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On the unlimited growth of a class of homogeneous multitype Markov chains

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We consider a homogeneous multitype Markov chain whose states have non-negative integer coordinates, and give criteria for deciding whether or not the chain grows indefinitely with positive probability. The results are applied to study the extinction problem in a general class of controlled multitype branching processes.

Keywords: homogeneous multitype Markov chains; multitype branching processes; unlimited growth

1. Introduction

The study of the extinction and/or the indefinite growth of certain biological (human, animal, cell, ...) and physical (particle, cosmic ray, ...) populations have given rise to a set of mathematical models known as branching processes. There have been so many major works on this topic that branching processes have acquired their own nomenclature related to population dynamics. Nevertheless, most branching processes can be considered as examples of a more general class of Markov processes, sometimes known as Markov population processes (see, for example, Klebaner 1994).

In the present work, we focus on homogeneous multitype Markov chains in discrete time that take values in the space of vectors with non-negative integer coordinates. We investigate the indefinite growth of these chains, providing one set of conditions for such an event to have positive probability and another set for it to have null probability. However, we try not to lose the perspective of branching processes and population dynamics, and we retain their special terminology. Indeed, we consider not only the classical multitype Galton–Watson branching process, but also other modified multitype branching processes, from the more general viewpoint of homogeneous multitype Markov chains. For such processes, indefinite growth and extinction are complementary phenomena, so that the two problems can be dealt with together.

To this end, we consider a homogeneous m-dimensional Markov chain (HMMC) $\{Z(n)\}_{n\geqslant 0}$ whose states have non-negative integer coordinates, $S \subseteq \mathbb{N}_0^m$, where S is the set of states. This chain can model a population where individuals of m different types coexist. In particular, the ith coordinate of $Z(n)$ can represent the number of individuals of type i n generations after the observation was started. The event we are interested in, known as *explosion of the chain*, is $\{\|Z(n)\| \to \infty\}$, where $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^m$. By the equivalence of norms on $\mathbb{R}^m$ we can also write the explosion event as $\{Z(n)\mathbf{1} \to \infty\}$, with

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1 the m-dimensional vector with all coordinates equal to unity. This simply means that the total number of individuals grows indefinitely in the explosion event.

We obtain sufficient conditions for the explosion event to have positive probability in Section 2 and null probability in Section 3. In Section 4 we introduce a new general class of controlled multitype branching processes, and apply the criteria obtained in the preceding sections to investigate the extinction or survival of populations modelled by such processes.

We shall consider, for each vector μ with positive coordinates ($\mu \in \mathbb{R}_+^m$), the sequence of linear functionals $\{Z(n)\mu\}_{n\geqslant 0}$. This process is not a Markov chain, but it has some remarkable properties. Indeed, $\{\|Z(n)\| \to \infty\} = \{Z(n)\mu \to \infty\}$, so that the explosion of the chain is equivalent to the unlimited growth of the sequence of functionals. In relation to this sequence of linear functionals we can introduce the variables ξ_n^μ, $n \geqslant 0$, and the functions $g_\mu(z)$ and $\sigma_\mu^2(z)$, defined for every non-null vector $z \in \mathbb{N}_0^m$ by:

$$\xi_{n+1}^\mu := Z(n+1)\mu - \mathrm{E}[Z(n+1)\mu | Z(n)],$$

$$g_\mu(z) := (z\mu)^{-1}\mathrm{E}[Z(n+1)\mu | Z(n) = z] - 1,$$

$$\sigma_\mu^2(z) := (z\mu)^{-1}\,\mathrm{var}[Z(n+1)\mu | Z(n) = z].$$

Note that, although they depend on the chosen vector μ, there is no ambiguity, so that we shall henceforth omit μ in the notation and write ξ_n, $g(z)$ and $\sigma^2(z)$ instead of ξ_n^μ, $g_\mu(z)$ and $\sigma_\mu^2(z)$, respectively.

2. Sufficient conditions for unlimited growth with positive probability

In this section we provide sufficient conditions for the process $\{Z(n)\}_{n\geqslant 0}$ to grow indefinitely, $P[\|Z(n)\| \to \infty] > 0$. The conditions proposed will depend on a process of the type $\{Z(n)\mu\}_{n\geqslant 0}$ for some $\mu \in \mathbb{R}_+^m$. In particular, we establish the following result:

Theorem 1. *Let $\{Z(n)\}_{n\geqslant 0}$ be an HMMC. Suppose there exists a vector $\mu \in \mathbb{R}_+^m$ such that*

$$\liminf_{\|z\|\to\infty} g(z) > 0 \tag{1}$$

and $\mathrm{E}\left[|\xi_{n+1}|^{1+\delta} | Z(n) = z\right] = O(\|z\|^\delta)$ for some $\delta \geqslant 0$. There then exists a constant $N \geqslant 0$ such that $P[\|Z(n)\| \to \infty | Z(0) = z^{(0)}] > 0$ if $\|z^{(0)}\| > N$.

Proof. From (1), there exist $r > 1$ and $N_0 > 0$ such that if $\|z\| > N_0$, then

$$(z\mu)^{-1}\mathrm{E}[Z(n+1)\mu | Z(n) = z] \geqslant r. \tag{2}$$

Since $\mu \in \mathbb{R}_+^m$, it suffices to prove that, for $\|z\|$ large enough,

$$P[Z(n)\mu \to \infty | Z(0) = z] > 0$$

or, more specifically, that there exist $N > 0$ and $\eta > 1$ such that if $\|z\| > N$, then

$$P\left[\bigcap_{n=0}^{\infty}\{Z(n+1)\mu > \eta Z(n)\mu\}|Z(0)=z\right] > 0.$$

Let $0 < \varepsilon < r-1$, and consider $\eta = r-\varepsilon$. For simplicity, we shall use the notation

$$A_n := \{Z(n+1)\mu > (r-\varepsilon)Z(n)\mu\}, \qquad n \geqslant 0.$$

From (2) and Markov's inequality, for each $z \in \mathbb{N}_0^m$ such that $\|z\| > N_0$ and for every $\delta \geqslant 0$,

$$\begin{aligned} P[A_n^c|Z(n)=z] &= P\left[\frac{Z(n+1)\mu}{Z(n)\mu} \leqslant r-\varepsilon|Z(n)=z\right] \\ &\leqslant P\left[\frac{Z(n+1)\mu}{Z(n)\mu} \leqslant \frac{\mathrm{E}[Z(n+1)\mu|Z(n)]}{Z(n)\mu} - \varepsilon|Z(n)=z\right] \\ &\leqslant P\left[\left|\frac{Z(n+1)\mu - \mathrm{E}[Z(n+1)\mu|Z(n)]}{Z(n)\mu}\right| \geqslant \varepsilon|Z(n)=z\right] \\ &\leqslant \frac{\mathrm{E}\left[|\xi_{n+1}|^{1+\delta}|Z(n)=z\right]}{\varepsilon^{1+\delta}(z\mu)^{1+\delta}}. \end{aligned}$$

By hypothesis, there exists some $\delta \geqslant 0$ satisfying $\mathrm{E}\left[|\xi_{n+1}|^{1+\delta}|Z(n)=z\right] = O(\|z\|^{\delta})$. For this value of δ it is possible to find constants $C', N_1 > 0$ such that if $\|z\| > N_1$, then $\mathrm{E}\left[|\xi_{n+1}|^{1+\delta}|Z(n)=z\right] \leqslant C'(z\mu)^{\delta}$. Hence, if $\|z\| > \max\{N_0, N_1\}$, and writing $C := \varepsilon^{-1-\delta}C'$, we obtain

$$P[A_n^c|Z(n)=z] \leqslant (z\mu)^{-1}C. \tag{3}$$

Also, for $\|z\| > \max\{N_0, N_1\}$ and $k \geqslant 1$,

$$P\left[\bigcap_{n=0}^{k} A_n|Z(0)=z\right] = P[A_0|Z(0)=z]\prod_{n=1}^{k} P\left[A_n|\bigcap_{l=0}^{n-1} A_l \cap \{Z(0)=z\}\right].$$

For every $\tilde{z} \in \mathbb{N}_0^m$, define $B_{\tilde{z}} := \bigcap_{l=0}^{n-1} A_l \cap \{Z(0)=z\} \cap \{Z(n)=\tilde{z}\}$. It is obvious that $\{B_{\tilde{z}}\}_{\tilde{z}\in\mathbb{N}_0^m}$ is a partition of the set $\bigcap_{l=0}^{n-1} A_l \cap \{Z(0)=z\}$, and, moreover, $Z(n)\mu > z\mu(r-\varepsilon)^n$ on $\bigcap_{l=0}^{n-1} A_l \cap \{Z(0)=z\}$. Consequently,

$$\begin{aligned} P\left[A_n|\bigcap_{l=0}^{n-1} A_l \cap \{Z(0)=z\}\right] &= P\left[A_n|\bigcup_{\tilde{z}} B_{\tilde{z}}\right] \geqslant \inf_{\tilde{z}\mu > z\mu(r-\varepsilon)^n} P[A_n|B_{\tilde{z}}] \\ &= \inf_{\tilde{z}\mu > z\mu(r-\varepsilon)^n} P[A_n|Z(n)=\tilde{z}]. \end{aligned} \tag{4}$$

Since $\tilde{z}\mu > z\mu(r-\varepsilon)^n$ and $r-\varepsilon > 1$, and from the equivalence of norms on $\mathbb{R}^m$, there exists a constant $N_2 > 0$ such that if $\|z\| > N_2$ then $\|\tilde{z}\| > \max\{N_0, N_1\}$. If $\|z\| > N := \max\{N_0, N_1, N_2\}$, then from (3) and (4) we obtain

$$P\left[A_n|\bigcap_{l=0}^{n-1} A_l \cap \{Z(0)=z\}\right] \geqslant 1 - \frac{C}{(z\mu)(r-\varepsilon)^n},$$

and therefore, if we choose $N \geqslant C$, we deduce for $\|z\| > N$ that

$$P\left[\bigcap_{n=0}^{\infty} A_n | Z(0) = z\right] = \lim_{k\to\infty} P\left[\bigcap_{n=0}^{k} A_n | Z(0) = z\right] \geqslant \prod_{n=0}^{\infty}\left(1 - \frac{C}{(z\mu)(r-\varepsilon)^n}\right).$$

The above product is positive because $\sum_{n=0}^{\infty}(r-\varepsilon)^{-n} < \infty$, which concludes the proof. □

For some Markov population processes, condition (1) is sufficient for the population to have unlimited growth with positive probability. For example, classical multitype branching processes, which are particular cases of HMMC, with irreducible matrix of means and associated Perron–Frobenius eigenvalue ρ, satisfy condition (1) if $\rho > 1$, with $\mu \in \mathbb{R}_+^m$ a right eigenvector associated with ρ (Seneta, 1981). Moreover, in this case $P[\|Z(n)\| \to \infty] > 0$ is equivalent to $\rho > 1$ (Mode, 1971). However, there exist HMMCs satisfying (1) for which the explosion event has null probability, giving rise to a richer behaviour than that of the classical multitype branching process, as is illustrated in the following example:

Example 1. Let $\{Z(n)\}_{n\geqslant 0}$ be an HMMC such that the null state is absorbing and the transition probabilities for each non-null vector z satisfy

$$P[Z(n+1) = \mathbf{0} | Z(n) = z] = \varepsilon^{1/\|z\|},$$
$$P[Z(n+1) = \lfloor(1-\varepsilon^{1/\|z\|})^{-1}\rfloor(a_1 z_1, \ldots, a_m z_m) | Z(n) = z] = 1 - \varepsilon^{1/\|z\|},$$

where $\mathbf{0}$ is the null vector, $0 < \varepsilon < 1$, $a_i \in \mathbb{N}$ for all $i \in \{1, \ldots, m\}$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and $\|z\| = z\mathbf{1}$.

If $a := \min\{a_i, i = 1, \ldots, m\}$ is greater than unity, then condition (1) holds by taking $\mu = \mathbf{1}$. However, it is easy to prove by induction on n that, for all $n \geqslant 1$ and $z \in \mathbb{N}_0^m$,

$$P[\|Z(n)\| > 0 | Z(0) = z] \leqslant (1-\varepsilon)^n, \; n \geqslant 1,$$

and hence that $P[\|Z(n)\| \to \infty] = 0$. □

Using the terminology of branching processes, we say that an HMMC satisfying condition (1) is *supercritical*. As we saw in Example 1, there are supercritical HMMCs which do not have unlimited growth. Unlike classical multitype branching processes, we can determine a wider set of situations for which the positive probability of the explosion of the chain is guaranteed. We posit the following result assuming the existence of moments of order greater than that required in Theorem 1:

Theorem 2. *Let* $\{Z(n)\}_{n\geqslant 0}$ *be an HMMC, and let* $z^{(0)} \in S$ *be such that,*

$$\text{for every } C > 0, \quad \text{there exists } n \geqslant 1 \text{ such that } P[Z(n)\mathbf{1} > C | Z(0) = z^{(0)}] > 0. \tag{5}$$

Assume there exists a vector $\mu \in \mathbb{R}_+^m$ *for which either*

$$\lim_{\|z\|\to\infty} g(z) = 0 \qquad \text{and} \qquad \liminf_{\|z\|\to\infty} \frac{2(z\mu)g(z)}{\sigma^2(z)} > 1 \tag{6}$$

or

$$\lim_{\|z\|\to\infty} g(z) > 0 \qquad \textit{and} \qquad \sigma^2(z) = O(\|z\|).$$

Assume further that, for some $0 < \delta \leqslant 1$ *and* $\alpha > 0$, *at least one of the following equalities holds:*

$$\mathrm{E}\left[|\xi_{n+1}|^{2+\delta}|Z(n) = z\right] = o\left(g(z)(z\mu)^{2+\delta}/(\log(z\mu))^{1+\alpha}\right),$$

$$\mathrm{E}\left[|\xi_{n+1}|^{2+\delta}|Z(n) = z\right] = o\left((z\mu)^{1+\delta}\sigma^2(z)/(\log(z\mu))^{1+\alpha}\right).$$

Then $P[\|Z(n)\| \to \infty | Z(0) = z^{(0)}] > 0$.

We omit the proof, which follows steps similar to those of Theorem 2 in Klebaner (1989) in the context of population-size-dependent multitype branching processes, given that the inequality

$$\mathrm{E}[(\log(Z(n+1)\mu + 3))^{-\alpha}|Z(n) = z] \leqslant (\log(z\mu + 3))^{-\alpha} - \alpha\frac{g(z)}{(\log(z\mu))^{\alpha+1}}(1 + o(1))$$
$$+ \frac{\alpha}{2}\frac{\sigma^2(z)}{(z\mu)(\log(z\mu))^{\alpha+1}}(1 + o(1))$$

holds for $\alpha > 0$ and $\|z\|$ large enough.

3. Sufficient conditions for unlimited growth with null probability

In this section we investigate conditions for the explosion of the chain to be a null event, $P[\|Z(n)\| \to \infty] = 0$. As was noted above, if $\mu \in \mathbb{R}_+^m$ then it suffices to obtain conditions guaranteeing that $P[Z(n)\mu \to \infty] = 0$. Hence, we formulate the following theorem:

Theorem 3. *Let* $\{Z(n)\}_{n\geqslant 0}$ *be an HMMC. If there exist a constant* $A > 0$ *and a vector* $\mu \in \mathbb{R}_+^m$ *such that*

$$\sup_{z\in\mathbb{N}_0^m : \|z\|\geqslant A} g(z) \leqslant 0, \tag{7}$$

then $P[\|Z(n)\| \to \infty] = 0$.

Proof. Let $A > 0$ and $\mu \in \mathbb{R}_+^m$ be such that condition (7) holds. It will suffice to show that

$$P[Z(n)\mu \to \infty | Z(0) = z^{(0)}] = 0$$

for every vector $z^{(0)} \in \mathbb{N}_0^m$. From the equality

$$\{Z(n)\mu \to \infty\} = \bigcup_{l=0}^{\infty}\left(\left\{\min_{n\geqslant l} Z(n)\mu \geqslant A\right\} \cap \{Z(n)\mu \to \infty\}\right)$$

and the fact that $\{Z(n)\}_{n\geqslant 0}$ is an HMMC, it suffices to prove that, for every $z \in \mathbb{N}_0^m$ such that $z\mu \geqslant A$,

$$P\left[\left\{\min_{n>0} Z(n)\mu \geqslant A\right\} \cap \{Z(n)\mu \to \infty\} | Z(0) = z\right] = 0. \tag{8}$$

To this end, define the auxiliary process $\{Z^*(n)\}_{n\geqslant 0}$ by $Z^*(0) := Z(0)$ and, for $n \geqslant 0$,

$$Z^*(n+1) := \begin{cases} \mathbf{0} & \text{if } Z^*(n)\mu < A, \\ Z(n+1) & \text{if } Z^*(n)\mu \geqslant A. \end{cases}$$

It is easy to verify that $\{Z^*(n)\}_{n\geqslant 0}$ is an HMMC and that condition (8) is equivalent to

$$P\left[\left\{\min_{n>0} Z^*(n)\mu \geqslant A\right\} \cap \{Z^*(n)\mu \to \infty\} | Z^*(0) = z\right] = 0.$$

Moreover, from (7), we obtain

$$\sup_{z\in\mathbb{N}_0^m : z\neq \mathbf{0}} \frac{\mathrm{E}[Z^*(n+1)\mu | Z^*(n) = z]}{z\mu} \leqslant 1,$$

and consequently $\{Z^*(n)\mu\}_{n\geqslant 0}$ is a non-negative supermartingale with respect to the sequence of σ-algebras $\{\mathcal{F}_n\}_{n\geqslant 0}$, defined by $\mathcal{F}_n := \sigma(Z^*(0), \ldots, Z^*(n))$. Applying the martingale convergence theorem, we obtain that $\{Z^*(n)\mu\}_{n\geqslant 0}$ converges almost surely to a finite limit, and therefore the proof is complete. □

For classical multitype branching processes with irreducible matrix of means and associated Perron–Frobenius eigenvalue ρ, condition (7) holds if and only if $\rho \leqslant 1$, with $\mu \in \mathbb{R}_+^m$ a right eigenvector associated with ρ and therefore $P[\|Z(n)\| \to \infty] = 0$ (Mode 1971). However, in the general context of HMMCs it is possible to obtain another set of conditions for non-explosion allowing $g(z)$ to be non-negative and assuming the existence of moments of order greater than 2. In this sense, we establish the following result:

Theorem 4. *Let $\{Z(n)\}_{n\geqslant 0}$ be an HMMC such that there exists a vector $\mu \in \mathbb{R}_+^m$ for which*

$$\lim_{\|z\|\to\infty} g(z) = 0. \tag{9}$$

Assume further that

$$\limsup_{\|z\|\to\infty} \frac{2(z\mu)g(z)}{\sigma^2(z)} < 1, \tag{10}$$

and, for some δ, $0 < \delta \leqslant 1$, either

$$\mathrm{E}\left[|\xi_{n+1}|^{2+\delta} | Z(n) = z\right] = o\left((z\mu)^{2+\delta} g(z)\right)$$

or

$$\mathrm{E}\big[|\xi_{n+1}|^{2+\delta}|Z(n)=z\big]=o\big((z\mu)^{1+\delta}\sigma^2(z)\big)$$

is satisfied. Then $P[\|Z(n)\|\to\infty]=0$.

The proof is similar to that of Theorem 3(1) in Klebaner (1991), again in the context of population-size-dependent multitype branching processes, given that if $\|z\|$ is large enough, then

$$\mathrm{E}[\log(Z(n+1)\mu+1)|Z(n)=z]\leqslant\log(z\mu+1)+g(z)(1+o(1))-\frac{\sigma^2(z)}{2(z\mu)}(1+o(1)).$$

Again using the nomenclature of branching processes, we say that an HMMC is *near-critical* if

$$\liminf_{\|z\|\to\infty}g(z)\leqslant 0\leqslant\limsup_{\|z\|\to\infty}g(z).$$

Hence, under condition (9), a near-critical HMMC may or may not have unlimited growth with positive probability (see Theorems 2 and 4, respectively).

Finally, we say that an HMMC is *subcritical* if

$$\limsup_{\|z\|\to\infty}g(z)<0.$$

In this case, we deduce from Theorem 3 that the explosion event has null probability.

A significant situation is when there is *extinction–explosion duality*, that is,

$$P[Z(n)\to\mathbf{0}]+P[\|Z(n)\|\to\infty]=1. \tag{11}$$

This behaviour is typical in branching process. Under this condition, $P[\|Z(n)\|\to\infty]=0$ implies $P[Z(n)\to\mathbf{0}]=1$, that is, the population becomes extinct almost surely. Using Markov chain theory (see Chung 1967), it is not hard to deduce that if the null state is absorbing and every non-null state is transient then (11) holds. Indeed, conditions such as $P[Z(1)=\mathbf{0}|Z(0)=z]>0$ for all $z\in\mathbb{N}_0^m$ are sufficient to guarantee the transience of every non-null state, obviously as long as $\mathbf{0}$ is absorbing. For example, non-singular irreducible classical multitype branching processes satisfy (11). However the results we have provided also apply to other processes whose behaviour is not determined by the extinction–explosion duality, such as branching models with immigration.

Remark 1. The proposed division of HMMCs into supercritical, near-critical, and subcritical leads to exhaustive categories with intersections that are empty for the process $\{Z(n)\mu\}_{n\geqslant 0}$ (for which obviously the division is also valid), but not necessarily empty for $\{Z(n)\}_{n\geqslant 0}$. However, it is not only fundamental in determining unlimited growth, but also the starting point for the study of the asymptotic behaviour of HMMCs, analogously to the case in branching processes.

4. On controlled multitype branching processes

In this section, we apply the results obtained for an HMMC to a general class of controlled multitype branching processes that have as yet not been investigated. The main points to focus on in order to obtain homogeneous branching models that more accurately describe real situations are: to consider population-size-dependent reproduction; to establish control of the number of each type of progenitor according to the population size; and to allow interaction between individuals of the same generation at reproduction time, that is, ‘dependent offspring’.

Control of the progenitors has been proposed by Sevast’yanov and Zubkov (1974) in a deterministic way, and population-size-dependent reproduction has been considered by Klebaner (1989). We introduce a new multitype model integrating control and size-dependent reproduction. This process generalizes to the one-dimensional model with control and reproduction dependent on population size considered by Küster (1985), who studied it for one particular situation only. In our proposed model, we assume that the number of each type of progenitor is controlled by a random mechanism, and consider possible dependence in the reproduction between individuals of the same generation. The introduction of dependence represents an important novelty with respect to classical branching models, since their implicit assumption of independence can only be considered to be a mere theoretical simplification of the more complex types of reproductive behaviour in nature. The following mathematical model is a possible description of such a situation:

Definition 1. *Let* $\{X^{i,n,j}(z) : i = 1, \ldots, m; n = 0, 1, \ldots; j = 1, 2, \ldots; z \in \mathbb{N}_0^m\}$ *and* $\{\phi^n(z) : n = 0, 1, \ldots; z \in \mathbb{N}_0^m\}$ *be two independent sequences of m-dimensional, non-negative, integer-valued random vectors satisfying the following conditions:*

(i) *For each* $z \in \mathbb{N}_0^m$, *the random vectors* $\{\phi^n(z)\}_{n\geqslant 0}$ *are independent and identically distributed.*
(ii) *If* $n, \tilde{n} \in \mathbb{N}_0$ *are such that* $n \neq \tilde{n}$, *then the random vectors* $\phi^n(z)$ *and* $\phi^{\tilde{n}}(\tilde{z})$ *are independent for every* $z, \tilde{z} \in \mathbb{N}_0^m$.
(iii) *For fixed* $z \in \mathbb{N}_0^m$, *the stochastic processes* $\{X^{i,n,j}(z) : i = 1, \ldots, m\}_{j\geqslant 1}$, $n = 0, 1, \ldots$ *are independent and identically distributed.*
(iv) *If* $n, \tilde{n} \in \mathbb{N}_0$ *are such that* $n \neq \tilde{n}$, *then for any* $z, \tilde{z} \in \mathbb{N}_0^m$ *the sequences* $\{X^{i,n,j}(z) : i = 1, \ldots, m; j = 1, 2, \ldots\}$ *and* $\{X^{i,\tilde{n},j}(\tilde{z}) : i = 1, \ldots, m; j = 1, 2, \ldots\}$ *are independent.*

The sequence of m-dimensional random vectors $\{Z(n)\}_{n\geqslant 0}$ *defined recursively as*

$$Z(0) = z \in \mathbb{N}_0^m, \qquad Z(n+1) = \sum_{i=1}^{m} \sum_{j=1}^{\phi_i^n(Z(n))} X^{i,n,j}(Z(n)), \quad n \geqslant 0,$$

will be referred to as a controlled multitype branching process with random control and population-size-dependent reproduction (CMPD).

The controlled multitype branching process proposed by Sevast’yanov and Zubkov (1974) and the population-size-dependent multitype branching process introduced by Klebaner (1989) can be deduced as particular cases of the CMPD. Moreover the process defined allows us to derive new multitype branching models as yet uninvestigated in the literature, and to extend the existing models by, for example, considering dependent offspring.

By definition, a CMPD is an HMMC, and the results of the previous sections can therefore be applied. First, we shall provide conditions for equality (11) to hold. Then, we shall analyse the particular form of the function $g(z)$ and give a classification for this type of process. Finally, we set suitable bounds for $\mathrm{E}[|\xi_{n+1}|^\gamma|Z(n)=z]$ when $\gamma \geqslant 1$, in order to determine whether or not there exists unlimited growth of the CMPD.

From the relationship between the control and reproduction vectors, it is not difficult to prove that the null state is absorbing if and only if, for each $i \in \{1, \ldots, m\}$,

$$P\big[\{\phi_i^0(\mathbf{0})=0\} \cup \{\phi_i^0(\mathbf{0})>0,\ X^{i,0,j}(\mathbf{0})=\mathbf{0},\ j=1, \ldots, \phi_i^0(\mathbf{0})\}\big]=1.$$

Also every non-null vector $z \in \mathbb{N}_0^m$ is transient if

$$P\left[\bigcap_{i=1}^{m}(\{\phi_i^0(z)=0\} \cup \{\phi_i^0(z)>0,\ X^{i,0,j}(z)=\mathbf{0},\ j=1, \ldots, \phi_i^0(z)\})\right]>0.$$

If these conditions are satisfied, then (11) holds. Hence, if explosion is a null event, then the process becomes extinct almost surely, or, equivalently, the only form of survival is the explosion of the process to infinity.

In order to obtain a useful expression for $g(z)$, we shall henceforth assume that, for every $z \in \mathbb{N}_0^m$ and $i \in \{1, \ldots, m\}$, the random vectors $X^{i,0,k}(z)$, $k \geqslant 0$, are identically distributed, and if $z_i=0$, then

$$P\big[\{\phi_i^0(z)=0\} \cup \{\phi_i^0(z)>0,\ X^{i,0,j}(z)=\mathbf{0},\ j=1, \ldots, \phi_i^0(z)\}\big]=1.$$

Hence we can define for each $z \in \mathbb{N}_0^m$ the matrix $M(z) := (m_{ij}(z))_{1\leqslant i,j\leqslant m}$, where

$$m_{ij}(z) := \begin{cases} \dfrac{\mathrm{E}[\phi_i^0(z)]\mathrm{E}[X_j^{i,0,1}(z)]}{z_i} & \text{if } z_i>0 \\ 0 & \text{if } z_i=0. \end{cases}$$

Intuitively, $m_{ij}(z)$ can be regarded as the average number of individuals of type j generated by each individual of group i in a generation, given that z individuals coexist in this generation.

Under these hypotheses, one obtains

$$\mathrm{E}[Z(n+1)|Z(n)=z]=zM(z), \qquad z \in \mathbb{N}_0^m. \tag{12}$$

We further assume that, for every $i, j \in \{1, \ldots, m\}$, the limit

$$m_{ij} := \lim_{\|z\|\to\infty: z_i\neq 0} \frac{\mathrm{E}[\phi_i^0(z)]\mathrm{E}[X_j^{i,0,1}(z)]}{z_i} \tag{13}$$

exists, and $M := (m_{ij})_{1\leqslant i,j\leqslant m}$ is an irreducible matrix with Perron–Frobenius eigenvalue ρ and an associated right eigenvector $\mu \in \mathbb{R}_+^m$.

From (12) and (13),

$$\lim_{\|z\|\to\infty} g(z) = \rho - 1,$$

and therefore we derive that a CMPD is *subcritical* if $\rho < 1$, *near-critical* if $\rho = 1$, and *supercritical* if $\rho > 1$. This classification is similar to the classical multitype branching process case, although with different associated behaviour, as we have already shown.

With the behaviour of $g(z)$ known, let us now set suitable bounds for the conditioned central moments $\mathrm{E}[|\xi_{n+1}|^\gamma | Z(n) = z]$ for $\gamma \geqslant 1$ and $z \in \mathbb{N}_0^m$. We have

$$\mathrm{E}[|\xi_{n+1}|^\gamma | Z(n) = z] = \mathrm{E}\left[\left|\sum_{i=1}^{m}\left(\sum_{j=1}^{\phi_i^n(z)} X^{i,n,j}(z)\mu - \mathrm{E}[\phi_i^n(z)]\mathrm{E}[X^{i,n,1}(z)\mu]\right)\right|^\gamma\right].$$

Since

$$\sum_{j=1}^{\phi_i^n(z)} X^{i,n,j}(z)\mu - \mathrm{E}[\phi_i^n(z)]\mathrm{E}[X^{i,n,1}(z)\mu] = \sum_{j=1}^{\phi_i^n(z)}(X^{i,n,j}(z)\mu - \mathrm{E}[X^{i,n,1}(z)\mu])$$
$$+ (\phi_i^n(z) - \mathrm{E}[\phi_i^n(z)])\mathrm{E}[X^{i,n,1}(z)\mu],$$

and using $|x+y|^r \leqslant C_r(|x|^r + |y|^r)$, $r > 0$, for some constant $C_r > 0$ (see Grimmett and Stirzaker, 1992, p. 287), we obtain

$$\mathrm{E}[|\xi_{n+1}|^\gamma | Z(n) = z] \leqslant A_1 \sum_{i=1}^{m} \mathrm{E}\left[\left|\sum_{j=1}^{\phi_i^n(z)}(X^{i,n,j}(z)\mu - \mathrm{E}[X^{i,n,1}(z)\mu])\right|^\gamma\right]$$
$$+ A_2 \sum_{i=1}^{m} \mathrm{E}[|\phi_i^n(z) - \mathrm{E}[\phi_i^n(z)]|^\gamma](\mathrm{E}[X^{i,n,1}(z)\mu])^\gamma,$$

for certain constants $A_1, A_2 > 0$. Moreover, in the general case, that is, when the random vectors $X^{i,n,j}(z)$, $j = 1, 2, \ldots$, $i = 1, \ldots, m$, are not necessarily independent for each fixed $z \in \mathbb{N}_0^m$ and $n \geqslant 0$, it can be shown that

$$\mathrm{E}\left[\left|\sum_{j=1}^{\phi_i^n(z)}(X^{i,n,j}(z)\mu - \mathrm{E}[X^{i,n,1}(z)\mu])\right|^\gamma\right] \tag{14}$$
$$\leqslant \mathrm{E}[\phi_i^n(z)^\gamma]\mathrm{E}\left[|X^{i,n,1}(z)\mu - \mathrm{E}[X^{i,n,1}(z)\mu]|^\gamma\right]. \tag{15}$$

On the other hand, if such vectors are independent, using von Bahr–Esseen and Marcinkiewicz–Zygmund inequalities (von Bahr and Esseen 1965; Chow and Teicher 1997), one finds that (14) can be bounded by either

$$\mathrm{E}[\phi_i^n(z)]\mathrm{E}\left[|X^{i,n,1}(z)\mu - \mathrm{E}[X^{i,n,1}(z)\mu]|^\gamma\right], \qquad \text{if } 1 \leqslant \gamma < 2, \tag{16}$$

or

$$\mathrm{E}[\phi_i^n(z)^{\gamma/2}]\mathrm{E}\left[|X^{i,n,1}(z)\mu - \mathrm{E}[X^{i,n,1}(z)\mu]|^\gamma\right], \qquad \text{if } \gamma \geqslant 2. \tag{17}$$

Hence, we can apply the HMMC results to CMPDs, taking the magnitude of ρ and the above bounds into account to investigate both the unlimited growth and the extinction of the population under (11). To summarize, we deduce that $P[\|Z(n)\| \to \infty] = 0$ if at least one of the following conditions is satisfied:

(i) $\rho < 1$;
(ii) $\rho = 1$, and for each $i, j \in \{1, \ldots, m\}$ and $\|z\|$ large enough,

$$\mathrm{E}[X_j^{i,0,1}(z)]\mathrm{E}[\phi_i^0(z)] \leqslant m_{ij} z_i\,;$$

(iii) $\rho = 1$, condition (10) holds, and for some $0 < \delta \leqslant 1$ and for every $i \in \{1, \ldots, m\}$, both (15) with $\gamma = 2 + \delta$ and $\mathrm{E}[|\phi_i^n(z) - \mathrm{E}[\phi_i^n(z)]|^{2+\delta}](\mathrm{E}[X^{i,n,1}(z)\mu])^{2+\delta}$ are either $o((z\mu)^{2+\delta} g(z))$ or $o((z\mu)^{1+\delta}\sigma^2(z))$.

On the other hand, $P[\|Z(n)\| \to \infty | Z(0) = z^{(0)}] > 0$ when at least one of the following conditions holds:

(iv) $\rho = 1$, $z^{(0)}$ satisfies (5), (6) holds, and, for some $0 < \delta \leqslant 1$, $\alpha > 0$ and for every $i \in \{1, \ldots, m\}$, both $\mathrm{E}[|\phi_i^n(z) - \mathrm{E}[\phi_i^n(z)]|^{2+\delta}](\mathrm{E}[X^{i,n,1}(z)\mu])^{2+\delta}$ and (15) with $\gamma = 2 + \delta$ are either

$$o((z\mu)^{2+\delta} g(z)/(\log(z\mu))^{1+\alpha})$$

or

$$o((z\mu)^{1+\delta}\sigma^2(z)/(\log(z\mu))^{1+\alpha});$$

(v) $\rho > 1$, $z^{(0)}$ is large enough, and, for some $\delta \geqslant 0$ and every $i \in \{1, \ldots, m\}$, both (15) with $\gamma = 1 + \delta$ and $\mathrm{E}[|\phi_i^n(z) - \mathrm{E}[\phi_i^n(z)]|^{1+\delta}](\mathrm{E}[X^{i,n,1}(z)\mu])^{1+\delta}$ are $O(\|z\|^\delta)$.

Remark 2. Under the independence assumption, (15) can be replaced by (16) or (17), depending on the value of γ.

Remark 3. The present study applies to the controlled multitype branching processes proposed by Sevast'yanov and Zubkov (1974), which have as yet not been investigated. Also, the results extend those of Klebaner (1989; 1991) relating to the extinction problem for a population-size-dependent multitype branching process.

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