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The Matsumoto–Yor property on trees

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Viewing the Matsumoto–Yor property as a bivariate property with respect to the simple tree with two vertices and one edge, we extend it to a p-variate property with respect to any tree with p vertices. The converse of the Matsumoto–Yor property, which characterizes the product of a gamma and a generalized inverse Gaussian distribution, is extended to characterize the product of a gamma and $p-1$ generalized inverse Gaussian distributions. A striking feature of this characterization is that we need the independence of the components of random vectors corresponding only to the leaves of the tree. We illustrate our results with two particular trees: the two-link chain and the three-branch 'daisy'.

Keywords: characterization of the product of a gamma and generalized inverse Gaussians; gamma; independence; inverse Gaussian; Matsumoto–Yor property; tree; Wishart

1. Introduction

The gamma $\gamma(p, a)$ and the generalized inverse Gaussian $GIG(q, b, c)$ distributions are respectively defined by the densities

$$f(x) \propto x^{p-1} \mathrm{e}^{-ax} I_{(0,\infty)}(x)$$

and

$$g(y) \propto y^{q-1} \mathrm{e}^{-by-c/y} I_{(0,\infty)}(y),$$

where p, a, b, c are positive numbers and q is a real. While studying properties of exponential Brownian motion, Matsumoto and Yor (2001) showed that if two random variables X and Y are independent and follow the $GIG(-q, a, b)$ and $\gamma(q, a)$ distributions respectively, then the two variables U and V defined as

$$U = \frac{1}{X+Y} \quad \text{and} \quad V = \frac{1}{X} - \frac{1}{X+Y} \tag{1.1}$$

are also independent and follow the $GIG(-q, b, a)$ and $\gamma(q, b)$ distributions, respectively. They actually originally proved this for $a = b$ only, but it was noticed in Letac and Wesołowski (2000) that the property holds also for $a \neq b$. Matsumoto and Yor (2003) interpreted this extension through properties of functionals of exponential Brownian motion.

Letac and Wesołowski (2000) proved the converse, that is, if X and Y are independent and U and V are also independent then $(X, Y) \sim GIG(-q, a, b) \otimes \gamma(q, a)$. Regression versions of this characterization were given in Seshadri and Wesołowski (2001) and Wesołowski (2002).

Both the direct Matsumoto–Yor (MY) property and its converse were proved for the matrix variate case in Letac and Wesołowski (2000) and Wesołowski (2002) for matrix variates X, Y, U and V having the same dimensions, and in Massam and Wesołowski (2003) for matrix variates having different dimensions. In the latter paper the MY property for matrix variates was obtained by identifying the joint distribution of (X, Y) with the conditional distribution of $(K_1^{-1}, K_2 - K_{21}K_1^{-1}K_{12})$ given $K_{12} = k_{12}$, where (K_1, K_{12}, K_2) is a block partitioning of a Wishart random matrix $\mathbf{K}$. Since the inverse of a generalized inverse Gaussian (GIG) random variable is also GIG there is no reason to work with $X = K_1^{-1}$ rather than $X = K_1$. So, if we identify the joint distribution of (X, Y) with the conditional distribution of $(K_1, K_2 - K_{21}K_1^{-1}K_{12})$ given $K_{12} = k_{12}$, the MY property can be expressed as follows. Let the two independent random variables X and Y follow the $GIG(q, a, b)$ and $\gamma(q, b)$ distributions respectively. Then the two variables U and V, defined by

$$U = X - \frac{1}{Y + 1/X} \quad \text{and} \quad V = Y + \frac{1}{X} \tag{1.2}$$

are also independent and follow the $\gamma(q, a)$ and $GIG(q, b, a)$ distributions, respectively. The form of (1.2) might not be as appealing as that of (1.1). However, the variables X, Y, U, V as given above are the 'right' variables and the natural object to work with is the Wishart random matrix $\mathbf{K}$, more precisely its conditional distribution given the off-diagonal elements.

Indeed, with this new identification of the distribution of (X, Y) the connection between K_1, K_2, X, Y, U and V can be represented by mappings defined graphically as follows. Let G be the simple tree with two vertices $\{1, 2\}$ and the edge (1, 2). Let $k_{12} \in \mathbb{R}$ be given. To each vertex i we assign a variable k_i, $i = 1, 2$, with

$$(k_1, k_2) \in \tilde{M}(G) = \{(k_1, k_2) : k_1 > 0,\ k_1k_2 > k_{12}^2\}.$$

We now choose vertex 1 as the root of the tree G and attach to the tree thus directed the mapping $\psi_1 : \tilde{M}(G) \mapsto (0, \infty)^2$ defined by

$$\psi_1(k_1, k_2) = \left(k_1,\ k_2 - \frac{k_{12}^2}{k_1}\right).$$

Similarly, when vertex 2 is the root of the tree we attach the mapping $\psi_2 : \tilde{M}(G) \mapsto (0, \infty)^2$ defined by

$$\psi_2(k_1, k_2) = \left(k_1 - \frac{k_{12}^2}{k_2},\ k_2\right),$$

so that the transformation of the pair (X, Y) into (U, V) is defined by means of the two relations

$$(x, y) = \psi_1(k_1, k_2) \quad \text{and} \quad (u, v) = \psi_2(k_1, k_2).$$

We will now formulate the MY property in a new way which allows for a natural multivariate generalization. It is not clear what such a generalization would be, had we started from the classical formulation (1.1) or even from (1.2).

We first define the $W_G^c(k_{12}, a, b, q)$ distribution on $\tilde{M}(G)$ as the distribution with density

$$f(k_1, k_2) \propto (k_1 k_2 - k_{12}^2)^{q-1} \mathrm{e}^{-ak_1 - bk_2},$$

where a, b and q are positive. This distribution can be viewed as the conditional distribution of (K_1, K_2) given $K_{12} = k_{12}$, when $\mathbf{K}$ is a Wishart matrix. (This is why we use the upper index c in the symbol W_G^c.) It has already been considered, for instance in Letac and Massam (2001) while proving the characterization of the quasi-Wishart distribution using the classical MY property. Having defined the W_G^c distribution, we can state the MY property as follows. Let (K_1, K_2) be a random vector following the $W_G^c(k_{12}, a, b, q)$ distribution. Then $(X, Y) = \psi_1(K_1, K_2) \sim GIG(q, a, b) \otimes \gamma(q, b)$ and $(U, V) = \psi_2(K_1, K_2) \sim \gamma(q, a) \otimes GIG(q, b, a)$.

Similarly, the characterization obtained in Letac and Wesołowski (2000) can be restated as follows. If both $(X, Y) = \psi_1(K_1, K_2)$ and $(U, V) = \psi_2(K_1, K_2)$ have positive non-degenerate independent components then $(K_1, K_2) \sim W_G^c(k_{12}, a, b, q)$ for some positive a, b and q.

In this paper the process by which we created the dual pairs (x, y) and (u, v), as described above, will be extended to any tree G with $p \geqslant 2$ vertices. We will build p vectors in $\mathbb{R}_+^p$ by transforming $(k_1, \ldots, k_p)$ through mappings ψ_r, $r = 1, \ldots, p$. These mappings are in one-to-one correspondence with the p directed trees created from G by moving the single root r through all possible vertices. We will also define a p-variate version of the W_G^c distribution. With these tools we will obtain a p-variate version of the MY property and its converse. In the next section we establish some preliminary results that we shall need to prove our main results. The main results are in Sections 3 and 4. In Section 3 we define the general W_G^c distribution, prove the p-variate MY property and illustrate it with two examples corresponding to the two basic trees; that is, the tree with three vertices in a line and the tree with four vertices and three leaves forming a 'daisy'. In Section 4 we give the converse of the MY property, that is, the characterization of the product of $p-1$ GIG distributions and one gamma distribution. We illustrate the converse with the same two examples.

2. Preliminaries

Let $G = (V, E)$ be a tree, where $V = \{1, \ldots, p\}$ is the set of vertices and the set of edges E is a set of unordered pairs (i, j) such that the distinct vertices i and j are linked in G. Let $L \subset V$ denote the set of leaves of G. For a given leaf $m \in L$ we write m_1 for its only neighbour. From an undirected tree G we can create a directed tree by choosing a single root. In this paper directed trees will have one root only, which we usually denote by r. For a vertex i in a directed tree G, we say that j is a child of i if there is a directed edge from i to j. Each vertex i has at most one child, which we denote $c(i)$. If i is a root, then $c(i)$ is empty. For a vertex i in a directed tree G, we say that j is a parent of i if there is a directed edge from j to i. The vertex i may have several parents. The set of parents of i is denoted $p(i)$. If i is a leaf then $p(i) = \emptyset$.

Let $\mathcal{V}_p^+$ be the cone of $p \times p$ positive definite symmetric matrices. We define

$$M(G, K_G) = \{k = (k_1, \dots, k_p) \in \mathbb{R}^p : \mathbf{k} = [k_{ij}] \in \mathcal{V}_p^+,\ k_{ii} = k_i,\ k_{ij} \in K_G,\ i \neq j\}, \quad (2.1)$$

where $K_G = \{k_{ij} \neq 0, (i, j) \in E, k_{ij} = 0, (i, j) \notin E\}$ is a given set of off-diagonal entries for the matrix $\mathbf{k} = [k_{ij}]$.

For a given leaf $m \in L$, let G^{-m} be the graph induced from G by the subset $V \backslash \{m\}$, i.e. $G^{-m} = (V^{-m}, E^{-m})$, where $V^{-m} = V \backslash \{m\}$ and $E^{-m} = E \backslash \{(m_1, m)\}$. (Henceforth, as in the previous sentence, the superindex $-m$ is, of course, never to be read as a power.) Finally, let $K_{G^{-m}}$ be the set of off-diagonal elements obtained from K_G by discarding $k_{m_1 m} \neq 0$ and $k_{im} = 0$, $i \in V \backslash \{m_1, m\}$.

Lemma 2.1. *For $k = (k_1, \dots, k_p) \in M(G, K_G)$ and $m \in L$, define the $(p-1)$-dimensional vector k^{-m} with components*

$$k_i^{-m} = k_i, \quad i \in V^{-m} \backslash \{m_1\}, \qquad k_{m_1}^{-m} = k_{m_1} - \frac{k_{m_1 m}^2}{k_m}. \quad (2.2)$$

This vector k^{-m} is in $M(G^{-m}, K_{G^{-m}})$. Furthermore, for any $k^{-m} \in M(G^{-m}, K_{G^{-m}})$ and any $k_m > 0$, the vector

$$\left(k_1^{-m}, \dots, k_{m_1 - 1}^{-m}, k_{m_1}^{-m} + \frac{k_{m_1 m}^2}{k_m}, k_m \right) \in M(G, K_G).$$

Proof. Without loss of generality we can assume that $p = m$ and $p - 1 = m_1$. We observe that the determinant of the matrix $\mathbf{k}$ is such that

$$|\mathbf{k}| = \begin{vmatrix} k_1 & * & \dots & * & * & 0 \\ * & k_2 & \dots & * & * & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ * & * & \dots & k_{m_1-1} & * & 0 \\ * & * & \dots & * & k_{m_1} & k_{m_1,m} \\ 0 & 0 & \dots & 0 & k_{m,m_1} & k_m \end{vmatrix} = k_m \begin{vmatrix} k_1 & * & \dots & * & * \\ * & k_2 & \dots & * & * \\ \dots & \dots & \dots & \dots & \dots \\ * & * & \dots & k_{m_1-1} & * \\ * & * & \dots & * & k_{m_1} - \dfrac{k_{m_1,m}^2}{k_m} \end{vmatrix}, \quad (2.3)$$

where the second matrix in the equation above, denoted $\mathbf{k}^{-m}$, is of dimension $(p-1) \times (p-1)$. Since $|\mathbf{k}| > 0$ and $k_m > 0$, the vector

$$\left(k_1, \dots, k_{m_1-1}, k_{m_1} - \frac{k_{m_1,m}^2}{k_m} \right) \in M(G^{-m}, K_{G^{-m}}). \quad (2.4)$$

Conversely, given the vector $k^{-m} = (k_1^{-m}, \dots, k_{m_1}^{-m}) \in M(G^{-m}, K_{G^{-m}})$ and $k_m > 0$, let us show that

$$(k_1, \ldots, k_m) = \left(k_1^{-m}, \ldots, k_{m_1}^{-m} + \frac{k_{m_1 m}^2}{k_m}, k_m \right) \in M(G, K_G).$$

To do so it is sufficient to show that all principal minors of the matrix $\mathbf{k}$ are positive. Clearly this is so for the first $m_1 - 1$. The m_1th is the determinant of the matrix which has all its entries equal to the respective entries of $\mathbf{k}^{-m}$ except for the (m_1, m_1)th entry which is $(\mathbf{k}^{-m})_{m_1 m_1} + k_{m_1 m}^2 / k_m$. Since $\mathbf{k}^{-m}$ is positive definite and $k_{m_1 m}^2 / k_m > 0$, the m_1th principal minor is positive. From (2.3) it is clear that the mth principal minor is also positive. □

For any $r \in V$, we direct the tree G by choosing r as the single root. For the tree thus directed we define the mapping $\psi_r : M(G, K_G) \mapsto \mathbb{R}_+^p$ by

$$\psi_r(k_1, \ldots, k_p) = (k_{1,(r)}, \ldots, k_{p,(r)}), \tag{2.5}$$

where, starting with the leaves and moving towards the root along the directed paths,

$$k_{i,(r)} = \begin{cases} k_i & \text{if } i \text{ is a leaf,} \\ k_i - \sum_{j \in p(i)} \dfrac{k_{ij}^2}{k_{j,(r)}} & \text{otherwise.} \end{cases} \tag{2.6}$$

We write $\psi(G) = \{\psi_r, r \in V\}$.

Lemma 2.2. *For any $j \in V^{-m}$,*

$$k_{j,(r)}^{-m} = k_{j,(r)} \text{ and } k_{j,(m_1)}^{-m} = k_{j,(m)}, \qquad r \in L \backslash \{m\}. \tag{2.7}$$

Proof. Observe that for any j on the path linking m and $r \in L \backslash \{m\}$, $j \neq m$, the quantity $k_{j,(r)}$ depends on k_m only through $k_{m_1} - k_{m_1 m}^2 / k_m$, and that for j not on this path $k_{j,(r)}$ does not depend on k_m. This proves the first equality. The second equality follows immediately from the fact that, for $j \neq m$, we have $k_{j,(m)} = k_{j,(m_1)}$. □

Lemma 2.3. *For any $r \in V$ the mapping $\psi_r : M(G, K_G) \mapsto \mathbb{R}_+^p$ defined by* (2.5) *and* (2.6) *is a bijection and its Jacobian is equal to one.*

Proof. From (2.6) it is clear that ψ_r is into. To prove that it is onto, we proceed by induction on the size p of the graph. We use our induction assumption on the graph G^{-m}, where m is a leaf. Without loss of generality we can assume that $p = m$ and $p - 1 = m_1$. Then the only thing to show is that if the $m_1 \times m_1$ submatrix of $\mathbf{k}$ as in (2.3), is positive definite, then $\mathbf{k}$ is also positive definite. This follows from an argument parallel to that in the proof of Lemma 2.1. Given the triangular form of (2.6), the Jacobian is clearly equal to one. □

Lemma 2.4. *Given any root $r \in V$, we have*

$$|\mathbf{k}| = \prod_{i \in V} k_{i,(r)}. \tag{2.8}$$

Proof. We will proceed by induction on the size p of the tree G. The statement is obvious for $p = 2$ (this reduces to $k_1(k_2 - k_{21}^2/k_1) = (k_1 - k_{12}^2/k_2)k_2$). Let us now assume that (2.8) is true for any tree G_{p-1} of size of $p-1$, for any set $K_{G_{p-1}}$ of off-diagonal elements and for any $k \in M(G_{p-1}, K_{G_{p-1}})$. Choose an arbitrary root $r \in V$ and a leaf $m \neq r$, which is always possible since a tree has at least two leaves. As usual, we write $m_1 = c(m)$. Without loss of generality, we can assume that $p = m$ and $p - 1 = m_1$.

Let ψ_r^{-m} be an element of $\psi(G^{-m})$. Note that for $i \in V\backslash\{m\}$, the ith component $(\psi_r^{-m})_i$ of

$$\psi_r^{-m}\left(k_1, \ldots, k_{m_1-1}, k_{m_1} - \frac{k_{m_1,m}^2}{k_m}\right)$$

is equal to the ith component $(\psi_r)_i$ of $\psi_r(k_1, \ldots, k_m)$. Since by the induction assumption

$$|\mathbf{k}^{-m}| = \prod_{i \in V\backslash\{m\}} (\psi_r^{-m})_i$$

and since $k_m = (\psi_r)_m$, it follows from (2.3) that

$$|\mathbf{k}| = k_m|\mathbf{k}^{-m}| = (\psi_r)_m \prod_{i \in V\backslash\{m\}} (\psi_r^{-m})_i = \prod_{i \in V}(\psi_r)_i,$$

which shows (2.8). □

Lemma 2.5. *For any root $r \in V$ and for any $a = (a_1, \ldots, a_p) \in \mathbb{R}^p$, we have*

$$(a, k) = \sum_{i \in V} a_i k_i = a_r k_{r,(r)} + \sum_{i \in V, i \neq r}\left(a_i k_{i,(r)} + \frac{k_{ic(i)}^2 a_{c(i)}}{k_{i,(r)}}\right) = \sum_{i \in V}\left(a_i k_{i,(r)} + \frac{k_{ic(i)}^2 a_{c(i)}}{k_{i,(r)}}\right). \tag{2.9}$$

Proof. The first equality holds by definition of the inner product in $\mathbb{R}^p$. From the definition of $k_{i,(r)}$ it is clear that

$$(a, k) = \sum_{i \in V} a_i k_i = \sum_{i \in V} a_i\left(k_{i,(r)} + \sum_{j \in p(i)} \frac{k_{ij}^2}{k_{j,(r)}}\right).$$

Since each $i \in V\backslash\{r\}$ has only one child, by changing the order of summation in the equation above we see that the second equality in (2.9) holds. The third equality follows immediately if we recall that the root does not have a child. □

3. The multivariate Matsumoto–Yor property

Let $G = (V, E)$ be a tree with p vertices. For K_G as defined in the previous section, for given positive q and $a = (a_1, a_p) \in I_+^p$, we define the p-variate $W_G^c(q, K_G, a)$ distribution as the distribution with density

$$f(k) \propto |\mathbf{k}|^{q-1} e^{-(a,k)}, \qquad k \in M(G, K_G), \tag{3.1}$$

and zero otherwise. Since $M(G, K_G)$ is a subset of $\mathbb{R}_+^p$ it is clear that f is a density. As in the two-dimensional case the upper index c in the symbol W_G^c refers to the fact that the distribution with density (3.1) is the conditional distribution of the diagonal elements of the G-Wishart random matrix (see Atay-Kayis and Massam 2004) given its off-diagonal elements.

With this distribution and the variables $X_r = \psi_r(K)$, $r \in V$, we now give a multivariate version of the MY property.

Theorem 3.1. *Let $G = (V, E)$ be a tree of size p, where p is any integer greater than or equal to 2. Let $K = (K_1, \ldots, K_p)$ be a random vector following the $W_G^c(q, K_G, a)$ distribution with $a = (a_1, \ldots, a_p) \in \mathbb{R}_+^p$ and positive q. Define $X_r = \psi_r(K)$, $r \in V$. Then for each $r \in V$ the components of $X_r = (X_{1,(r)}, \ldots, X_{p,(r)})$ are independent. Moreover,*

$$X_{r,(r)} \sim \gamma(q, a_r), \qquad \text{and } X_{i,(r)} \sim GIG(q, a_i, k_{ic(i)}^2 a_{c(i)}), \quad i \in V \backslash \{r\}. \tag{3.2}$$

Proof. Using (2.8) and (2.9), we can split the density (3.1) of K in p different ways corresponding to the p different possible choices of $r \in V$:

$$f(k) \propto k_{r,(r)}^{q-1} e^{-a_r k_{r,(r)}} \prod_{i \in V, i \neq r} k_{i,(r)}^{q-1} \exp\left\{-\left(a_i k_{i,(r)} + \frac{k_{ic(i)}^2 a_{c(i)}}{k_{i,(r)}}\right)\right\}.$$

This is clearly the product of the gamma and the $p-1$ GIG densities as stated in the theorem. Since by Lemma 2.3 the mappings ψ_r are bijections with Jacobian equal to one, the result follows. □

We illustrate this theorem with two examples.

Example 3.1. Let G be the tree with $V = \{1, 2, 3\}$, $E = \{(1, 2), (2, 3)\}$ and $K_G = \{k_{12} = 1, k_{23} = 1\}$. The mappings ψ_r, $r \in V$, are therefore

$$\psi_1(k_1, k_2, k_3) = \left(k_1 - \frac{1}{k_2 - 1/k_3}, k_2 - \frac{1}{k_3}, k_3\right),$$

$$\psi_2(k_1, k_2, k_3) = \left(k_1, k_2 - \frac{1}{k_1} - \frac{1}{k_3}, k_3\right),$$

$$\psi_3(k_1, k_2, k_3) = \left(k_1, k_2 - \frac{1}{k_1}, k_3 - \frac{1}{k_2 - 1/k_1}\right).$$

The decompositions in (2.8) and (2.9) are in this case

$$|\mathbf{k}| = k_1k_2k_3 - k_1 - k_3 = \left(k_1 - \frac{1}{k_2 - 1/k_3}\right)\left(k_2 - \frac{1}{k_3}\right)k_3$$

$$= k_1\left(k_2 - \frac{1}{k_1} - \frac{1}{k_3}\right)k_3 = k_1\left(k_2 - \frac{1}{k_1}\right)\left(k_3 - \frac{1}{k_2 - 1/k_1}\right)$$

and

$$a_1k_1 + a_2k_2 + a_3k_3 = a_1\left(k_1 - \frac{1}{k_2 - 1/k_3}\right) + a_2\left(k_2 - \frac{1}{k_3}\right) + a_1\frac{1}{k_2 - 1/k_3} + a_3k_3 + a_2\frac{1}{k_3}$$

$$= a_1k_1 + a_2\frac{1}{k_1} + a_2\left(k_2 - \frac{1}{k_1} - \frac{1}{k_3}\right) + a_3k_3 + a_2\frac{1}{k_3}$$

$$= a_1k_1 + a_2\frac{1}{k_1} + a_2\left(k_2 - \frac{1}{k_1}\right) + a_3\frac{1}{k_2 - 1/k_1} + a_3\left(k_3 - \frac{1}{k_2 - 1/k_1}\right).$$

If $K = (K_1, K_2, K_3)$ follows the $W_G^c(q, K_G, a)$ distribution with $a = (a_1, a_2, a_3)$ then

$$(X_1, X_2, X_3) = \psi_3(K) \sim GIG(q, a_1, a_2) \otimes GIG(q, a_2, a_3) \otimes \gamma(q, a_3),$$

$$(Y_1, Y_2, Y_3) = \psi_1(K) \sim \gamma(q, a_1) \otimes GIG(q, a_2, a_1) \otimes GIG(q, a_3, a_2),$$

$$(Z_1, Z_2, Z_3) = \psi_2(K) \sim GIG(q, a_1, a_2) \otimes \gamma(q, a_2) \otimes GIG(q, a_3, a_2).$$

We wish to emphasize here the analogy between the classical bivariate MY property given in (1.2) and our present three-dimensional result. We rewrite (1.2) with $X = X_1$, $Y = X_2$, $U = Y_2$ and $V = Y_1$. If $(X_1, X_2) \sim GIG(q, a_1, a_2) \otimes \gamma(q, a_2)$ then

$$(Y_1, Y_2) = \left(X_1 - \frac{1}{X_2 + 1/X_1}, X_2 + \frac{1}{X_1}\right) \sim \gamma(q, a_1) \otimes GIG(q, a_2, a1).$$

The three-dimensional MY property given above can be rephrased as follows. If $(X_1, X_2, X_3) \sim GIG(q, a_1, a_2) \otimes GIG(q, a_2, a_3) \otimes \gamma(q, a_3)$ then

$$(Y_1, Y_2, Y_3) = \left(X_1 - \frac{1}{X_2 + 1/X_1 - 1/(X_3 + 1/X_2)}, X_2 + \frac{1}{X_1} - \frac{1}{X_3 + 1/X_2}, X_3 + \frac{1}{X_2}\right)$$

$$\sim \gamma(q, a_1) \otimes GIG(q, a_2, a_1) \otimes GIG(q, a_3, a_2)$$

and

$$(Z_1,\ Z_2,\ Z_3) = \left(X_1,\ X_2 - \frac{1}{X_3 + 1/X_2},\ X_3 + \frac{1}{X_2}\right)$$

$$\sim GIG(q,\ a_1,\ a_2) \otimes \gamma(q,\ a_2) \otimes GIG(q,\ a_3,\ a_2).$$

Example 3.2. Let G be the tree with $V = \{1, 2, 3, 4\}$, $E = \{(1, 2), (1, 3), (1, 4)\}$ and $K_G = \{k_{12} = 1,\ k_{13} = 1,\ k_{14} = 1\}$. The mappings ψ_r, $r \in V$, are therefore

$$\psi_1(k_1,\ k_2,\ k_3,\ k_4) = \left(k_1 - \frac{1}{k_2} - \frac{1}{k_3} - \frac{1}{k_4},\ k_2,\ k_3,\ k_4\right),$$

$$\psi_2(k_1,\ k_2,\ k_3,\ k_4) = \left(k_1 - \frac{1}{k_3} - \frac{1}{k_4},\ k_2 - \frac{1}{k_1 - 1/k_3 - 1/k_4},\ k_3,\ k_4\right),$$

$$\psi_3(k_1,\ k_2,\ k_3,\ k_4) = \left(k_1 - \frac{1}{k_2} - \frac{1}{k_4},\ k_2,\ k_3 - \frac{1}{k_1 - 1/k_2 - 1/k_4},\ k_4\right),$$

$$\psi_4(k_1,\ k_2,\ k_3,\ k_4) = \left(k_1 - \frac{1}{k_2} - \frac{1}{k_3},\ k_2,\ k_3,\ k_4 - \frac{1}{k_1 - 1/k_2 - 1/k_3}\right).$$

The decompositions in (2.8) and (2.9) are in this case

$$|\mathbf{k}| = k_1k_2k_3k_4 - k_2k_3 - k_3k_4 - k_2k_4 = \left(k_1 - \frac{1}{k_2} - \frac{1}{k_3} - \frac{1}{k_4}\right)k_2k_3k_4$$

$$= \left(k_1 - \frac{1}{k_3} - \frac{1}{k_4}\right)\left(k_2 - \frac{1}{k_1 - 1/k_3 - 1/k_4}\right)k_3k_4$$

$$= \left(k_1 - \frac{1}{k_2} - \frac{1}{k_4}\right)k_2\left(k_3 - \frac{1}{k_1 - 1/k_2 - 1/k_4}\right)k_4$$

$$= \left(k_1 - \frac{1}{k_2} - \frac{1}{k_3}\right)k_2k_3\left(k_4 - \frac{1}{k_1 - 1/k_2 - 1/k_3}\right)$$

and

$$
\begin{aligned}
&a_1k_1+a_2k_2+a_3k_3+a_4k_4\\
&=a_1\left(k_1-\frac{1}{k_2}-\frac{1}{k_3}-\frac{1}{k_4}\right)+a_2k_2+a_1\frac{1}{k_2}+a_3k_3+a_1\frac{1}{k_3}+a_4k_4+a_1\frac{1}{k_4}\\
&=a_1\left(k_1-\frac{1}{k_3}-\frac{1}{k_4}\right)+a_2\frac{1}{k_1-1/k_3-1/k_4}+a_2\left(k_2-\frac{1}{k_1-1/k_3-1/k_4}\right)\\
&\quad+a_3k_3+a_1\frac{1}{k_3}+a_4k_4+a_1\frac{1}{k_4}\\
&=a_1\left(k_1-\frac{1}{k_2}-\frac{1}{k_4}\right)+a_3\frac{1}{k_1-1/k_2-1/k_4}+a_2k_2+a_1\frac{1}{k_2}+a_3\left(k_3-\frac{1}{k_1-1/k_2-1/k_4}\right)\\
&\quad+a_4k_4+a_1\frac{1}{k_4}\\
&=a_1\left(k_1-\frac{1}{k_2}-\frac{1}{k_3}\right)+a_4\frac{1}{k_1-1/k_2-1/k_3}+a_2k_2+a_1\frac{1}{k_2}+a_3k_3+a_1\frac{1}{k_3}\\
&\quad+a_4\left(k_4-\frac{1}{k_1-1/k_2-1/k_3}\right).
\end{aligned}
$$

If $K=(K_1, K_2, K_3, K_4)$ follows the $W_G^c(q, K_G, a)$ distribution with $a=(a_1, a_2, a_3, a_4)$ then

$$
\begin{aligned}
(X_1, X_2, X_3, X_4)=\psi_3(K)&\sim GIG(q, a_1, a_3)\otimes GIG(q, a_2, a_1)\otimes\gamma(q, a_3)\otimes GIG(q, a_4, a_1),\\
(Y_1, Y_2, Y_3, Y_4)=\psi_1(K)&\sim \gamma(q, a_1)\otimes GIG(q, a_2, a_1)\otimes GIG(q, a_3, a_1)\otimes GIG(q, a_4, a_1),\\
(Z_1, Z_2, Z_3, Z_4)=\psi_2(K)&\sim GIG(q, a_1, a_2)\otimes\gamma(q, a_2)\otimes GIG(q, a_3, a_1)\otimes GIG(q, a_4, a_1),\\
(T_1, T_2, T_3, T_4)=\psi_4(K)&\sim GIG(q, a_1, a_4)\otimes GIG(q, a_2, a_1)\otimes GIG(a_3, a_1)\otimes\gamma(q, a_4).
\end{aligned}
$$

4. Characterization

In this section we will show that if the components of X_r, $r\in L$, are independent, then the distribution of K is W_G^c, which implies that, for all $r\in V$, $\psi_r(K)$ follows a distribution which is a product of gamma and GIGs as given in (3.2).

Theorem 4.1. *Let $G=(V, E)$ be a tree of size p. Let $L\subset V$ be its set of leaves. Let the set K_G be given and let K be a random vector taking its values in $M(G, K_G)$. Let $X_r=\psi_r(K)$, $r\in V$. If, for any root $r\in L$, the components of $X_r=(X_{1,(r)}, \dots, X_{p,(r)})$ are mutually independent then there exist $q>0$ and $a=(a_1, \dots, a_p)$ with positive components such that $K\sim W_G^c(q, K_G, a)$, which implies that* (3.2) *holds.*

Proof. Our proof is in three steps. In step 1 we will prove that there exist $q > 0$ such that, for any root $r \in L$, there exists $a_r > 0$ such that $X_{r,(r)} \sim \gamma(q, a_r)$. In step 2 using two arbitrarily chosen leaves $m, n \in L$, we will identify the densities of the random variables $X_{j,(l)}$ for $l = m, n$ and $j \in V$ as gammas and GIGs with parameters q^{-l}, a_j^{-l}, $j \in V$, $l = m, n$. In step 3 we will show that the two sets of parameters, for $l = m$ and $l = n$, are identical.

Step 1. In this step our method of proof is an extension of the method used in Letac and Wesołowski (2000, Theorem 4.1).

For any positive integer α and any root $r \in L$, let us define A_r^α as follows

$$A_r^\alpha = \mathrm{E}\left[\left(\prod_{i\in V} X_{i,(r)}\right)^\alpha \exp\left\{\sum_{i\in V}(s_i X_{i,(r)} + k_{ic(i)}^2 s_{c(i)} X_{i,(r)}^{-1})\right\}\right] = \prod_{i\in V} A_{i,(r)}^\alpha, \tag{4.1}$$

where

$$A_{i,(r)}^\alpha = \mathrm{E}[(X_{i,(r)})^\alpha \mathrm{e}^{s_i X_{i,(r)} + k_{ic(i)}^2 s_{c(i)} X_{i,(r)}^{-1}}].$$

By (2.8) and (2.9) it is clear that, for all $r \in L$,

$$A_r^\alpha = \mathrm{E}\left[|\mathbf{K}|^\alpha \exp\left(\sum_{i\in V} s_i K_i\right)\right], \tag{4.2}$$

where $\mathbf{K}$ is the matrix with random diagonal elements $K_{ii} = K_i$, $i \in V$, and constant off-diagonal elements $K_{ij} = k_{ij} \in K_G$, $i \neq j$.

Let us consider two roots $m, n \in L$. There is a unique path $\mathcal{P} \subset V$ in G linking m and n. Consider $i, j \in \mathcal{P}$, which are adjacent. Let us now differentiate the equality

$$\begin{aligned}\log(A_m^\alpha) &= \sum_{l\in V} \log\left[\mathrm{E}\left(X_{l,(m)}^\alpha \mathrm{e}^{s_l X_{l,(m)} + k_{lc(l)}^2 s_{c(l)} X_{l,(m)}^{-1}}\right)\right] \\ = \log(A_n^\alpha) &= \sum_{l\in V} \log\left[\mathrm{E}\left(X_{l,(n)}^\alpha \mathrm{e}^{s_l X_{l,(n)} + k_{lc(l)}^2 s_{c(l)} X_{l,(n)}^{-1}}\right)\right],\end{aligned}$$

which is a consequence of (4.2), with respect to s_i and s_j. Since the pair (s_i, s_j) appears only in one summand in each one of the expressions above, the differentiation leads to

$$1 - \frac{A_{i,(m)}^{\alpha-1} A_{i,(m)}^{\alpha+1}}{(A_{i,(m)}^\alpha)^2} = 1 - \frac{A_{j,(n)}^{\alpha-1} A_{j,(n)}^{\alpha+1}}{(A_{j,(n)}^\alpha)^2},$$

where we have assumed, without loss of generality, that $j = c(i)$ in the tree with root m and thus $i = c(j)$ in the tree with root n. For $\alpha = 1$ this yields

$$\frac{A_{i,(m)}^0 A_{i,(m)}^2}{(A_{i,(m)}^1)^2} = \frac{A_{j,(n)}^0 A_{j,(n)}^2}{(A_{j,(n)}^1)^2}. \tag{4.3}$$

Using $A^\alpha_m = A^\alpha_n$ for $\alpha = 0, 1, 2$, we immediately obtain

$$\prod_{l\in V} \frac{A^0_{l,(m)} A^2_{l,(m)}}{(A^1_{l,(m)})^2} = \prod_{l\in V} \frac{A^0_{l,(n)} A^2_{l,(n)}}{(A^1_{l,(n)})^2}. \tag{4.4}$$

We have

$$A^\alpha_{l,(m)} = A^\alpha_{l,(n)} \qquad \text{for } l \notin \mathcal{P}. \tag{4.5}$$

For any $i \in \mathcal{P}$ different from m on the directed path from n to m, there exists $j = c(i) \neq n$ such that (4.3) holds. Similarly, for any $j \in \mathcal{P}$ different from n on the directed path from m to n, there exists $i = c(j) \neq n$ such that (4.3) holds. Using the identities (4.5) and the equalities (4.3), we can simplify (4.4) to obtain

$$\frac{A^0_{m,(m)} A^2_{m,(m)}}{(A^1_{m,(m)})^2} = \frac{A^0_{n,(n)} A^2_{n,(n)}}{(A^1_{n,(n)})^2}.$$

By the principle of separation of variables the two sides of the equation above are constant. Through a standard argument (see Letac and Wesołowski 2000), it follows that there exist q, a_m and a_n positive such that $X_{m,(m)} \sim \gamma(q, a_m)$ and $X_{n,(n)} \sim \gamma(q, a_n)$. Since m and n were chosen arbitrarily in L it follows that there exists $q > 0$ such that, for all $r \in L$, there exist $a_r > 0$ such that $X_{r,(r)} \sim \gamma(q, a_r)$.

Step 2. Since by (2.9), for any $s = (s_1, \ldots, s_p) \in \mathbb{R}^p$,

$$(s, K) = s_r X_{r,(r)} + \sum_{j\in V, j\neq r} s_j X_{j,(r)}$$

with $X_{j,(r)}$, $j \in V$, $j \neq r$, and $X_{r,(r)}$ independent, and since, as we have just proved, $X_{r,(r)}$ has a density, it follows that K and thus all $X_{j,(r)}$, $j \in V$, $r \in L$, have densities.

We will now prove that the distributions of $X_{j,(r)}$, $j \in V\setminus\{r\}$, $r \in L$, are GIGs. This will be done by induction on the size p of the graph G. Since we now know that the densities of $X_{j,(r)}$ exist, we can rewrite the independence assumption as

$$\prod_{i\in V} f_{i,(r)}(k_{i,(r)}) = h(k) \tag{4.6}$$

almost surely with respect to the Lebesgue measure for $k = (k_1, \ldots, k_p) \in M(G, K_G)$, where $k_{i,(r)}$ is the ith component of the vector $\psi_r(k)$, $i \in V$, and where $f_{i,(r)}$ is the density of $X_{i,(r)}$, $i \in V$, $r \in L$, and h is a function independent of r.

Let us fix $m \in L$. Let $m_1 \in V$ be the only vertex adjacent to m in the tree G. Then (4.6) can be written as

$$\prod_{l\in V} f_{l,(r)}(k_{l,(r)}) = f_{m,(m)}\left(k_m - \frac{k_{mm_1}^2}{k_{m_1} - \sum_{j\in p(m_1)}(k_{m_1 j}^2/k_{j,(m)})}\right) f_{m_1,(m)}\left(k_{m_1} - \sum_{j\in p(m_1)}\frac{k_{m_1 j}^2}{k_{j,(m)}}\right) \times \prod_{i\in V\setminus\{m,m_1\}} f_{i,(m)}(k_{i,(m)}) \tag{4.7}$$

for any $r \in L\setminus\{m\}$.

We now fix k_m and consider the $(p-1)$-dimensional vector k^{-m} as defined in (2.2). According to Lemma 2.1, this vector belongs to $M(G^{-m}, K_{G^{-m}})$. Consequently, since $k_m = k_{m,(r)}$ for any $r \in L\setminus\{m\}$, by Lemma 2.2 equation (4.7) can be rewritten as

$$f_{m,(r)}(k_m)\prod_{l\in V^{-m}} f_{l,(r)}(k_{l,(r)}^{-m}) \tag{4.8}$$

$$= f_{m,(m)}\left(k_m - \frac{k_{mm_1}^2}{k_{m_1}^{-m} + k_{m_1 m}^2/k_m - \sum_{j\in p(m_1)}(k_{m_1 j}^2/k_{j,(m_1)}^{-m})}\right) f_{m_1,(m)}\left(k_{m_1}^{-m} + \frac{k_{m_1 m}^2}{k_m} - \sum_{j\in p(m_1)}\frac{k_{m_1 j}^2}{k_{j,(m_1)}^{-m}}\right) \times \prod_{i\in V\setminus\{m,m_1\}} f_{i,(m)}(k_{i,(m_1)}^{-m}).$$

Since, when m is not the root, the set $p(m_1)\setminus\{m\}$ in G is identical to the set $p(m_1)$ in G^{-m}, it follows that

$$k_{m_1,(m_1)}^{-m} = k_{m_1}^{-m} - \sum_{j\in p(m_1)}\frac{k_{m_1 j}^2}{k_{j,(m_1)}^{-m}}.$$

For $r = n \in L\setminus\{m\}$, (4.8) thus becomes

$$\prod_{i\in V^{-m}} f_{i,(n)}(k_{i,(n)}^{-m}) = g(k_{m_1,(m_1)}^{-m})\prod_{i\in V^{-m}\setminus\{m_1\}} f_{i,(m)}(k_{i,(m_1)}^{-m}), \tag{4.9}$$

where

$$g(x) = f_{m,(m)}\left(k_m - \frac{k_{m,m_1}^2}{x + k_{m_1,m}^2/k_m}\right) f_{m_1,(m)}\left(x + \frac{k_{m_1,m}^2}{k_m}\right) \Big/ f_{m,(n)}(k_m).$$

We can always choose k_m in such a way that the above equation holds almost surely with respect to the Lebesgue measure for $k^{-m} \in M(G^{-m}, K_G^{-m})$. Since all the functions on the left-hand side of (4.9), apart from g, are densities of $X_{j,(n)} = X_{j,(n)}^{-m}$ it follows that g is also a density, in fact it is the density of $X_{m_1,(m_1)}^{-m}$. Since $L^{-m} \subset (L\setminus\{m\}) \cup \{m_1\}$, it follows from (4.9) and our induction assumption that there exist $q^{-m} > 0$ and $a_i^{-m} > 0$, $i \in V^{-m}$, such that

$$X_{n,(n)}^{-m} = X_{n,(n)} \sim \gamma(q^{-m}, a_n^{-m}),$$

$$X_{j,(n)}^{-m} = X_{j,(n)} \sim GIG(q^{-m}, a_j^{-m}, k_{jc(j)}^2 a_{c(j)}^{-m}), \qquad j \in V^{-m}\setminus\{n\}, \tag{4.10}$$

$$X_{j,(m_1)}^{-m} = X_{j,(m)} \sim GIG(q^{-m}, a_j^{-m}, k_{jc(j)}^2 a_{c(j)}^{-m}), \qquad j \in V^{-m}\setminus\{m_1\}. \tag{4.11}$$

Note that $c(j)$ in (4.11) denotes the child of j in the tree with root m, while in (4.10) $c(j)$ denotes the child of j in the tree with root n.

Swapping the roles of m and n, we have that there exist $q^{-n} > 0$ and $a_j^{-n} > 0$, $j \in V^{-n}$, such that

$$X_{m,(m)}^{-n} = X_{m,(m)} \sim \gamma(q^{-n}, a_m^{-n}),$$

$$X_{j,(m)}^{-n} = X_{j,(m)} \sim GIG(q^{-n}, a_j^{-n}, k_{jc(j)}^2 a_{c(j)}^{-n}), \qquad j \in V^{-n}\setminus\{m\}, \tag{4.12}$$

$$X_{j,(n_1)}^{-n} = X_{j,(n)} \sim GIG(q^{-n}, a_j^{-n}, k_{jc(j)}^2 a_{c(j)}^{-n}), \qquad j \in V^{-n}\setminus\{n_1\}, \tag{4.13}$$

where, as before, $c(j)$ in (4.13) refers to the tree with root n and in (4.12) to the tree with root m. Thus we know the distributions of $X_{j,(l)}$ for any $j \in V$, $l = m, n$.

Step 3. We know from step 1 that $X_{m,(m)}$ and $X_{n,(n)}$ are respectively $\gamma(q, a_m)$ and $\gamma(q, a_n)$, from which it follows immediately that $q^{-m} = q^{-n} = q$, $a_n^{-m} = a_n$ and $a_m^{-n} = a_m$. Then the independence equation (4.6) for $r = m, n$, can be written as

$$\left(\prod_{j\in V} k_{j,(m)}\right)^{q-1} \exp\left[\sum_{j\in V\setminus\{n\}} \left(a_j^{-n} k_{j,(m)} + a_{c(j)}^{-n} \frac{k_{jc(j)}^2}{k_{j,(m)}}\right) + a_n^{-m} k_{n,(m)} + a_{c(n)}^{-m} \frac{k_{nc(n)}^2}{k_{n,(m)}}\right]$$
$$= \left(\prod_{j\in V} k_{j,(n)}\right)^{q-1} \exp\left[\sum_{j\in V\setminus\{m\}} \left(a_j^{-m} k_{j,(n)} + a_{c(j)}^{-m} \frac{k_{jc(j)}^2}{k_{j,(n)}}\right) + a_m^{-n} k_{m,(n)} + a_{c(m)}^{-n} \frac{k_{mc(m)}^2}{k_{m,(n)}}\right],$$

where on the left-hand side $c(j)$ denotes the child of j in the tree with root m and on the right it denotes the child of j in the tree with root n. Define

$$a_n^{-n} = a_n^{-m} \quad \text{and} \quad a_m^{-m} = a_m^{-n}.$$

Then, by (2.8) and (2.9), the equality above yields

$$\sum_{j\in V} a_j^{-m} k_j + (a_{c(m)}^{-n} - a_{c(m)}^{-m}) \frac{k_{mc(m)}^2}{k_m} = \sum_{j\in V} a_j^{-n} k_j + (a_{c(n)}^{-m} - a_{c(n)}^{-n}) \frac{k_{nc(n)}^2}{k_n}. \tag{4.14}$$

Let $k_m \to 0$. If $a_{c(m)}^{-n}$ and $a_{c(m)}^{-m}$ are different then the left-hand side of (4.14) tends to ∞ while the right-hand side remains finite, which is impossible and therefore they are equal. Similarly, $a_{c(n)}^{-m} = a_{c(n)}^{-n}$. Therefore (4.14) becomes

$$\sum_{j\in V} a_j^{-n} k_j = \sum_{j\in V} a_j^{-m} k_j.$$

It is now obvious that $a_j^{-m} = a_j^{-n} = a_j$, $j \in V$. This completes the proof. $\square$

Example 4.1. Let G and K_G be as in Example 3.1, so that $L = \{1, 3\}$. Let $K = (K_1, K_2, K_3)$ be a random vector taking its values in $M(G, K_G)$. Let

$$X = (X_1, X_2, X_3) = \psi_3(K) = \left(K_1, K_2 - \frac{1}{K_1}, K_3 - \frac{1}{K_2 - 1/K_1}\right),$$

$$Y = (Y_1, Y_2, Y_3) = \psi_1(K) = \left(K_1 - \frac{1}{K_2 - 1/K_3}, K_2 - \frac{1}{K_3}, K_3\right).$$

If the components of X are independent and the components of Y are independent, then there exist q, a_1, a_2, a_3 positive such that $K \sim W_G^c(q, K_G, a_1, a_2, a_3)$. Consequently, the distributions of X, Y and $Z = \psi_2(K)$ are products of one gamma and two GIGs as given in Example 3.1.

Example 4.2. Let G and K_G be as in Example 3.2. So $L = \{2, 3, 4\}$. Let $K = (K_1, K_2, K_3, K_4)$ be a random vector taking its values in $M(G, K_G)$. Let

$$X = (X_1, X_2, X_3, X_4) = \psi_3(K) = \left(K_1 - \frac{1}{K_2} - \frac{1}{K_4}, K_2, K_3 - \frac{1}{K_1 - 1/K_2 - 1/K_4}, K_4\right),$$

$$Z = (Z_1, Z_2, Z_3, Z_4) = \psi_2(K) = \left(K_1 - \frac{1}{K_3} - \frac{1}{K_4}, K_2 - \frac{1}{K_1 - 1/K_3 - 1/K_4}, K_3, K_4\right)$$

$$T = (T_1, T_2, T_3, T_4) = \psi_4(K) = \left(K_1 - \frac{1}{K_2} - \frac{1}{K_3}, K_2, K_3, K_4 - \frac{1}{K_1 - 1/K_2 - 1/K_3}\right).$$

If the components of X are independent, the components of Z are independent and the components of T are independent, then there exist q, a_1, a_2, a_3, a_4 positive such that $K \sim W_G^c(q, K_G, a_1, a_2, a_3, a_4)$. Consequently, the distributions of X, $Y = \psi_1(K)$, Z and T are products of one gamma and three GIGs as given in Example 3.2.

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