthermore, if $\gamma \in \Theta_{I,\mu}$, then*

$$\mathcal{H}_{I,\mu,\gamma} = \{\sigma \in \mathcal{C}(I) : \exists x \in (l, \xi), \exists y \in (l, \xi) \text{ such that}$$
$$\sigma(\xi) = 0 \wedge \sigma(w) = \frac{\mu(w)}{\psi_{x,y,\gamma}(w)} \forall w \in I\backslash\{\xi\}\},$$

where $\psi_{x,y,\gamma}$ is given in (4.4) and x and y in the representation of $\sigma \in \mathcal{H}_{I,\mu,\gamma}$ are uniquely determined by μ and σ.

Proof. The proof is a refinement of the proof of Proposition 4.4. It relies on three observations. Firstly, (4.6) holds for all $\gamma \in \Gamma_{I,\mu}$ and $\tilde{\gamma} \in (0, \gamma)$; this implies the chain of inclusions. Secondly, if the limit in (4.7) exists and vanishes for some $x \in (l, \xi)$ and $y \in (\xi, r)$, then it exists and vanishes for all $x \in (l, \xi)$ and all $y \in (\xi, r)$. This is a consequence of formula (4.4). Finally, the roots of $\psi(x) = -\psi(y) = 1$ are uniquely determined for the extension ψ of μ/σ to I; consequently, uniqueness of the representation holds. □

The remaining case to investigate concerns the situation when $\gamma_{I,\mu} \in \Gamma_{I,\mu}$ but $\gamma_{I,\mu} \notin \Theta_{I,\mu}$.

Proposition 4.6. *Let $I \subseteq \mathbb{R}$ be an open interval and $\mu \in \mathcal{C}(I)$ with centre ξ. Suppose that $\Gamma_{I,\mu}$ is non-empty and bounded and that $\gamma_{I,\mu} \notin \Theta_{I,\mu}$.*

(i) *The following assertions are equivalent:*
   (a) $\gamma_{I,\mu} \in \Gamma_{I,\mu}$.
   (b) *ξ is an isolated point of $Z(\mu)$. For all $\bar{x} \in (l, \xi)$ and all $\bar{y} \in (\xi, r)$, the following limits exist and we have*

$$\lim_{w\uparrow\xi}\lim_{x\uparrow\xi} \frac{\mu(w)}{\mu(x)} \psi_{w,\bar{y},\gamma_{I,\mu}}(x) = \lim_{w\downarrow\xi}\lim_{x\downarrow\xi} \frac{\mu(w)}{\mu(x)} |\psi_{\bar{x},w,\gamma_{I,\mu}}(x)| = 1. \tag{4.8}$$

(ii) *Suppose that either* (a) *or* (b) *of* (i) *holds. For $\bar{x} \in (l, \xi)$* and *$\bar{y} \in (\xi, r)$, let*

$$\sigma_{\gamma_{I,\mu}}(w) = \begin{cases} \lim_{x\uparrow\xi} \dfrac{\mu(w)}{\mu(x)} \psi_{w,\bar{y},\gamma_{I,\mu}}(x), & \text{for } x \in (l, \xi), \\ 1, & \text{for } x = \xi, \\ \lim_{x\downarrow\xi} \dfrac{\mu(w)}{\mu(x)} |\psi_{\bar{x},w,\gamma_{I,\mu}}(x)|, & \text{for } x \in (\xi, r). \end{cases} \tag{4.9}$$

Then $\sigma_{\gamma_{I,\mu}}$ is well defined and $\sigma_{\gamma_{I,\mu}} \in \mathcal{C}(I)$. The representation does not depend on $\bar{x}$ and $\bar{y}$. Furthermore, $\mathcal{H}_{I,\mu,\gamma_{I,\mu}} = \{c\,\sigma_{\gamma_{I,\mu}} : c \in \mathbb{R}^+\}$.

Proof. (i) We first show that (a) ⇒ (b). There exists $\sigma \in \mathcal{C}(I)$ such that (I, μ, σ) is proper with the corresponding FC $\gamma_{I,\mu}$. Let ψ be the absolutely continuous extension of μ/σ to I. Analogously, we find $\bar{x} \in (l, \xi)$ and $\bar{y} \in (\xi, r)$ such that $\psi(w) = \psi_{\bar{x},\bar{y},\gamma_{I,\mu}}(w)$ for all $w \in I\backslash\{\xi\}$.

Let $w \leqslant x < \xi$. From (4.4) we conclude that $\psi(x) = \psi_{\bar{x},\bar{y},\gamma_{I,\mu}}(x) = \psi(w)\psi_{w,\bar{y},\gamma_{I,\mu}}(x)$. But then

$$\sigma(\xi) = \lim_{x\uparrow\xi, x\notin Z(\sigma)} \sigma(x) = \lim_{x\uparrow\xi, x\notin Z(\sigma)} \frac{\mu(x)}{\psi(x)} = \frac{1}{\psi(w)} \lim_{x\uparrow\xi, x\notin Z(\sigma)} \frac{\mu(x)}{\psi_{w,\bar{y},\gamma_{I,\mu}}(x)}. \tag{4.10}$$

Thus, the inner limit on the left-hand side of (4.8) exists for $x\uparrow\xi$, where $x \in I\backslash Z(\sigma)$. (P2) implies $\psi(w) > 0$ for all $w \in (l, \xi)$. Moreover, $\gamma \notin \Theta_{I,\mu}$. By an observation made in the proof of Proposition 4.5, the limit on the right-hand side of (4.10) is necessarily non-vanishing for all $w \in (l, \xi)$ and $\bar{y} \in (\xi, r)$. Consequently, $\sigma(\xi) > 0$.

As $\sigma(\xi)$ is strictly positive and $\sigma \in \mathcal{C}(I)$, it is strictly positive in a neighbourhood of ξ. As $Z(\mu) = \{\xi\} \cup Z(\sigma)$ by Lemma 3.1, ξ is an isolated point of $Z(\mu)$. Consequently, there exists a neighbourhood U of ξ in I such that $\mu(x) \neq 0$ and $\psi(x) = \mu(x)/\sigma(x)$ for all $x \in U\backslash\{\xi\}$. Therefore, we can drop the condition $x \notin Z(\sigma)$ in all limits of (4.10). Directly from (4.10),

$$\lim_{w\uparrow\xi} \lim_{x\uparrow\xi} \frac{\mu(w)}{\mu(x)} \psi_{w,\bar{y},\gamma_{I,\mu}}(x) = \frac{1}{\sigma(\xi)} \lim_{w\uparrow\xi} \frac{\mu(w)}{\psi(w)} = \frac{1}{\sigma(\xi)} \lim_{w\uparrow\xi} \sigma(w) = 1.$$

Therefore, (4.8) is established on (l, ξ) and the left-hand-side does not depend on $\bar{y}$. As the right-hand side limit in (4.10) is strictly positive, this implies that for all $w \in (l, \xi)$,

$$\frac{1}{\psi(w)} = \sigma(\xi) \lim_{x\uparrow\xi} \frac{\psi_{w,\bar{y},\gamma_{I,\mu}}(x)}{\mu(x)}.$$

Therefore, the limit on the right-hand side is a continuous function in $w \in (l, \xi)$.

Rewriting the last equation on $I\backslash Z(\sigma)$ yields

$$\sigma(w) = \sigma(\xi) \lim_{x\uparrow\xi} \frac{\mu(w)}{\mu(x)} \psi_{w,\bar{y},\gamma_{I,\mu}}(x).$$

The set $I\backslash Z(\sigma)$ is dense and all functions are continuous in w on $I\backslash Z(\sigma)$. Consequently, the identity extends to all $w \in (l, \xi)$. Thus, σ has (necessarily) the form described in (2). The representation does not depend on the choice of $\bar{y}$.

As analogous reasoning holds for (ξ, r), we obtain $\mathcal{H}_{I,\mu,\gamma_{I,\mu}} \subseteq \{c\,\sigma_{\gamma_{I,\mu}} : c \in \mathbb{R}^+\}$ for $\sigma_{\gamma_{I,\mu}}$ given in (4.9).

We now show that (b) $\Rightarrow$ (a). We show that $(I, \mu, \sigma_{\gamma_{I,\mu}})$ is proper with friction coefficient $\gamma_{I,\mu}$. By Proposition 4.3, $\mathcal{H}_{I,\mu,\gamma_{I,\mu}}$ is a cone; this also completes the proof of (ii), i.e.,

$$\{c\sigma_{\gamma_{I,\mu}} : c \in \mathbb{R}^+\} \subseteq \mathcal{H}_{I,\mu,\gamma_{I,\mu}}.$$

As $1/\mu \in \mathcal{L}_C(I\backslash\{\xi\})$, $\psi_{x,y,\gamma_{I,\mu}}(z)$ is well defined and non-vanishing for all $z \in I\backslash\{\xi\}$, $x \in I\backslash\{\xi\}$, $y \in (\xi, r)$. Let $\bar{x} \in (l, \xi)$ and $\bar{y} \in (\xi, r)$. Set

$$\psi(w) = \begin{cases} \lim_{x\uparrow\xi} \dfrac{\mu(x)}{\psi_{w,\bar{y},\gamma_{I,\mu}}(x)}, & \text{for } w \in (l, \xi), \\[2ex] \lim_{x\downarrow\xi} \dfrac{\mu(x)}{|\psi_{\bar{x},w,\gamma_{I,\mu}}(x)|}, & \text{for } w \in (\xi, r). \end{cases}$$

Define $\psi(\xi) = 0$. Then ψ is well defined. This is a consequence of (4.4) and (4.8). For $l < w_1 \leqslant w_2 < \xi$,

$$\psi(w_2) = \psi(w_1)\psi_{w_1,\bar{y},\gamma_{I,\mu}}(w_2). \tag{4.11}$$

Consequently, $\psi \in \mathcal{AC}((l, \xi))$ holds and ψ is strictly decreasing on (l, ξ) by (4.4). Since $\Gamma_{I,\mu} \neq \emptyset$, assertion (4.5) holds, and by the definition of $\psi_{w_1,\bar{y},\gamma_{I,\mu}}(w_2)$ in (4.4), we have

$$\lim_{w_1 \downarrow l} \psi(w_1) = \psi(w_2)\exp\left[\gamma_{I,\mu}\int_l^{w_2} \frac{\mathrm{d}z}{\mu(z)}\right] = \infty,$$

$$\lim_{w_2 \uparrow \xi} \psi(w_2) = \psi(w_1)\exp\left[-\gamma_{I,\mu}\int_r^{\xi} \frac{\mathrm{d}z}{\mu(z)}\right] = 0.$$

The second line implies left-continuity of ψ in ξ; thus, $\psi \in \mathcal{C}((l, \xi])$; the same argument as in Lemma 4.2 yields $\psi \in \mathcal{AC}((l, \xi])$.

By definition, $\sigma_{\gamma_{I,\mu}} = \mu/\psi$ on (l, ξ). As ψ is strictly increasing on $(l, \xi]$ we know that $\psi(w) > 0$ for all $w \in (l, \xi)$. Hence, $\sigma_{\gamma_{I,\mu}} \in \mathcal{C}(l, \xi)$. Left continuity in ξ follows from (4.8) and the definition of $\sigma_{\gamma_{I,\mu}}$. Therefore (P1) is satisfied on $(l, \xi]$.

Since $\sigma_{\gamma_{I,\mu}}(\xi) = 1$ and $\mu(\xi) = 0$ we obtain by definition $\psi(\xi) = \mu(\xi)/\sigma_{\gamma_{I,\mu}}(\xi)$. As $\sigma_{\gamma_{I,\mu}} = \mu/\psi$ on (l, ξ) this implies $\psi = \mu/\sigma_{I,\gamma}$ on $(l, \xi]\backslash Z(\sigma_{I,\gamma})$. Thus, ψ is a function such that (P2) holds for μ and $\sigma_{\gamma_{I,\mu}}$ on $(l, \xi]$. To show (P3) observe that (4.11) implies a.e. on $(l, \xi]$ that

$$\psi' = -\gamma_{I,\mu}\frac{\psi}{\mu} = -\gamma_{I,\mu}\frac{1}{\sigma_{\gamma_{I,\mu}}}.$$

As analogous reasoning holds for $[\xi, r)$, the triple $(I, \mu, \sigma_{\gamma_{I,\mu}})$ is proper with corresponding FC $\gamma_{I,\mu}$. □

5. Parametric models

In this section we present some new models given by proper triples and derive their stationary densities. In all models parameters are chosen from sets in finite-dimensional spaces.

Every proper triple (I, μ, σ) is complemented by a centre ξ of μ, an SST f, an FC γ (Definition 3.3, Lemma 3.1). If (I, μ, σ) is H-proper, then the stationary solution $X = f(O^{H,\gamma})$ of (1.4) has marginal density p given in (3.12).

5.1. Affine drift

In this subsection we apply the results of Propositions 4.4–4.6 to $\mu : \mathbb{R} \to \mathbb{R}$ given by $\mu(x) = \alpha + \beta x$ for $\alpha, \beta \in \mathbb{R}$.

Proposition 5.1. *Let $\mu : \mathbb{R} \to \mathbb{R}$ be given by $\mu(x) = \alpha + \beta x$ for α, $\beta \in \mathbb{R}$. There exist $I \subseteq \mathbb{R}$ and $\sigma \in \mathcal{C}(I)$ such that (I, μ, σ) is proper if and only if $\beta < 0$. In this case*

$$I = \mathbb{R}, \qquad \Gamma_{I,\mu} = (0, |\beta|], \qquad \Theta_{I,\mu} = (0, |\beta|), \qquad \xi = -\frac{\alpha}{\beta}.$$

Proof. We have to check the conditions of Proposition 4.4. There exists a centre ξ for μ if and only if $\beta < 0$. Thus, suppose that $\beta < 0$. Conditions (a) and (b) in Proposition 4.4 are satisfied for $I = \mathbb{R}$ and $\xi = -\alpha/\beta$. They also imply that $\mathbb{R}$ is the minimal state space. To obtain $\Gamma_{I,\mu}$, we have to calculate $\Theta_{I,\mu}$; secondly, $\gamma_{I,\mu} \in \Gamma_{I,\mu}$ is verified if either $\gamma_{I,\mu} \in \Theta_{I,\mu}$ or $\gamma_{I,\mu}$ satisfies (4.8) of Proposition 4.6. For γ and $x < \xi < y$, we obtain from (4.4),

$$\frac{\mu(w)}{\psi_{x,y,\gamma}(w)} = \begin{cases} (\alpha + \beta x)^{-\gamma/\beta}(\alpha + \beta w)^{1+\gamma/\beta}, & \text{for } w < \xi, \\ -|\alpha + \beta y|^{-\gamma/\beta}|\alpha + \beta w|^{1+\gamma/\beta}, & \text{for } w < \xi. \end{cases} \tag{5.1}$$

Thus, $\Theta_{I,\mu} = (0, |\beta|)$ and $\gamma_{I,\mu} = |\beta|$ by Proposition 4.5. The FC $\gamma_{I,\mu}$ is not an element of $\Theta_{I,\mu}$, but it satisfies (4.8) in Proposition 4.6. In particular, $\Gamma_{I,\mu} = (0, |\beta|]$. □

As $\Theta_{I,\mu}$ differs from $\Gamma_{I,\mu}$, two types of cones $\mathcal{H}_{\mathbb{R},\mu,\gamma}$ will appear – one-dimensional and two-dimensional ones.

Proposition 5.2. *Let $H \in (0, 1)$ and $\beta < 0$. Then every $\sigma \in \mathcal{H}_{\mathbb{R},\mu,|\beta|}$ leads to a* Vasicek model, *i.e.,*

$$\mathcal{H}_{\mathbb{R},\mu,|\beta|} = \{\sigma \in \mathcal{C}(\mathbb{R}) : \sigma(x) \equiv \sigma_0, \, x \in \mathbb{R} \, \sigma_0 > 0\}.$$

Moreover,

(i) *(I, μ, σ) is H-proper with the SST $f(x) = \sigma_0 x + \xi = \sigma_0 x - \alpha/\beta$.*
(ii) *The stationary pathwise solution X of* (1.4) *is unique and a Gaussian process with mean $\mathrm{E}X_t = \xi$ and variance $\mathrm{E}(X_t - \xi)^2 = 2^{-1}|\beta|^{-2H}\sigma_0^2\Gamma(2H + 1)$.*

Proof. This is an application of Proposition 4.6, and it is verified by (4.9) there. For a function $\sigma \equiv \sigma_0$, we obtain $\psi(x) = \mu(x)/\sigma_0 = \alpha/\sigma_0 + \beta x/\sigma_0$. The SST f has to be calculated by $f(x) = \psi^{-1}(\beta x)$, whereas p is straightforward from (3.12). As f is affine, (I, μ, σ) is always H-proper. Moreover, as X is the image of $O^\gamma(B^H)$ under an affine mapping it is Gaussian. Clearly, $Z(\sigma)$ is empty. Thus, X is the unique stationary pathwise solution of (1.4). □

Recall from Remark 3.1(ii) that H-proper triples, which differ from the Vasicek model, only occur for $H \in (1/2, 1)$.

Proposition 5.3. *Let $\beta < 0$ and $\delta \in (0, 1)$. Then $(1 - \delta)|\beta| \in \Theta_{I,\mu}$ and the following assertions hold.*

(i) *$\sigma \in \mathcal{H}_{\mathbb{R},\mu,(1-\delta)|\beta|}$ if and only if there exist constants σ_1, $\sigma_2 > 0$ such that*

$$\sigma(x) = \begin{cases} \sigma_1|\alpha+\beta x|^\delta, & \textit{for } x<\xi, \\ \sigma(x) = \sigma_2|\alpha+\beta x|^\delta, & \textit{for } x \geqslant \xi. \end{cases} \tag{5.2}$$

For such σ *and* $f_i = |\beta|^{\delta/(1-\delta)}\sigma_i^{1/(1-\delta)}(1-\delta)^{1/(1-\delta)}$, $i=1,2$, *the SST* f *is given by*

$$f(x) = \begin{cases} -f_1|x|^{1/(1-\delta)}+\xi & \textit{for } x<0, \\ f_2|x|^{1/(1-\delta)}+\xi & \textit{for } x \geqslant 0. \end{cases}$$

(ii) (I, μ, σ) *is* H*-proper if and only if* $\delta \in (1-H, 1)$ *and* $H \in (1/2, 1)$. *In this case the stationary density is given by*

$$p(x) = \bar{p}(x|\sigma_1)1_{(-\infty,\xi)}(x) + \bar{p}(x|\sigma_2)1_{(\xi,\infty)}(x),$$

$$\bar{p}(x|\sigma_i) = \frac{|\beta|^{H-\delta}(1-\delta)^H}{\sqrt{\pi\,\Gamma(2H+1)\sigma_i^2}}|x-\xi|^{-\delta}\exp\left[-\frac{|\beta|^{2(H-\delta)}(1-\delta)^{2H-2}}{\sigma_i^2\,\Gamma(2H+1)}|x-\xi|^{2(1-\delta)}\right].$$

Remark 5.1. By Lemma 3.3(ii), $\sigma \in \mathcal{C}^{((1-H)/H)+}(I)$ implies that a proper triple is also H-proper. In the situation of Proposition 5.3(i) this is satisfied if $\delta > (1-H)/H$ and $H \in (1/2, 1)$. As stated in (ii), a triple is already H-proper if $\delta > 1-H$ and $H \in (1/2, 1)$, which is clearly a weaker condition. This refers to Remark 2.3.

Proof of Proposition 5.3. This is an application of Proposition 4.5 to $\gamma = (1-\delta)|\beta|$. For $\bar{x} \in (-\infty, \xi)$ and $\bar{y} \in (\xi, \infty, \xi)$, set

$$\sigma_{\bar{x},\bar{y},\gamma}(x) = \begin{cases} |\alpha+\beta\bar{x}|^{-\gamma/\beta}|\alpha+\beta x|^{1+\gamma/\beta}, & \text{for } x<\xi, \\ 0, & \text{for } x=\xi, \\ |\alpha+\beta\bar{y}|^{-\gamma/\beta}|\alpha+\beta x|^{1+\gamma/\beta}, & \text{for } x>\xi. \end{cases}$$

By Proposition 4.5 and (5.1), $\mathcal{H}_{I,\mu,\gamma} = \{\sigma_{\bar{x},\bar{y},\gamma} : \bar{x} \in (-\infty, \xi), \bar{y} \in (\xi, \infty)\}$; furthermore, the following mappings are bijections:

$$(-\infty, \xi), (\xi, \infty) \ni w \mapsto \sigma_i = |\alpha+\beta w|^{-\gamma/\beta} \in \mathbb{R}^+, \qquad (0, |\beta|) \ni \gamma \mapsto \delta = 1+\gamma/\beta \in (0, 1).$$

$\mathcal{H}_{I,\mu,\gamma}$ is a cone; thus, $\sigma \in \mathcal{H}_{I,\mu,\gamma}$ holds if and only if σ is of the form stated in (5.2). The SST is calculated by $f(x) = \psi^{-1}(-(1-\delta)|\beta|x)$, where $\psi(x) = \mu(x)/\sigma(x)$ for $x \neq \xi$ and $\psi(\xi) = 0$. The formula for p follows from (3.12).

For $\delta \in (0, 1)$, we find $\delta/(1-\delta) > (1-H)/H$ if and only if $\delta > 1-H$. By Remark 3.1(ii) and Lemma 3.3(i), (I, μ, σ) is H-proper if and only if $f' \in \mathcal{C}^{((1-H)/H)+}(\mathbb{R})$ and $H \in (1/2, 1)$; this is true if $\delta > 1-H$ and $H \in (1/2, 1)$. □

5.2. Power volatility

We apply Proposition 4.1 to the function $\sigma : \mathbb{R} \to [0, \infty)$ given by $\sigma(x) = \sigma_0|x|^\delta$ for $\sigma_0 > 0$ and $\delta \in \mathbb{R}$.

Proposition 5.4. *Let $\sigma : \mathbb{R} \to [0, \infty)$ be given by $\sigma(x) = \sigma_0 |x|^\delta$ for $\sigma_0 > 0$ and $\delta \in \mathbb{R}$. We set $\sigma(0) = \infty$ for $\delta < 0$. There exist $I \subseteq \mathbb{R}$ and $\mu \in \mathcal{C}(I)$ such that (I, μ, σ) is proper if and only if $\delta \in [0, 1]$.*

Proof. If $\delta < 0$ then $1/\sigma \in \mathcal{L}_C(\mathbb{R})$; thus, $I = \mathbb{R}$ would be the minimal state space in this case. However (P1) is violated as σ cannot be extended to a continuous function on $\mathbb{R}$. If $\delta > 1$ then $1/\sigma \in \mathcal{L}_C(\mathbb{R}\backslash\{0\})$, but $\int_{-\infty}^{-\epsilon} 1/\sigma(z)\mathrm{d}z$ and $\int_{\epsilon}^{\infty} 1/\sigma(z)\mathrm{d}z$ are finite for all $\epsilon > 0$, violating (4.2) in Proposition 4.1. For $\delta \in [0, 1]$, there always exists a choice for a state space I satisfying the conditions in Proposition 4.1(ii)(b). □

Next we derive representations of the cones $\mathcal{K}_{I,\sigma}$ for all $\delta \in [0, 1]$. The choice of $\delta = 0$ leads to a Vasicek model and has already been discussed in Proposition 5.2.

Proposition 5.5. *Let $\delta = 1$.*

(i) *$\mathcal{K}_{I,\sigma}$ is non-empty if and only if either $I = (0, \infty)$ or $I = (-\infty, 0)$.*
(ii) *If $I = (0, \infty)$ or $I = (-\infty, 0)$, then (I, μ, σ) is H-proper for all $(\gamma, \mu) \in \mathcal{K}_{I,\sigma}$ and $H \in (1/2, 1)$. In this case the stationary pathwise solution X of* (1.4) *is unique.*
(iii) *$\mathcal{K}_{(0,\infty),\sigma}$ has representation*

$$\mathcal{K}_{(0,\infty),\sigma} = \{(|\beta|, \mu) \in \mathbb{R}^+ \times \mathcal{C}(I) : \mu(x) = \alpha x + \beta x \log x,\ \alpha \in \mathbb{R},\ \beta < 0,\ x \in I\}.$$

(iv) *If $\mu(x) = \alpha x + \beta x \log x$ and $\mu \in \mathcal{K}_{(0,\infty),\sigma}$, then the following formulae hold for the SST f, the centre ξ and the stationary density p for $((0, \infty), \mu, \sigma)$:*

$$f(x) = \mathrm{e}^{\sigma_0 x - \alpha/\beta}, \qquad x \in (0, \infty),$$

$$\xi = \mathrm{e}^{-\alpha/\beta};$$

$$p(x) = \pi^{-1/2} C x^{-1} \exp\left[-C^2\left(\log x + \frac{\alpha}{\beta}\right)^2\right], \qquad x \in (0, \infty);$$

$$C = \frac{|\beta|^H}{\sqrt{\Gamma(2H+1)\sigma_0^2}}.$$

p is the density of a lognormal random variable. Analogous formulae hold for $I = (-\infty, 0)$.

Remark 5.2. A similar construction was used in Comte and Renault (1998). They define, for $\gamma > 0$ and $H \in (1/2, 1)$, a stationary process of the form $Y_t^{H,\gamma} = \int a^{H,\gamma}(t-s)\mathrm{d}B_s^{1/2}$, where

$$a^{H,\gamma}(x) = \frac{1}{\Gamma(H+1/2)}\left(x^{H-1/2} - \mathrm{e}^{-\gamma x}\int_0^x \mathrm{e}^{\gamma u} u^{H-1/2}\,\mathrm{d}u\right), \qquad x > 0.$$

Up to a norming constant, this process is equivalent to an FOUP – cf. formulae (2.5) in Comte and Renault (1998) and (6.1) in Hult (2003), respectively. They consider a model

$f(Y^{H,\gamma})$ for $f = \exp$, which shows a certain similarity to the model defined in Proposition 5.5.

Proof of Proposition 5.5. Condition (b) in Proposition 4.1(i) is satisfied for both $I = (0, \infty)$ and $I = (-\infty, 0)$. By Proposition 4.1, the cone $\mathcal{K}_{(0,\infty),\sigma}$ contains precisely the pairs (γ, μ), where $\mu(x) = -\gamma\sigma(x)\int_\xi^x 1/\sigma(z)\mathrm{d}z$ and $\gamma > 0$; $\xi \in (0, \infty)$ is the corresponding centre of μ. We calculate $\mu(x) = \gamma x \log \xi - \gamma x \log x$ for $x, \xi, \gamma > 0$. The mapping $\mathbb{R}^+ \times \mathbb{R}^+ \ni (\xi, \gamma) \mapsto (\gamma \log(\xi), -\gamma) \in \mathbb{R} \times (-\infty, 0)$ is a bijection; thus, $\mathcal{K}_{(0,\infty),\sigma}$ is precisely of the form stated in (iii). The formulae for f and p are calculated from $\psi = \mu/\sigma$ and $f(x) = \psi^{-1}(-|\beta|x)$. □

Proposition 5.7. *Let* $\delta \in (0, 1)$.

(i) $\mathcal{K}_{I,\sigma}$ *is non-empty if and only if* $I = \mathbb{R}$.
(ii) $\mathcal{K}_{\mathbb{R},\sigma}$ *has representation*

$$\mathcal{K}_{\mathbb{R},\sigma} = \{((1-\delta)|\beta|, \mu) \in \mathbb{R}^+ \times \mathcal{C}(\mathbb{R}) : \mu(x) = \alpha|x|^\delta + \beta x, \alpha \in \mathbb{R}, \beta < 0, x \in \mathbb{R}\}.$$

(iii) *If* $(\gamma, \mu) \in \mathcal{K}_{\mathbb{R},\sigma}$ *then* (I, μ, σ) *is* H*-proper if and only if* $\delta \in (1-H, 1)$ *and* $H \in (1/2, 1)$.
(iv) *If* $\mu(x) = \alpha|x|^\delta + \beta x$ *and* $\mu \in \mathcal{K}_{\mathbb{R},\sigma}$ *then the following formulae hold for the SST* f, *the centre* ξ *and the stationary density* p *for* $(\mathbb{R}, \mu, \sigma)$:

$$f(x) = \operatorname{sign}\,[\sigma_0(1-\delta)x - \alpha/\beta]|\sigma_0(1-\delta)x - \alpha/\beta|^{1/(1-\delta)}, \quad x \in \mathbb{R};$$

$$\xi = \operatorname{sign}(\alpha)|\alpha/\beta|^{1/(1-\delta)};$$

$$p(x) = C_1|x|^{-\delta}\exp(-C_2(x|x|^{-\delta} + \alpha/\beta)), \qquad x \in \mathbb{R},$$

where

$$C_1 = \frac{|\beta|^H(1-\delta)^H}{\sqrt{\pi\Gamma(2H+1)}\sigma_0^2}, \qquad C_2 = \frac{|\beta|^{2H}(1-\delta)^{2H-2}}{\sigma_0^2\Gamma(2H+1)}.$$

Proof. Condition (b) of Proposition 4.1(i) holds for $I = \mathbb{R}$. For $\xi \in \mathbb{R}$ and $\gamma > 0$, we obtain $\mu \in \mathcal{K}_{\mathbb{R},\sigma}$ via

$$\mu(x) = -\gamma\,\sigma(x)\int_\xi^x \frac{\mathrm{d}z}{\sigma(z)} = \frac{\gamma}{1-\delta}\ \operatorname{sign}\,(\xi)|\xi|^{1-\delta}|x|^\delta - \frac{\gamma}{1-\delta x}, \qquad x \in \mathbb{R}.$$

Analogous reasoning yields the formulae stated in (iv). A proper triple is H-proper if and only if $f \in \mathcal{C}^{((1-H)/H)+}(\mathbb{R})$ for the SST f; this holds if $\delta \in (1-H, 1)$. □

5.3. Bounded state space

In this section we present two models with a bounded state space I. According to Remark 3.1(ii) such models exist only for $H \in (1/2, 1)$.

Example 5.1 *Bounded state space*. We construct an H-proper triple with state space $I = (l, r)$, where $l < r$ are finite real numbers. For $x \in (l, r)$, set

$$\sigma(x) = \sigma_0(x - l)(r - x), \qquad \mu(x) = \sigma_0\alpha(x - l)(r - x) + \sigma_0\beta(x - l)(r - x)\log\left[\frac{x - l}{r - x}\right].$$

The triple (I, μ, σ) is H-proper. We obtain the following quantities:

$$\xi = \frac{1}{1 + \exp(-\alpha/\beta)} l + \frac{\exp(-\alpha/\beta)}{1 + \exp(-\alpha/\beta)} r;$$

$$\gamma = -\beta\sigma_0;$$

$$f(x) = \frac{1}{1 + \exp(\sigma_0 x - \alpha/\beta)} l + \frac{\exp(\sigma_0 x - \alpha/\beta)}{1 + \exp(\sigma_0 x - \alpha/\beta)} r, \qquad x \in (l, r);$$

$$p(x) = C_1(H, \beta, \sigma_0)(x - l)^{-1}(r - x)^{-1} \times \exp\left[-C_2(\mathrm{H}, \beta, \sigma_0)\left(\alpha + \beta\log\left[\frac{x - l}{r - x}\right]\right)^2\right], \qquad x \in (l, r),$$

where

$$C_1(H, \beta, \sigma_0) = \frac{|\beta|^H \sigma_0^{H-1}}{\sqrt{\pi}\Gamma(2H + 1)}, \qquad C_2(H, \beta, \sigma_0) = \frac{|\beta|^{2H-2}\sigma_0^{2H-2}}{\Gamma(2H + 1)}.$$

Note that f and ξ are obtained by logistic convex combinations from l and r.

Example 5.2 *Construction based on* ψ. Let $I = (-\pi/2, \pi/2)$. We start with $\gamma > 0$ and define $\psi_\gamma(x) = -\gamma \tan x$. Then $\psi_\gamma \in \mathcal{C}^\infty(I)$ is strictly decreasing, satisfying (3.1) of (P2) in I. We seek functions $\mu, \sigma \in \mathcal{C}(I)$ such that (I, μ, σ) is proper with FC γ. By (P3), $\psi'_\gamma = -\gamma/\sigma$; thus, $\sigma(x) = \cos^2 x$. Moreover, $\mu = \sigma\psi_\gamma$, hence $\mu(x) = -\gamma \cos x \sin x$. Then μ has centre $\xi = 0$ and the SST is $f(x) = \arctan x$. Clearly, (I, μ, σ) is H-proper. Moreover, X is the unique pathwise solution of (1.4) and we find

$$p(x) = \frac{|\gamma|^H}{\sqrt{\pi}\Gamma(2H + 1)} \frac{1}{\cos^2 x} \exp\left[-\frac{\gamma^{2H}}{\Gamma(2H + 1)} \tan^2 x\right], \qquad x \in I.$$

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