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On roughness indices for fractional fields

ALBERT BENASSI[1], SERGE COHEN[2] and JACQUES ISTAS[3]

[1]*Université Blaise Pascal (Clermont-Ferrand II), LaMP, CNRS UPRESA 6016, F-63177 Aubière Cedex, France. E-mail: benassi@opgc.univ-bpclermont.fr*
[2]*Université Paul Sabatier, UFR MIG, Laboratoire de Statistique et de Probabilités, 118 route de Narbonne, F-31062 Toulouse, France. E-mail: scohen@cict.fr*
[3]*LMC-IMAG and LabSad, Département IMSS BSHM, Université Pierre Mendès-France, F-38000 Grenoble, France. E-mail: jacques-istas@upmf.grenable.fr*

The class of moving-average fractional Lévy motions (MAFLMs), which are fields parameterized by a d-dimensional space, is introduced. MAFLMs are defined by a moving-average fractional integration of order H of a random Lévy measure with finite moments. MAFLMs are centred d-dimensional motions with stationary increments, and have the same covariance function as fractional Brownian motions. They have $H - d/2$ Hölder-continuous sample paths. When the Lévy measure is the truncated random stable measure of index α, MAFLMs are locally self-similar with index $\tilde{H} = H - d/2 + d/\alpha$. This shows that in a non-Gaussian setting these indices (local self-similarity, variance of the increments, Hölder continuity) may be different. Moreover, we can establish a multiscale behaviour of some of these fields. All the indices of such MAFLMs are identified for the truncated random stable measure.

Keywords: identification; local asymptotic self-similarity; second-order fields; stable fields

1. Introduction

The concept of self-similarity is often used to give a mathematical meaning to the heuristic concept of roughness. In this domain the fractional Brownian motion (FBM) $B_H(t)$ of fractional index $0 < H < 1$, introduced by Kolmogorov (1940), is certainly the most famous model. Recall that the FBM is the only centred d-dimensional Gaussian process such that

$$\mathbb{E}(B_H(t) - B_H(s))^2 = \|t - s\|^{2H}, \qquad t, s \in \mathbb{R}^d,$$

$$B_H(0) = 0 \text{ (almost surely).}$$

The FBM has stationary increments, is self-similar of index H, is almost surely H-Hölderian, and H may be identified in an efficient way using generalized quadratic variations (Istas and Lang 1997; Coeurjolly and Istas 2001). In summary, H describes without ambiguity the roughness of the FBM, and this roughness is identifiable. Classical representations (Mandelbrot and Van Ness 1968) of the FBM are the harmonizable representation,

$$B_H(t) = \int_{\mathbb{R}^d} \frac{e^{it.\lambda} - 1}{\|\lambda\|^{H+d/2}} \, dW(\lambda),$$

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and the moving-average representation,

$$B_H(t) = \int_{\mathbb{R}^d} (\|t-s\|^{H-d/2} - \|s\|^{H-d/2}) \mathrm{d}W(s),$$

where the W are the d-dimensional random Brownian measures. In outline, the construction is as follows: one performs a fractional integration, either harmonizable or moving-average, of the random Brownian measure. The stochastic integrals with respect to random Brownian measures are isometries that map deterministic functions in $L^2(\mathbb{R})$ to Gaussian random variables in $L^2(\Omega, \mathbb{P})$, where $(\Omega, \mathbb{P})$ is the underlying probability space (see Neveu 1968). The following question arises: what happens when the 'Gaussian' condition is replaced by a more general condition? The stable case has been widely studied (Samorodnitsky and Taqqu 1994), and we will consider the case where the 'Gaussian' condition is replaced by the existence of all moments. Let us be more precise. Let M be a random Lévy measure all of whose moments are finite: typically, M will be a truncated stable random measure of index α. We consider the following two processes:

$$X_{HA}(t) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{\mathrm{i}t.\lambda} - 1}{\|\lambda\|^{H+d/2}} \mathrm{d}M(\lambda),$$

and

$$X_{MA}(t) = \int_{\mathbb{R}^d} (\|t-s\|^{H-d/2} - \|s\|^{H-d/2}) \mathrm{d}M(s).$$

Because of the isometry property of such random Lévy measure, the second-order structures of X_{HA}, X_{MA} and FBM are the same:

$$\begin{aligned} \mathbb{E}(X_{HA}(t) - X_{HA}(s))^2 &= \mathbb{E}(X_{MA}(t) - X_{MA}(s))^2 \\ &= \|t-s\|^{2H}. \end{aligned} \tag{1}$$

The process X_{HA}, called real harmonizable fractional Lévy motion (RHFLM), was studied in Benassi *et al.* (2002). X_{HA} still has stationary increments but is no longer self-similar. X_{HA} is locally self-similar with an FBM as tangent field:

$$\lim_{\varepsilon \to 0^+} \left(\frac{X_{HA}(t+\varepsilon u) - X_{HA}(t)}{\varepsilon^H} \right)_{u \in \mathbb{R}^d} \stackrel{(d)}{=} (B_H(u))_{u \in \mathbb{R}^d},$$

where $\stackrel{(d)}{=}$ stands for the limit in distribution. RHFLMs are almost surely H-Hölderian. The index H may be identified with the same tools as for FBM (Benassi *et al.* 2002). Once again, H describes without ambiguity the roughness of the process and this roughness is identifiable. One therefore should wonder whether the index H of the second-order structure (cf. (1)) always describes the roughness of a process. The study of processes X_{MA}, called moving-average fractional Lévy motion (MAFLM), the subject of this paper, clearly indicates that the answer is no.

It is known (Falconer 2002; 2003) that the tangent field, if it exists, of a process is almost everywhere a self-similar field with stationary increments. We show in this paper that the tangent field of some MAFLMs is not an FBM, but is a moving-average stable motion (defined in Samorodnitsky and Taqqu 1994). To our knowledge, these MAFLMs are the first known second-order processes having non-Gaussian tangent fields. Moreover, the index of local self-similarity of these MAFLMs is not H, but another index, $\tilde{H}$, which we give in the paper. It follows that the roughness is no longer described by a single index. We then prove that MAFLMs have continous sample paths if and only if $d = 1$ and $H > 1/2$. Thus, MAFLMs are almost surely $(H - 1/2)$-Hölderian.

The indices H and $\tilde{H}$ of MAFLMs are then identified thanks to the observation of a single sample path on a bounded interval. The local self-similarity suggests the use of log-variations to identify $\tilde{H}$, as was done in Abry *et al.* (2000) and Dury (2001) for stable processes in a wavelet setting and generalized by Cohen and Istas (2003). β-variations are used for the identification of H. Actually it is shown that β-variations behave differently according to whether $\beta < \alpha$ or $\beta > \alpha$. This fact is reminiscent of the multiscale behaviour of MAFLMs that we will describe in this paper.

The paper is organized as follows. In Section 2 the MAFLMs are constructed by means of a Poisson representation. The properties of general MAFLMs (asymptotic self-similarity, smoothness of the sample paths) are given in Section 3. The property of local self-similarity, where we need to restrict the class of MAFLMs, is studied in Section 4. The identification is carried out in Section 5.

2. Construction of moving-average fractional Lévy motions

In this section MAFLMs are obtained by means of a Poisson representation of the random Lévy measure $M(\mathrm{d}s)$ that integrates the classical moving-average kernel:

$$G(t, s) = \|t - s\|^{H-d/2} - \|s\|^{H-d/2}. \tag{2}$$

As in Benassi *et al.* (2002), a real-valued field is obtained that has moments of second order $\mathbb{E}(X_H^2(t)) < +\infty$ for all $t \in \mathbb{R}^d$. Since the kernel is itself real-valued, the construction of MAFLMs is even easier than that of RHFLMs since the random measure $M(\mathrm{d}s)$ can be chosen real-valued.

Let us consider a random Poisson measure $N(\mathrm{d}s, \mathrm{d}u)$ in the sense of Section 3.12 of Samorodnitsky and Taqqu (1994) but with a control measure that has moments of every order (see also Benassi *et al.* 2002). More precisely, let $N(\mathrm{d}s, \mathrm{d}u)$ be a random Poisson measure on $\mathbb{R}^d \times \mathbb{R}$ for which the mean measure $n(\mathrm{d}s, \mathrm{d}u) = \mathbb{E}N(\mathrm{d}s, \mathrm{d}u) = \mathrm{d}s\nu(\mathrm{d}u)$ satisfies,

$$\forall p \geqslant 2, \qquad \int_{\mathbb{R}} |u|^p \nu(\mathrm{d}u) < +\infty. \tag{3}$$

A control measure is said to be finite if

$$\int_{\mathbb{R}} \nu(\mathrm{d}u) < +\infty.$$

Denoting by $\widetilde{N} = N - n$ the compensated random Poisson measure, the characteristic function of the stochastic integral is, for all $\phi \in L^2$, for all $v \in \mathbb{R}$,

$$\mathbb{E}\exp\left(\mathrm{i}v\int\varphi\,\mathrm{d}\widetilde{N})\right) = \exp\left[\int_{\mathbb{R}^d\times\mathbb{R}}[\exp(\mathrm{i}v\varphi) - 1 - \mathrm{i}v\varphi]\mathrm{d}s\nu(\mathrm{d}u)\right], \tag{4}$$

where the integral on the right-hand side is convergent since

$$|\exp(\mathrm{i}x) - 1 - \mathrm{i}x| \leqslant C|x|^2 \qquad \forall x \in \mathbb{R}. \tag{5}$$

Let us now define the random Lévy measure $M(\mathrm{d}s)$ by

$$\int_{\mathbb{R}^d} f(s)M(\mathrm{d}s) \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^d\times\mathbb{R}} f(s)u\widetilde{N}(\mathrm{d}s, \mathrm{d}u) \tag{6}$$

for every function $f : \mathbb{R}^d \to \mathbb{R}$ where $f \in L^2(\mathbb{R}^d)$. Moreover, an isometry property for the random Lévy measure $M(\mathrm{d}s)$ holds:

$$\mathbb{E}\left|\int_{\mathbb{R}^d} f(s)M(\mathrm{d}s)\right|^2 = \|f\|^2_{L^2(\mathbb{R}^d)}\int_{\mathbb{R}} u^2\nu(\mathrm{d}u) \tag{7}$$

Since $G(t, \cdot)$ is in $L^2(\mathbb{R}^d)$ for every $t \in \mathbb{R}^d$, the MAFLM can now be defined.

Definition 2.1. *Let us call a real-valued field $(X_H(t))_{t\in\mathbb{R}^d}$ which admits a well-balanced moving-average representation*

$$X_H(t) = \int_{\mathbb{R}^d}\left(\|t-s\|^{H-d/2} - \|s\|^{H-d/2}\right)M(\mathrm{d}s),$$

where $M(\mathrm{d}s)$ is a random Lévy measure defined by (6) *that satisfies the finite-moment assumption* (3)*, a moving-average fractional Lévy motion with parameter H.*

In this paper, for the sake of simplicity, we omit the case $d = 1$, $H = 1/2$: $X_{1/2}(t)$ is equal in distribution to $\int_0^t M(\mathrm{d}s)$, which is a Lévy process, and this case is of no interest in this paper.

Since $n(\mathrm{d}s, \mathrm{d}u)$ is translation-invariant with respect to the variable s, it is straightforward to show that MAFLMs have stationary increments.

Let us illustrate this construction with a simple example: $d = 1$ and $\nu(\mathrm{d}u) = \frac{1}{2}(\delta_{-1}(\mathrm{d}u) + \delta_1(\mathrm{d}u))$, where the δs are Dirac masses. In this case $M(\mathrm{d}s)$ is a compound random Poisson measure and can be written as an infinite sum of random Dirac masses,

$$M(\mathrm{d}s) = \sum_{n\in\mathbb{Z}}\delta_{S_n}(\mathrm{d}s)\varepsilon_n,$$

where $S_{n+1} - S_n$ are identically independent random variables with an exponential law, and ε_n are identically distributed independent Bernoulli random variables such that $\mathbb{P}(\varepsilon_n = 1) = \mathbb{P}(\varepsilon_n = -1) = 1/2$. The ε_n are independent of the S_n. Since the measure ν is finite and $\int_{\mathbb{R}} u\nu(\mathrm{d}u) = 0$, the corresponding MAFLM is, in this special case,

$$X(t) = \sum_{n=-\infty}^{+\infty}\varepsilon_n(|t - S_n|^{H-1/2} - |S_n|^{H-1/2}). \tag{8}$$

Even if this equality is in the L^2 sense, it suggests that the regularity of the sample path can be governed by $H - 1/2$. In the following section we need other tools to prove this fact, but it turns out to be true.

3. Properties of general moving-average fractional Lévy motions

3.1. Asymptotic self-similarity

Proposition 3.1. *MAFLMs are asymptotically self-similar with parameter H,*

$$\lim_{R\to+\infty}\left(\frac{X_H(Rt)}{R^H}\right)_{t\in\mathbb{R}^d} \overset{(d)}{=} \int_{\mathbb{R}} u^2\nu(\mathrm{d}u) \times (B_H(t))_{t\in\mathbb{R}^d}, \tag{9}$$

where the convergence is that of the finite-dimensional margins and B_H is a standard FBM of index H.

Proof. Let us consider the multivariate function

$$g_{t,v,H}(R, s, u) = \mathrm{i}u\sum_{k=1}^{n} v_k \frac{\|Rt_k - s\|^{H-d/2} - \|s\|^{H-d/2}}{R^H}, \tag{10}$$

where $t = (t_1, \dots, t_n)$ and $v = (v_1, \dots, v_n)$ are in $\mathbb{R}^n$. Then

$$\mathbb{E}\exp\left(\mathrm{i}\sum_{k=1}^{n} v_k \frac{X_H(Rt_k)}{R^H}\right) = \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}} [\exp(g_{t,v,H}(R, s, u)) - 1 - g_{t,v,H}(R, s, u)]\mathrm{d}s\nu(\mathrm{d}u)\right). \tag{11}$$

The change of variable $s = R\sigma$ is applied to the integral on the right-hand side to give:

$$\int_{\mathbb{R}^d\times\mathbb{R}} [\exp(R^{-d/2}g_{t,v,H}(1, \sigma, u)) - 1 - R^{-d/2}g_{t,v,H}(1, \sigma, u)]R^d\,\mathrm{d}\sigma\,\nu(\mathrm{d}u). \tag{12}$$

Then, as $R \to +\infty$, a dominated convergence argument yields that

$$\lim_{R\to+\infty}\mathbb{E}\exp\left(\mathrm{i}\sum_{k=1}^{n} v_k \frac{X_H(Rt_k)}{R^H}\right) = \exp\left(\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}} g^2_{t,v,H}(1, \sigma, u)\mathrm{d}\sigma\,\nu(\mathrm{d}u)\right). \tag{13}$$

Therefore the logarithm of this limit is

$$-\frac{1}{2}\int_0^{+\infty} u^2\nu(\mathrm{d}u)\int_{\mathbb{R}^d}\left(\sum_{k=1}^{n} v_k(\|t_k - \sigma\|^{H-d/2} - \|\sigma\|^{H-d/2})\right)^2 \mathrm{d}\sigma, \tag{14}$$

the second integral of which is the variance of $\sum_{k=1}^{n} v_k B_H(t_k)$, which concludes the proof of the convergence of finite-dimensional margins. □

3.2. Regularity of the sample paths

In this subsection general MAFLMs are considered again. To investigate the regularity of the sample paths of MAFLMs one can use the Kolmogorov theorem to show that the sample paths are locally Hölder-continuous for every exponent $H' < H - d/2$ when $H > d/2$. This is a direct application of the isometry property. The question is then what happens when $H < d/2$ or if $H - d/2 > 0$. Can we show that the 'true' exponent is strictly larger than $H - d/2$? If we consider the integrand $G(t, s) = \|t - s\|^{H-d/2} - \|s\|^{H-d/2}$ it is clear that, when $H - d/2 < 0$, $G(\cdot, s)$ is not locally bounded, and when $H > d/2$, it is not H'-Hölderian if $H' > H - d/2$ in a neighbourhood of s. Following Rosinski's rule of the thumb (Rosinski 1989), it is known that the simple paths of the integral defining $X_H(t)$ cannot be 'smoother' than the integrand G.

Let us now make some precise statements.

Proposition 3.2. *If $H > d/2$, for every $H' < H - d/2$ there exists a continuous modification of the MAFLM X_H such that*

$$\mathbb{P}\left[\omega; \sup_{0<\|s-t\|<\epsilon(\omega),\|s\|\leqslant 1,\|t\|\leqslant 1} \left(\frac{X_H(s) - X_H(t)}{\|s - t\|^{H'}}\right) \leqslant \delta\right] = 1, \tag{15}$$

where $\epsilon(\omega)$ is an almost surely positive random variable and $\delta > 0$. Moreover, for every $H' > H - d/2$, $\mathbb{P}(X_H \notin \mathcal{C}^{H'}) > 0$, where $\mathcal{C}^{H'}$ is the space of Hölder-continous functions on $[0, 1]^d$. Furthermore, if the control measure ν of the random measure M is not finite, $\mathbb{P}(X_H \notin \mathcal{C}^{H'}) = 1$.

Proof. Since

$$\mathbb{E}(X_H(s) - X_H(t))^2 = C\|t - s\|^{2H},$$

property (15) is a direct consequence of the Kolmogorov theorem. To prove the second part of the proposition, Theorem 4 in Rosinski (1989) will be applied to X_H. First, we take a separable modification of X_H with a separable representation. The next step is to use the symmetrization argument of Rosinski (1989, Section 5) if ν is not already symmetric. Then we can remark that the kernel $t \to \|t - s\|^{H-d/2} - \|s\|^{H-d/2} \notin \mathcal{C}^{H'}$ for every $H' > H - d/2$, and the conclusion of Theorem 4 is applied to the measurable linear subspace $\mathcal{C}^{H'}$ to give $\mathbb{P}(X_H \notin \mathcal{C}^{H'}) > 0$. To show that this probability is actually one, we rely on a zero–one law. The process X_H can be viewed as an infinitely divisible law on the Banach space $\mathcal{C}[0, 1]$ of the continous functions endowed with the supremum norm. Let us consider the map

$$\begin{aligned} \varphi : \mathbb{R} \times \mathbb{R}^d &\mapsto \mathcal{C}[0, 1] \\ (u, s) &\to u(\|\cdot - s\|^{H-d/2} - \|s\|^{H-d/2}\|); \end{aligned}$$

the random Lévy measure $F(\mathrm{d}f)$ of the infinitely divisible law defined by X_H is now given by $\varphi(\nu^{\mathrm{sym}}(\mathrm{d}u) \times \mathrm{d}s) = F(\mathrm{d}f)$, where ν^{sym} is the control measure of the symmetrized process.

Hence $F(\mathcal{C}[0, 1]\backslash C^{H'}) = +\infty$ if $\nu^{\mathrm{sym}}(\mathbb{R}) = +\infty$. Corollary 11 in Janssen (1982) and $\mathbb{P}(X_H \notin \mathcal{C}^{H'}) > 0$ yield the last result of the proposition. $\square$

Now let us return to the case $H < d/2$.

Proposition 3.3. *If $H < d/2$, for every compact interval $K \subset \mathbb{R}^d$,*

$$\mathbb{P}(X_H \notin \mathcal{B}(K)) > 0,$$

where $\mathcal{B}(K)$ is the space of bounded functions on K.

Proof. In this case, we remark that $t \to \|t - s\|^{H-d/2} - \|s\|^{H-d/2} \notin \mathcal{B}(K)$ for every $s \in K$. The proposition is then proved by applying Theorem 4 in Rosinski (1989) to $\mathcal{B}(K)$.

4. Local self-similarity of MAFLMs

We now investigate local self-similarity for MAFLMs. It should be noted that MAFLMs generally do not have a tangent field. In this section we focus on the truncated stable case. In view of Propositions 3.1 and 4.1, the truncated stable case can be viewed as a bridge between FBM and moving-average stable motion. Let

$$\nu(\mathrm{d}u) = \frac{\mathbf{1}_{\{|u| \leqslant 1\}}\,\mathrm{d}u}{|u|^{1+\alpha}}.$$

Denote the corresponding MAFLM by

$$X_{H,\alpha}(t) = \int_{\mathbb{R}^d} G(t, s)\mathrm{d}M(s).$$

The behaviour of MAFLMs at small scales $\epsilon \to 0^+$ is similar to the behaviour of RHFLMs at large scales $R \to +\infty$. For instance, the limit field is a moving-average stable motion (cf. Samorodnitsky and Taqqu 1994) with parameter $\tilde{H}$. However, the relationship between $\tilde{H}$ and H is slightly different in the setting of moving averages, as shown in the following proposition.

Proposition 4.1. *Let us assume that $\tilde{H}$, defined by $\tilde{H} - d/\alpha = H - d/2$, is such that $0 < \tilde{H} < 1$. The MAFLM $X_{H,\alpha}$ with control measure*

$$\nu(\mathrm{d}u) = \frac{\mathbf{1}_{\{|u| \leqslant 1\}}\,\mathrm{d}u}{|u|^{1+\alpha}}$$

is locally self-similar with parameter $\tilde{H}$. For every fixed $t \in \mathbb{R}^d$,

$$\lim_{\epsilon \to 0^+} \left(\frac{X_{H,\alpha}(t + \epsilon x) - X_{H,\alpha}(t)}{\epsilon^{\tilde{H}}} \right)_{x \in \mathbb{R}^d} \stackrel{(d)}{=} \left(Y_{\tilde{H}}(x) \right)_{x \in \mathbb{R}^d}, \tag{16}$$

where the limit is in distribution for all finite-dimensional margins of the field and the limit is a moving-average fractional stable motion that has a representation

$$Y_{\tilde{H}}(x) = \int_{\mathbb{R}^d} (\|x-\sigma\|^{\tilde{H}-d/\alpha} - \|\sigma\|^{\tilde{H}-d/\alpha}) M_\alpha(\mathrm{d}\sigma), \tag{17}$$

where $M_\alpha(\mathrm{d}\xi)$ is a stable α-symmetric random measure.

Proof. Since the MAFLM has stationary increments we need only prove the convergence for $t=0$. As in the previous proposition, we consider a multivariate function

$$g_{t,v,H}(\epsilon, s, u) = \mathrm{i}u \sum_{k=1}^{n} v_k \frac{\|\epsilon t_k - s\|^{H-d/2} - \|s\|^{H-d/2}}{\epsilon^{\tilde{H}}}, \tag{18}$$

where t and v are in $\mathbb{R}^n$. Then

$$\mathbb{E}\exp\left(\mathrm{i}\sum_{k=1}^{n} v_k \frac{X_{\tilde{H}}(\epsilon u_k)}{\epsilon^{\tilde{H}}}\right) = \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}} [\exp(g_{t,v,H}(\epsilon, s, u)) - 1 - g_{t,v,H}(\epsilon, s, u)] \mathrm{d}s\nu(\mathrm{d}u)\right). \tag{19}$$

Then the change of variable $\sigma = s/\epsilon$ is applied, and $\tilde{H}$ has been chosen such that the integral in (19) is now

$$\int_{\mathbb{R}^d\times\mathbb{R}} [\exp(g_{t,v,H}(1, \sigma, \epsilon^{-d/\alpha}u)) - 1 - g_{t,v,H}(1, \sigma, \epsilon^{-d/\alpha}u)] \mathbf{1}(|u| < 1)\epsilon^d \,\mathrm{d}\sigma \frac{\mathrm{d}u}{|u|^{1+\alpha}}. \tag{20}$$

Let us set $w = \epsilon^{-d/\alpha}u$. The integral becomes

$$I(\epsilon) = \int_{\mathbb{R}^d\times\mathbb{R}} [\exp(g_{t,v,H}(1, \sigma, w)) - 1 - g_{t,v,H}(1, \sigma, w)] \mathbf{1}(|w| < \epsilon^{-d/\alpha}) \,\mathrm{d}\sigma \frac{\mathrm{d}w}{|w|^{1+\alpha}}. \tag{21}$$

Recall that

$$-C(\alpha)|x|^\alpha = \int_{\mathbb{R}} [\mathrm{e}^{\mathrm{i}xr} - 1 - \mathrm{i}xr\mathbf{1}(|r| \leqslant \epsilon^{-d/\alpha})] \frac{\mathrm{d}r}{|r|^{1+\alpha}} \tag{22}$$

for every $\epsilon > 0$, where $C(\alpha) = 2\int_0^{+\infty} (1 - \cos(r)) \mathrm{d}r/r^{1+\alpha}$. Let us write

$$J_\epsilon = \int_{\mathbb{R}} [\mathrm{e}^{\mathrm{i}xr} - 1 - \mathrm{i}xr] \mathbf{1}(|r| \leqslant \epsilon^{-d/\alpha}) \frac{\mathrm{d}r}{|r|^{1+\alpha}}.$$

Then

$$\lim_{\epsilon\to 0^+} (J_\epsilon + C(\alpha)|x|^\alpha) = \lim_{\epsilon\to 0^+} \int_{\mathbb{R}} [1 - \mathrm{e}^{\mathrm{i}xr}] \mathbf{1}(|r| > \epsilon^{-d/\alpha}) \frac{\mathrm{d}r}{|r|^{1+\alpha}} = 0.$$

Hence,

$$\lim_{\epsilon\to 0^+} I(\epsilon) = -C(\alpha) \int_{\mathbb{R}^d} |g_{t,v,H}(1, \sigma, 1)|^\alpha \,\mathrm{d}\sigma. \tag{23}$$

Since this last expression is the logarithm of

$$\mathbb{E}\exp\left(\mathrm{i}\sum_{k=1}^{n} v_k Y_{\tilde{H}}(t_k)\right),$$

the proof is complete.

5. Identification of the fractional indices

We now carry out our identification for MAFLMs with truncated stable control measures. M is therefore a random Lévy measure associated with $\nu(\mathrm{d}u) = \mathbf{1}_{\{|u|\leqslant 1\}}\,\mathrm{d}u/|u|^{1+\alpha}$. Recall that the corresponding MAFLM is denoted by

$$X_{H,\alpha}(t) = \int_{\mathbb{R}^d} G(t, s)\mathrm{d}M(s).$$

For $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ and $n \in \mathbb{N}^\star$, define

$$\frac{\mathbf{k}}{2^n} = \left(\frac{k_1}{2^n}, \ldots, \frac{k_d}{2^n}\right),$$

$$X_{H,\alpha}\left(\frac{\mathbf{k}}{2^n}\right) = X_{H,\alpha}\left(\frac{k_1}{2^n}, \ldots, \frac{k_d}{2^n}\right).$$

The aim of this section is to perform the identification of the fractional indices H and $\tilde{H}$, or equivalently the indices H and α, with discrete observations of the field $X_{H,\alpha}$ on $[0, 1]^d$. $X_{H,\alpha}$ is observed at times $(k_1/2^n, \ldots, k_d/2^n)$, $0 \leqslant k_i \leqslant 2^n$, $i = 1, \ldots, d$.

Let (a_ℓ), $\ell = 0, \ldots, K$ be a real-valued sequence such that

$$\sum_{\ell=0}^{K} a_\ell = 0, \qquad \sum_{\ell=0}^{K} \ell a_\ell = 0. \tag{24}$$

From now on, multi-indices are written with bold letters. For $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$, define

$$a_{\mathbf{k}} = a_{k_1} \ldots a_{k_d}.$$

Define the increments of $X_{H,\alpha}$ associated with the sequence a:

$$\Delta X_{\mathbf{p},n} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} X_{H,\alpha}\left(\frac{\mathbf{k}+\mathbf{p}}{2^n}\right)$$

$$\stackrel{\text{def}}{=} \sum_{k_1,\ldots,k_d=0}^{K} a_{k_1} \ldots a_{k_d} X_{H,\alpha}\left(\frac{k_1+p_1}{2^n}, \ldots, \frac{k_d+p_d}{2^n}\right);$$

one can, for instance, take $K = 2$, $a_0 = 1$, $a_1 = -2$, $a_2 = 1$.

For $\beta > 0$, define the β-variations by

$$V_{n,\beta} = \frac{1}{(2^n - K)^d} \sum_{\mathbf{p}=\mathbf{1}}^{\mathbf{2^n-K}} |\Delta X_{\mathbf{p},n}|^\beta.$$

Define the log-variations by:

$$V_{n,0} = \frac{1}{(2^n - K)^d} \sum_{\mathbf{p}=\mathbf{1}}^{\mathbf{2^n-K}} \log|\Delta X_{\mathbf{p},n}|.$$

Variations of processes are classical tools for identifying parameters: quadratic variations were introduced some time ago for Gaussian processes, and log-variations were introduced in Abry *et al.* (2000) for stable processes in a wavelet setting. The main result of this section concerns the asymptotic behaviour of the log- and β-variations:

Theorem 5.1. *For the convergence of* log-*variations, we have*

$$\lim_{n\to+\infty} -\frac{1}{n\log 2} V_{n,0} \overset{(\mathbb{P})}{=} \tilde{H}.$$

For the convergence of β-variations, there are two cases. For $0 < \beta < \alpha$, there exists a constant $C_\beta > 0$ such that

$$\lim_{n\to+\infty} 2^{n\beta\tilde{H}} V_{n,\beta} \overset{(a.s.)}{=} C_\beta.$$

For $\alpha < \beta < 2$, there exists a constant $C_\beta > 0$ such that

$$\lim_{n\to+\infty} 2^{n\beta(H+d/\beta-d/2)} V_{n,\beta} \overset{(a.s.)}{=} C_\beta.$$

The fractional indices H and $\tilde{H}$ can then be identified as follows. A consistent estimator of $\tilde{H}$ is given by

$$\tilde{H}_n = -\frac{1}{\log 2^n} V_{n,0}. \tag{25}$$

To estimate H, we have to assume weak a priori knowledge on α, for instance that α belongs to the interval $]0, \alpha_{\sup}[$, with $\alpha_{\sup} < 2$ known. For any $\alpha_{\sup} \leqslant \beta < 2$, a consistent estimator of H is then given by

$$H_n = \frac{1}{\beta}\left(\log_2 \frac{V_{n-1,\beta}}{V_{n,\beta}} + \frac{\beta d}{2} - d\right). \tag{26}$$

Using (25) and (26), a consistent estimator of α is of course

$$\alpha_n = \frac{d}{\tilde{H}_n - H_n + d/2}.$$

Note that we could have estimated α using the results on the convergence of β-variations. Actually, if we assume that we know $(\beta, \log_2(V_{n-1,\beta}/V_{n,\beta}))$ for different values of β then α is the point at which the slope is changing. Although this method does not theoretically require any a priori knowledge for α, we believe it is not numerically feasable to determine a sampling design for the βs without this a priori knowledge.

Proof of Theorem 5.1. The convergence of the log-variations is a particular case of a more general result due to Cohen and Istas (2003) and will be omitted.

Integral representations of power functions are used extensively and are given in the following. For all $\beta \in (0, 2)$, for all $x \in \mathbb{R}$,

$$|x|^\beta = \left(\int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}y} - 1 - \mathrm{i}y\mathbf{1}_{|y|\leqslant 1}}{|y|^{1+\beta}} \mathrm{d}y \right)^{-1} \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}xy} - 1 - \mathrm{i}xy\mathbf{1}_{|y|\leqslant 1}}{|y|^{1+\beta}} \mathrm{d}y.$$

Because of this integral representation the process

$$S_n(y) = \frac{1}{(2^n - K)^d} \sum_{\mathbf{p}=\mathbf{1}}^{\mathbf{2^n-K}} \exp\left(\mathrm{i}y2^{n\tilde{H}} \Delta X_{\mathbf{p},n}\right), \qquad y \in \mathbb{R},$$

is introduced for the the study of the β-variations and log-variations. Let

$$\Delta G_{\mathbf{p},n}(s) = \sum_{\ell=0}^{K} a_\ell G\left(\frac{\mathbf{p}+\ell}{2^n}, s\right),$$

where G is defined in (2) and

$$S(y) = \exp\left\{ |y|^\alpha \int_{\mathbb{R}^d \times \mathbb{R}} [\exp(\mathrm{i}v\Delta G_{\mathbf{0},1}(\sigma)) - 1 - \mathrm{i}v\Delta G_{\mathbf{0},1}(\sigma)\mathbf{1}_{|v|\leqslant 1}]\mathrm{d}\sigma \frac{\mathrm{d}v}{|v|^{1+\alpha}} \right\}.$$

We first prove the following intermediate lemma on $S_n(y)$.

Lemma 5.2.

$$\lim_{n\to+\infty} S_n(y) \overset{(a.s.)}{=} S(y).$$

Proof. We first prove the convergence of $\mathbb{E}S_n(y)$. By (4),

$$\mathbb{E}S_n(y) = \exp\left\{ \int_{\mathbb{R}^d \times \mathbb{R}} [\exp(\mathrm{i}uy2^{n\tilde{H}}\Delta G_{\mathbf{0},n}(s)) - 1 - \mathrm{i}uy2^{n\tilde{H}}\Delta G_{\mathbf{0},n}(s)]\mathrm{d}s\nu(\mathrm{d}u) \right\}.$$

The change of variables $s = \sigma/2^n$, $v = uy2^{nd/\alpha}$ leads to

$$\mathbb{E}S_n(y) = \exp\left\{ |y|^\alpha \int_{\mathbb{R}^d \times \mathbb{R}} [\exp(\mathrm{i}v\Delta G_{\mathbf{0},1}(\sigma)) - 1 - \mathrm{i}v\Delta G_{\mathbf{0},1}(\sigma)]\mathrm{d}\sigma \frac{\mathrm{d}v}{|v|^{1+\alpha}} \mathbf{1}_{|v|\leqslant |y|2^{nd/\alpha}} \right\}.$$

and the convergence of $\mathbb{E}S_n(y)$ toward $S(y)$ is proved by using the same arguments as in (23).

Let us now study the variance of $S_n(y)$. We have

$$\operatorname{var} S_n(y) = \frac{1}{(2^n - K)^{2d}} \sum_{\mathbf{p},\mathbf{p}'=\mathbf{1}}^{\mathbf{2^n-K}} I_{\mathbf{p},\mathbf{p}'},$$

with

$$I_{\mathbf{p},\mathbf{p}'} = \mathbb{E}\exp(\mathrm{i}y2^{n\tilde{H}}(\Delta X_{\mathbf{p},n} - \Delta X_{\mathbf{p}',n})) - \mathbb{E}\exp(\mathrm{i}y2^{n\tilde{H}}\Delta X_{\mathbf{p},n})\mathbb{E}\exp(-\mathrm{i}y2^{n\tilde{H}}\Delta X_{\mathbf{p}',n}).$$

Because of (4), $I_{\mathbf{p},\mathbf{p}'}$ can also be written

$$I_{\mathbf{p},\mathbf{p}'} = \exp\left\{\int_{\mathbb{R}^d\times\mathbb{R}}[\exp(\mathrm{i}uy2^{n\tilde{H}}(\Delta G_{\mathbf{p},n}(s) - \Delta G_{\mathbf{p}',n}(s))) - 1 \right.$$
$$\left. - \mathrm{i}uy2^{n\tilde{H}}(\Delta G_{\mathbf{p},n}(s) - \Delta G_{\mathbf{p}',n}(s))]\mathrm{d}s\nu(\mathrm{d}u)\right\}$$
$$- \exp\left\{\int_{\mathbb{R}^d\times\mathbb{R}}[\exp(\mathrm{i}uy2^{n\tilde{H}}\Delta G_{\mathbf{p},n}(s)) + \exp(-\mathrm{i}uy2^{n\tilde{H}}\Delta G_{\mathbf{p}',n}(s)) \right.$$
$$\left. - 2 - \mathrm{i}uy2^{n\tilde{H}}\Delta G_{\mathbf{p},n}(s) - \mathrm{i}uy2^{n\tilde{H}}\Delta G_{\mathbf{p}',n}(s)]\mathrm{d}s\nu(\mathrm{d}u)\right\}.$$

Hence

$$I_{\mathbf{p},\mathbf{p}'} = A_{\mathbf{p},\mathbf{p}'} \times B_{\mathbf{p},\mathbf{p}'}$$

with

$$A_{\mathbf{p},\mathbf{p}'} = \exp\left\{\int_{\mathbb{R}^d\times\mathbb{R}}[\exp(\mathrm{i}uy2^{n\tilde{H}}\Delta G_{\mathbf{p},n}(s)) + \exp(-\mathrm{i}uy2^{n\tilde{H}}\Delta G_{\mathbf{p}',n}(s)) - 2 \right.$$
$$\left. - \mathrm{i}uy2^{n\tilde{H}}\Delta G_{\mathbf{p},n}(s) - \mathrm{i}uy2^{n\tilde{H}}\Delta G_{\mathbf{p}',n}(s)]\mathrm{d}s\nu(\mathrm{d}u)\right\}$$

and

$$B_{\mathbf{p},\mathbf{p}'} = \exp\left\{\int_{\mathbb{R}^d\times\mathbb{R}}[\exp(\mathrm{i}uy2^{n\tilde{H}}\Delta G_{\mathbf{p},n}(s)) - 1] \right.$$
$$\left. [\exp(-\mathrm{i}uy2^{n\tilde{H}}\Delta G_{\mathbf{p}',n}(s)) - 1]\mathrm{d}s\nu(\mathrm{d}u)\right\} - 1.$$

Clearly,

$$|A_{\mathbf{p},\mathbf{p}'}| \leqslant 1.$$

The change of variables $s = \sigma/2^n$, $v = uy2^{nd/\alpha}$ leads to

$$B_{\mathbf{p},\mathbf{p}'} = \exp\left\{|y|^\alpha\int_{\mathbb{R}^d\times\mathbb{R}}[\exp(\mathrm{i}v\Delta G_{\mathbf{p},1}(\sigma)) - 1] \right.$$
$$\left. [\exp(-\mathrm{i}v\Delta G_{\mathbf{p}',1}(\sigma)) - 1]\mathrm{d}\sigma\frac{\mathrm{d}v}{|v|^{1+\alpha}}\mathbf{1}_{|v|\leqslant 2^{nd/\alpha}}\right\} - 1.$$

Define

$$C_{\mathbf{p},\mathbf{p}'} = \int_{\mathbb{R}^d\times\mathbb{R}}[\exp(\mathrm{i}v\Delta G_{\mathbf{p}-\mathbf{p}',1}(\sigma)) - 1][\exp(-\mathrm{i}v\Delta G_{\mathbf{0},1}(\sigma)) - 1]\mathrm{d}\sigma\frac{\mathrm{d}v}{|v|^{1+\alpha}}.$$

This leads to

$$B_{\mathbf{p},\mathbf{p}'} = \big(\exp(C_{\mathbf{p},\mathbf{p}'}|y|^\alpha) - 1\big)(1 + o(1)).$$

We split $C_{\mathbf{p},\mathbf{p}'}$ into two parts,

$$T_1 = \int_{\mathbb{R}^d\times\{|v|\leqslant A\}} [\exp(\mathrm{i}v\Delta G_{\mathbf{p}-\mathbf{p}',1}(\sigma)) - 1][\exp(-\mathrm{i}v\Delta G_{\mathbf{0},1}(\sigma)) - 1]\mathrm{d}\sigma\,\frac{\mathrm{d}v}{|v|^{1+\alpha}},$$

$$T_2 = \int_{\mathbb{R}^d\times\{|v|\geqslant A\}} [\exp(\mathrm{i}v\Delta G_{\mathbf{p}-\mathbf{p}',1}(\sigma)) - 1][\exp(-\mathrm{i}v\Delta G_{\mathbf{0},1}(\sigma)) - 1]\mathrm{d}\sigma\,\frac{\mathrm{d}v}{|v|^{1+\alpha}},$$

with A to be chosen later.

$$\begin{aligned}|T_1| &\leqslant \int_{\mathbb{R}^d\times|v|\leqslant A}\left|\sum_{\ell=0}^{K} a_\ell\|\ell+\mathbf{p}-\mathbf{p}'-\sigma\|^{H-d/2}\sum_{\ell=0}^{K} a_\ell\|\ell-\sigma\|^{H-d/2}\right|\mathrm{d}\sigma\,\frac{\mathrm{d}v}{|v|^{\alpha-1}}\\ &\leqslant CA^{2-\alpha}\int_{\mathbb{R}^d}\left|\sum_{\ell=0}^{K} a_\ell\|\ell+\mathbf{p}-\mathbf{p}'-\sigma\|^{H-d/2}\sum_{\ell=0}^{K} a_\ell\|\ell-\sigma\|^{H-d/2}\right|\mathrm{d}\sigma.\end{aligned}$$

This is identical to the term obtained in the Gaussian case (cf. Benassi *et al.* 1998). Therefore, a Taylor expansion of order 2 around $\mathbf{p}-\mathbf{p}'$ is used:

$$|T_1| \leqslant CA^{2-\alpha}\|\mathbf{p}-\mathbf{p}'\|^{H-d/2-2}.$$

Moreover, for $\delta > 0$ arbitrarily small,

$$|T_2| \leqslant \frac{C}{A^{\alpha-\delta}}\int_{\mathbb{R}^d\times\mathbb{R}}\left|[\exp(\mathrm{i}v\Delta G_{\mathbf{p}-\mathbf{p}',1}(\sigma)) - 1][\exp(-\mathrm{i}v\Delta G_{\mathbf{0},1}(\sigma)) - 1]\right|\mathrm{d}\sigma\,\frac{\mathrm{d}v}{|v|^{1+\delta}}.$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}&\left(\int_{\mathbb{R}^d}\left|[\exp(\mathrm{i}v\Delta G_{\mathbf{p}-\mathbf{p}',1}(\sigma)) - 1][\exp(-\mathrm{i}v\Delta G_{\mathbf{0},1}(\sigma)) - 1]\right|\mathrm{d}\sigma\right)^2\\ &\quad\leqslant \int_{\mathbb{R}^d}\left|[\exp(\mathrm{i}v\Delta G_{\mathbf{p}-\mathbf{p}',1}(\sigma)) - 1]\right|^2\mathrm{d}\sigma\int_{\mathbb{R}^d}\left|[\exp(-\mathrm{i}v\Delta G_{\mathbf{0},1}(\sigma)) - 1]\right|^2\mathrm{d}\sigma\\ &\quad= \left(\int_{\mathbb{R}^d}\left|[\exp(-\mathrm{i}v\Delta G_{\mathbf{0},1}(\sigma)) - 1]\right|^2\mathrm{d}\sigma\right)^2,\end{aligned}$$

so that

$$|T_2| \leqslant \frac{C'}{A^{\alpha-\delta}}.$$

We choose A such that $A^{2+\delta} = \|\mathbf{p}-\mathbf{p}'\|^{2+d/2-H}$. Therefore, as $\|\mathbf{p}-\mathbf{p}'\| \to +\infty$:

$$|B_{\mathbf{p},\mathbf{p}'}| \leqslant C\|\mathbf{p}-\mathbf{p}'\|^{(\delta-\alpha)(2+d/2-H)/(2+\delta)}|y|^\alpha.$$

We choose δ small enough, so that $\sum_n \operatorname{var} S_n(y)$ is convergent. For every $\epsilon > 0$,

$$\sum_{n=1}^{+\infty}\mathbb{P}(|S_n(y) - \mathbb{E}S_n(y)| > \epsilon) \leqslant \epsilon^{-2}\sum_n \operatorname{var} S_n(y).$$

Hence, by the Borel–Cantelli lemma,

$$\lim_{n\to+\infty}(S_n(y) - \mathbb{E}S_n(y)) \overset{\text{(a.s.)}}{=} 0$$

and

$$\lim_{n\to+\infty} S_n(y) \overset{\text{(a.s.)}}{=} S(y)$$

and the convergence of $\mathbb{E}S_n(y)$ concludes the proof of the lemma. $\square$

We can now prove the convergence of the β-variations for $0 < \beta < \alpha$. The integral representation of power functions leads to

$$\frac{2^{n\beta\tilde{H}}}{(2^n - K)^d}\sum_{\mathbf{p}=\mathbf{1}}^{\mathbf{2^n-K}} |\Delta X_{\mathbf{p},n}|^\beta = \int_{\mathbb{R}} \frac{S_n(y) - 1 - iy\mathbf{1}_{|y|\leqslant 1}(1/(2^n-K)^d)\sum_{\mathbf{p}=\mathbf{1}}^{\mathbf{2^n-K}} 2^{n\tilde{H}}\Delta X_{\mathbf{p},n}}{|y|^{1+\beta}}\,\mathrm{d}y.$$

The sequence $\sum_{\mathbf{p}=\mathbf{1}}^{\mathbf{2^n-K}}\Delta X_{\mathbf{p},n}$ is a telescopic one: $\mathbb{E}(\sum_{\mathbf{p}=\mathbf{1}}^{\mathbf{2^n-K}}\Delta X_{\mathbf{p},n})^2$ converges to zero and can be overestimated by a constant. By the Borel–Cantelli lemma, $(2^{n\tilde{H}}/(2^n-K)^d)\sum_{\mathbf{p}=\mathbf{1}}^{\mathbf{2^n-K}}\Delta X_{\mathbf{p},n}$ converges (almost surely) to 0.

An application of the dominated convergence theorem leads to

$$\lim_{n\to+\infty}\frac{2^{n\beta\tilde{H}}}{(2^n - K)^d}\sum_{\mathbf{p}=\mathbf{1}}^{\mathbf{2^n-K}} |\Delta X_{\mathbf{p},n}|^\beta \overset{\text{(a.s.)}}{=} \int_{\mathbb{R}}\frac{S(y)-1}{|y|^{1+\beta}}\,\mathrm{d}y. \tag{27}$$

We now study the β-variations for $\alpha < \beta < 2$. The integral on the right-hand side of (27) is divergent, so of course the dominated convergence theorem can no longer be applied. First, recall that

$$\int_{\mathbb{R}}\frac{\mathbb{E}S_n(y) - 1}{|y|^{1+\beta}}\,\mathrm{d}y = \int_{\mathbb{R}}\frac{\exp\left\{|y|^\alpha\int_{\mathbb{R}^d\times\mathbb{R}} E(v,\sigma)\mathrm{d}\sigma\,\mathrm{d}v|v|^{-(1+\alpha)}\mathbf{1}_{|v|\leqslant|y|2^{nd/\alpha}}\right\} - 1}{|y|^{1+\beta}}\,\mathrm{d}y,$$

where

$$E(v,\sigma) = \exp(\mathrm{i}v\Delta G_{\mathbf{0},1}(\sigma)) - 1 - \mathrm{i}v\Delta G_{\mathbf{0},1}(\sigma).$$

The previous integral is split into three terms,

$$\int_{|y|\leqslant 2^{-nd/\alpha}}\dots + \int_{2^{-nd/\alpha}<|y|\leqslant 1/n}\dots + \int_{|y|>1/n}\dots$$

For the sake of brevity, the integrand with respect to y has been suppressed in the following when no confusion is possible. For the first term, the change of variables $z = y2^{nd/\alpha}$ leads to:

$$\int_{|y|\leqslant 2^{-nd/\alpha}}\frac{\exp\left\{|y|^\alpha\int_{\mathbb{R}^d\times\mathbb{R}} E(v,\sigma)\mathrm{d}\sigma\,\mathrm{d}v|v|^{-(1+\alpha)}\mathbf{1}_{|v|\leqslant|y|2^{nd/\alpha}}\right\} - 1}{|y|^{1+\beta}}\,\mathrm{d}y$$

$$= 2^{nd\beta/\alpha} \int_{|z| \leqslant 1} \frac{\exp\left\{2^{-nd}|z|^{\alpha} \int_{\mathbb{R}^d \times \mathbb{R}} E(v, \sigma)\mathrm{d}\sigma\, \mathrm{d}v |v|^{-(1+\alpha)} \mathbf{1}_{|v| \leqslant |z|}\right\} - 1}{|z|^{1+\beta}} \mathrm{d}z.$$

Since $2^{-nd} \to 0$ as $n \to +\infty$, a Taylor expansion of order 1 is used:

$$\exp\left\{2^{-nd}|z|^{\alpha} \int_{\mathbb{R}^d \times \mathbb{R}} E(v, \sigma)\mathrm{d}\sigma\, \mathrm{d}v |v|^{-(1+\alpha)} \mathbf{1}_{|v| \leqslant |z|}\right\} - 1 = 2^{-nd}|z|^{\alpha} \int_{\mathbb{R}^d \times \mathbb{R}} E(v, \sigma)\mathrm{d}\sigma\, \mathrm{d}v |v|^{-(1+\alpha)} \mathbf{1}_{|v| \leqslant |z|}(1 + o(1)).$$

Note that, because of the term $\mathbf{1}_{|v| \leqslant |z|}$, the integral

$$\int_{|z| \leqslant 1} \frac{\mathrm{d}z}{|z|^{1+\beta-\alpha}} \int_{\mathbb{R}^d \times \mathbb{R}} E(v, \sigma)\mathrm{d}\sigma \frac{\mathrm{d}v}{|v|^{1+\alpha}} \mathbf{1}_{|v| \leqslant |z|}$$

is convergent. It follows that:

$$\int_{|y| \leqslant 2^{-nd/\alpha}} \frac{\exp\left\{|y|^{\alpha} \int_{\mathbb{R}^d \times \mathbb{R}} E(v, \sigma)\mathrm{d}\sigma\, \mathrm{d}v |v|^{-(1+\alpha)} \mathbf{1}_{|v| \leqslant |y 2^{nd/\alpha}|}\right\} - 1}{|y|^{1+\beta}} \mathrm{d}y$$
$$= 2^{n(-d+d\beta/\alpha)} \int_{|z| \leqslant 1} \frac{\mathrm{d}z}{|z|^{1+\beta-\alpha}} \int_{\mathbb{R}^d \times \mathbb{R}} E(v, \sigma)\mathrm{d}\sigma \frac{\mathrm{d}v}{|v|^{1+\alpha}} \mathbf{1}_{|v| \leqslant |z|}(1 + o(1)).$$

We now turn to the third term. Because of the symmetry of $\mathrm{d}v/|v|^{1+\alpha}$, the integral $\int_{\mathbb{R}^d \times \mathbb{R}} E(v, \sigma)\mathrm{d}\sigma \mathrm{d}v/|v|^{1+\alpha} \mathbf{1}_{|v| \leqslant |y 2^{nd/\alpha}|}$ is negative.

We can bound $\exp\{|y|^{\alpha} \int_{\mathbb{R}^d \times \mathbb{R}} E(v, \sigma)\mathrm{d}\sigma\, \mathrm{d}v |v|^{-(1+\alpha)} \mathbf{1}_{|v| \leqslant |y 2^{nd/\alpha}|}\}$ by 1, so that

$$\int_{|y| \geqslant 1/n} \frac{\exp\left\{|y|^{\alpha} \int_{\mathbb{R}^d \times \mathbb{R}} E(v, \sigma)\mathrm{d}\sigma\, \mathrm{d}v |v|^{-(1+\alpha)} \mathbf{1}_{|v| \leqslant |y 2^{nd/\alpha}|}\right\} - 1}{|y|^{1+\beta}} \mathrm{d}y \leqslant C n^{\beta}.$$

Finally, we consider the second term. Since $1/n \to 0$, a Taylor expansion of order 1 leads to:

$$\int_{2^{-nd/\alpha} \leqslant |y| \leqslant 1/n} \frac{\exp\left\{|y|^{\alpha} \int_{\mathbb{R}^d \times \mathbb{R}} E(v, \sigma)\mathrm{d}\sigma\, \mathrm{d}v |v|^{-(1+\alpha)} \mathbf{1}_{|v| \leqslant |y 2^{nd/\alpha}|}\right\} - 1}{|y|^{1+\beta}} \mathrm{d}y$$
$$= \int_{2^{-nd/\alpha} \leqslant |y| \leqslant 1/n} \frac{\mathrm{d}y}{|y|^{1+\beta-\alpha}} \int_{\mathbb{R}^d \times \mathbb{R}} E(v, \sigma)\mathrm{d}\sigma \frac{\mathrm{d}v}{|v|^{1+\alpha}} \mathbf{1}_{|v| \leqslant |y 2^{nd/\alpha}|}(1 + o(1)).$$

The change of variable $z = y 2^{nd/\alpha}$ leads to:

$$\int_{2^{-nd/\alpha} \leqslant |y| \leqslant 1/n} \frac{\mathrm{d}y}{|y|^{1+\beta-\alpha}} \int_{\mathbb{R}^d \times \mathbb{R}} E(v, \sigma)\mathrm{d}\sigma \frac{\mathrm{d}v}{|v|^{1+\alpha}} \mathbf{1}_{|v| \leqslant |y 2^{nd/\alpha}|}$$
$$= 2^{n(-d+d\beta/\alpha)} \int_{1 \leqslant |z| \leqslant 2^{nd/\alpha}/n} \frac{\mathrm{d}z}{|z|^{1+\beta-\alpha}} \int_{\mathbb{R}^d \times \mathbb{R}} E(v, \sigma)\mathrm{d}\sigma \frac{\mathrm{d}v}{|v|^{1+\alpha}} \mathbf{1}_{|v| \leqslant |z|}.$$

Since $\beta > \alpha$, the integral

$$\int_{1\leqslant|z|\leqslant 2^{nd/\alpha}/n} \frac{\mathrm{d}z}{|z|^{1+\beta-\alpha}} \int_{\mathbb{R}^d\times\mathbb{R}} E(v, \sigma)\mathrm{d}\sigma \frac{\mathrm{d}v}{|v|^{1+\alpha}} \mathbf{1}_{|v|\leqslant|z|}$$

converges to

$$\int_{1\leqslant|z|} \frac{\mathrm{d}z}{|z|^{1+\beta-\alpha}} \int_{\mathbb{R}^d\times\mathbb{R}} E(v, \sigma)\mathrm{d}\sigma \frac{\mathrm{d}v}{|v|^{1+\alpha}} \mathbf{1}_{|v|\leqslant|z|},$$

so that

$$\int_{2^{-nd/\alpha}\leqslant|y|\leqslant 1/n} \frac{\exp\left\{|y|^\alpha \int_{\mathbb{R}^d\times\mathbb{R}} E(v, \sigma)\mathrm{d}\sigma\, \mathrm{d}v|v|^{-(1+\alpha)} \mathbf{1}_{|v|\leqslant|y|2^{nd/\alpha}}\right\} - 1}{|y|^{1+\beta}} \mathrm{d}y$$
$$= 2^{n(-d+d\beta/\alpha)}(1 + o(1)) \int_{|z|\geqslant 1} \frac{\mathrm{d}z}{|z|^{1+\beta-\alpha}} \int_{\mathbb{R}^d\times\mathbb{R}} E(v, \sigma)\mathrm{d}\sigma \frac{\mathrm{d}v}{|v|^{1+\alpha}} \mathbf{1}_{|v|\leqslant|z|}.$$

To summarize, the first term is equivalent to $C2^{n(-d+d\beta/\alpha)}$, the third is equivalent to $C2^{n(-d+d\beta/\alpha)}$ and the second is negligible as compared to the two others. We have proved that:

$$2^{n(d-d\beta/\alpha)} \int_{\mathbb{R}} \frac{\mathbb{E}S_n(y) - 1}{|y|^{1+\beta}} \mathrm{d}y \to C.$$

From Lemma 5.2, $S_n(y) = \mathbb{E}S_n(y)(1 + o_{(a.s.)}(1))$. We have therefore proved that $2^{n\beta\tilde{H}}2^{n(d-d\beta/\alpha)}V_{n,\beta}$ converges, as $n \to +\infty$ to a constant. Since $\beta\tilde{H} + d - d\beta/\alpha) = \beta(H - d/2 + d/\beta)$, Theorem 5.1 is proved. □

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