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Martingale-type stochastic calculus for anticipating integral processes

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We prove that the class of Skorohod integral processes coincides with a class of Itô integrals. Using the techniques of the classical Itô stochastic calculus, we develop a new stochastic calculus for Skorohod integral processes, different from that introduced by Nualart and Pardoux.

Keywords: Malliavin calculus; stochastic integrals

1. Introduction

The anticipating integral, or the Skorohod integral, is an extension of the Itô integral to non-adapted integrands and coincides with the Itô integral when the integrand is adapted.

The aim of this work is to present a different approach to the anticipating stochastic calculus, extending the results of the martingale theory. We study anticipating integral processes $(X_t)_{t\in T}$, with $X_t = \int_0^t u_s \, \mathrm{d}W_s$. These processes are not martingales, since they are not adapted to the Brownian filtration $(\mathcal{F}_t)_{t\in T}$. But they satisfy an interesting property, that is, if $\mathbb{F}_{[s,t]^c}$ denotes the σ-algebra generated by the increments of the Wiener process W on $T\backslash[s, t]$, then

$$\mathbb{E}\big(X_t - X_s | \mathbb{F}_{[s,t]^c}\big) = 0. \tag{1}$$

Relation (1) implies immediately that the projection of X_t on the Brownian filtration is a martingale. Indeed, by (1), $\mathbb{E}(X_t|\mathbb{F}_s) = \mathbb{E}(X_s|\mathbb{F}_s)$ and, if we put $Z_t = \mathbb{E}[X_t|\mathbb{F}_t]$, we have

$$\mathbb{E}(Z_t/\mathbb{F}_s) = \mathbb{E}(\mathbb{E}[X_t|\mathbb{F}_t]|\mathbb{F}_s) = \mathbb{E}[X_t|\mathbb{F}_s] = \mathbb{E}[X_s|\mathbb{F}_s] = Z_s.$$

We will see that every Skorohod integral process X_t can be written as an 'Itô' integral of the form $Y_t = \int_0^t \mathbb{E}[v_\alpha|\mathbb{F}_{[\alpha,t]^c}]\mathrm{d}W_\alpha$, where v is a square-integrable process depending on u. The integral Y is an isometry since, by the mean square formula for the Skorohod integral,

$$\mathbb{E}\left(\int_0^t \mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t]^c}\big]\mathrm{d}W_\alpha\right)^2 = \int_0^t \big(\mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t]^c}\big]\big)^2 \, \mathrm{d}\alpha,$$

and it has another important property, that it can be viewed as a limit almost surely of a certain uniformly integrable martingale. This fact will allow us to develop a parallel stochastic calculus for anticipating integrals, different from that introduced in Nualart and Pardoux (1988) but coinciding with it in the adapted case. We believe that our construction

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gives a more probabilistic approach to the anticipating stochastic calculus, opening up the possibility of defining stopping times or almost sure limits for Skorohod integrals. We plan to study these problems further.

Section 2 contains some preliminaries on the Malliavin calculus. In Section 3 we show the connection between the class of Skorohod integrals and a class of isometric 'Itô' integrals. We show that, at every instant t, the integral X_t is the last element of a uniformly integrable martingale. We derive an anticipating Itô formula from the classical Itô formula and we introduce a generalized local time for anticipating integral processes. Some immediate consequences – Burkholder inequalities for anticipating processes and an Itô formula for the product of a martingale and a backward martingale – are given in Section 4.

2. Preliminaries

Let $T = [0, 1]$, the unit interval. Denote by$(W_t)_{t\in T}$ the standard Wiener process on the canonical Wiener space $(\Omega, \mathbb{F}, P)$, and let $(\mathbb{F}_t)_{t\in T}$ be the natural filtration generated by W.

A functional of the Brownian motion of the form

$$F = f(W_{t_1}, \dots, W_{t_n}) \qquad \text{with } t_1, \dots, t_n \in T,\ f \in C_b^\infty(R^n), \tag{2}$$

is called a smooth random variable, and this class is denoted by $\mathcal{S}$.

The Malliavin derivative is defined on $\mathcal{S}$ as

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W_{t_1}, \dots, W_{t_n}) 1_{[0,t_i]}(t), \qquad t \in T,$$

if F has the form (2). The operator D is closable and can be extended to the closure of $\mathcal{S}$ with respect to the seminorm

$$\|F\|_{k,p}^p = \mathbb{E}|F|^p + \sum_{j=1}^{k} \mathbb{E}\|D^{(j)}F\|_{L^p(T^j)}^p$$

($D^{(i)}$ denoting the ith iterated derivative). Note that if F is $\mathbb{F}_A$-measurable (A being a Borel subset of $\mathbb{R}$), then $DF = 0$ on $A^c \times \Omega$.

The adjoint of D is denoted by δ and is called the Skorohod integral. That is, δ is defined on its domain

$$\mathrm{Dom}(\delta) = \left\{ u \in L^2(T \times \Omega) \middle| \left| \mathbb{E}\int_0^T u_s D_s F \mathrm{d}s \right| \leqslant C\|F\|_{L^2(\Omega)} \right\}$$

and is given by the duality relationship

$$\mathbb{E}(F\delta(u)) = \mathbb{E}\int_0^T u_s D_s F \mathrm{d}s, \qquad u \in \mathrm{Dom}(\delta),\ F \in \mathcal{S}.$$

Recall the formula for the covariance of two Skorohod integrals,

$$\mathbb{E}(\delta(u)\delta(v)) = \mathbb{E}\int_0^T u_\alpha v_\alpha \, \mathrm{d}\alpha + \mathbb{E}\int_0^T\int_0^T D_\beta u_\alpha D_\alpha v_\beta \, \mathrm{d}\alpha \, \mathrm{d}\beta. \tag{3}$$

Also recall the commutativity relationship between the derivative operator and the Skorohod integral: if $u \in \mathrm{Dom}(\delta)$ with $D_t u \in \mathrm{Dom}(\delta)$, then

$$D_t\delta(u) = u_t + \delta(D_t u), \qquad t \in T. \tag{4}$$

By $\mathbb{L}^{k,p}$ we denote the set $L^p(T; \mathbb{D}^{k,p})$, for $k \geqslant 1$ and $p \geqslant 2$, and $\mathbb{L}^{\infty,p} = \bigcap_{k\geqslant 1} \mathbb{L}^{k,p}$. Note that $\mathbb{L}^{k,p}$ is a subset of the domain of δ. Meyer's inequality implies

$$\mathbb{E}|\delta(u)|^p \leqslant \|u\|_{1,p}^p. \tag{5}$$

The following generalized Ocone–Clark formula was given in Nualart and Pardoux (1988):

$$F = \mathbb{E}\big(F|\mathbb{F}_{[s,t]^c}\big) + \int_s^t \mathbb{E}\big(D_\alpha F|\mathbb{F}_{[\alpha,t]^c}\big)\mathrm{d}W_\alpha, \qquad \text{for } F \in \mathbb{D}^{1,2}. \tag{6}$$

If the process $u \in \mathbb{L}^{1,2}$, then $u1_{[0,t]}$ belongs to $\mathrm{Dom}(\delta)$ for every t, and we can consider the indefinite Skorohod integral $X_t = \delta\,(u1_{[0,t]}) = \int_0^t u_s \, \mathrm{d}W_s$.

3. Martingale-type stochastic calculus for anticipating integral processes

3.1. Representation for anticipating integrals and Itô formula

Our construction is based on the following observation. Let $X_t = \int_0^t u_\alpha \, \mathrm{d}W_\alpha$ be a Skorohod integral process with integrand regular enough in the Malliavin sense (e.g., $u \in \mathbb{L}^{2,2}$), and let us apply (6) to the integrand u. By interchanging the two Skorohod integrals (see Nualart and Zakai 1988), we obtain

$$\begin{aligned} X_t &= \int_0^t \mathbb{E}\big[u_\alpha|\mathbb{F}_{[\alpha,t]^c}\big]\mathrm{d}W_\alpha + \int_0^t\int_\alpha^t \mathbb{E}\big[D_\beta u_\alpha|\mathbb{F}_{[\beta,t]^c}\big]\mathrm{d}W_\beta \, \mathrm{d}W_\alpha \\ &= \int_0^t \mathbb{E}\big[u_\alpha|\mathbb{F}_{[\alpha,t]^c}\big]\mathrm{d}W_\alpha + \int_0^t\left(\int_0^\beta \mathbb{E}\big[D_\beta u_\alpha|\mathbb{F}_{[\beta,t]^c}\big]\mathrm{d}W_\alpha\right)\mathrm{d}W_\beta \\ &= \int_0^t \mathbb{E}\big[u_\alpha|\mathbb{F}_{[\alpha,t]^c}\big]\mathrm{d}W_\alpha + \int_0^t\left[\mathbb{E}\left(\int_0^\beta D_\beta u_\alpha \, \mathrm{d}W_\alpha\right)|\mathbb{F}_{[\beta,t]^c}\right]\mathrm{d}W_\beta. \end{aligned}$$

Let $r_\beta = \int_0^\beta D_\beta u_\alpha \, \mathrm{d}W_\alpha$. Thus, if $v = u + r$, X_t can be written as

$$X_t = \int_0^t \mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t]^c}\big]\mathrm{d}W_\alpha.$$

Let us define, for $k \geqslant 1$ and $p \geqslant 2$, the sets of processes

$$\mathcal{M}^{k,p} = \left\{ X = (X_t)_{t\in T},\, X_t = \int_0^t u_s \,\mathrm{d}W_s,\, u \in \mathbb{L}^{k,p} \right\}$$

and

$$\mathcal{N}^{k,p} = \left\{ Y = (Y_t)_{t\in T},\, Y_t = \int_0^t \mathbb{E}\big[v_s | \mathbb{F}_{[s,t]^c}\big] \mathrm{d}W_s,\, v \in \mathbb{L}^{k,p} \right\}.$$

We refer to the elements of $\mathcal{N}^{k,p}$ as *Itô–Skorohod integral processes.*

The following two propositions will show that there exists a strong relation between the class of Skorohod integrals and the class of Itô–Skorohod integrals.

Proposition 1. *Let $u = (u_t)_{t\in T}$ be a stochastic process belonging to the Sobolev space $\mathbb{L}^{k,p}$, with $k \geqslant 3$, $p > 2$. Then there exists a unique process $v \in \mathbb{L}^{k-2,p}$ such that $X_t = \int_0^t \mathbb{E}\big[v_\alpha | \mathbb{F}_{[\alpha,t]^c}\big] \mathrm{d}W_\alpha$ for every $t \in T$. Moreover, $v_\cdot = u_\cdot + \int_0^\cdot D_\cdot u_s \,\mathrm{d}W_s$.*

Proof. The existence of v follows from computations given above. Let us show that it belongs to $\mathbb{L}^{k-2,p}$. Write $r_t = \int_0^t D_t u_s \,\mathrm{d}W_s$. By Meyer's inequality and the properties (4) and (5) of the Skorohod integral,

$$\begin{aligned}
\|r\|_{1,p}^p &\leqslant \mathbb{E}\int_T |\delta\big(D_t u.1_{[0,t]}(\cdot)\big)|^p \,\mathrm{d}t + \mathbb{E}\int_T\int_T |D_s\delta\big(D_t u.1_{[0,t]}(\cdot)\big)|^p \,\mathrm{d}s\,\mathrm{d}t \\
&\leqslant C_p\left(\mathbb{E}\int_T\int_T |D_t u_s|^p \,\mathrm{d}s\,\mathrm{d}t + \mathbb{E}\int_T\int_T |\delta\big(D_s D_t u.1_{[0,t]}(\cdot)\big)|^p \,\mathrm{d}s\,\mathrm{d}t\right) \\
&\leqslant C_p\left(\mathbb{E}\int_T\int_T |D_t u_s|^p \,\mathrm{d}s\,\mathrm{d}t + \mathbb{E}\int_T\int_T\int_T |D_s D_t u_\alpha|^p \,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\alpha \right.\\
&\quad \left. + \mathbb{E}\int_T\int_T\int_T\int_T |D_r D_s D_t u_\alpha|^p \,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\alpha\right) \leqslant C_p \|u\|_{3,p}^p.
\end{aligned}$$

In general, it is no more difficult to prove that $\|r\|_{k-2,p} \leqslant C_p\|u\|_{k,p}$.

Suppose now that there exists another process $v' \in \mathbb{L}^{k-2,p}$ satisfying $X_t = \int_0^t \mathbb{E}[v'_\alpha|_{\mathbb{F}}[\alpha,_{t]^c}]\mathrm{d}W_\alpha$ for every $t \in T$, and let $z = v - v'$. Then $\int_0^t \mathbb{E}\big[z_\alpha | \mathbb{F}_{[\alpha,t]^c}\big] \mathrm{d}W_\alpha = 0$ and, taking the Malliavin derivative, we obtain, by (4),

$$1_{[0,t]}(s)\mathbb{E}\big[z_s|\mathbb{F}_{[s,t]^c}\big] + \int_s^t \mathbb{E}\big[D_s z_\alpha | \mathbb{F}_{[\alpha,t]^c}\big] \mathrm{d}W_\alpha = 0.$$

We know (see Nualart 1995) that a Skorohod integral process has a continuous modification if the integrand belongs to $\mathbb{L}^{1,p}$ with $p > 2$. Therefore, taking the limit as $t \to s$ in the above identity, we obtain $z = 0$ on $T \times \Omega$. □

We show that also the Itô–Skorohod integral processes can be written as a Skorohod integral process.

Proposition 2. *Let $(v_t)_{t\in T}$ be a stochastic process in $\mathbb{L}^{k,p}$, with $k \geqslant 3$ and $p > 2$. Then the process $Y = (Y_t)_{t\in T}$, $Y_t = \int_0^t \mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t]^c}\big]\,\mathrm{d}W_\alpha$, admits a Skorohod integral representation, that is, there exists a unique process $u \in \mathbb{L}^{k-2,p}$ such that $Y_t = \int_0^t u_s\,\mathrm{d}W_s$, for every $t \in T$. Moreover, u is given by*

$$u_t = v_t - \int_0^t \mathbb{E}\big[D_t v_s|\mathbb{F}_{[s,t]^c}\big]\,\mathrm{d}W_s. \tag{7}$$

Proof. *Existence.* We will use the criteria of Minh Duc and Nualart (1990) to show that the process Y admits a Skorohod integral representation. By Proposition 2.1 of Minh Duc and Nualart (1990) we have to prove that the following properties are satisfied:

(a) $Y_0 = 0$.
(b) $Y_t \in L^2(\Omega)$ for every $t \in T$.
(c) $\mathbb{E}\big(Y_t - Y_s|\mathbb{F}_{[s,t]^c}\big) = 0$.
(d) If $\Delta:\ 0 = t_0 < t_1 < \ldots < t_n = 1$ denotes a partition of T, the quadratic variation $V(Y) := \sup_\Delta \mathbb{E}\sum_{i=0}^{n-1}(Y_{t_{i+1}} - Y_{t_i})^2$ is finite.

Clearly, conditions (a) and (b) are satisfied. To see (c), we note that

$$\begin{aligned}\mathbb{E}\big(Y_t|\mathbb{F}_{[s,t]^c}\big) &= \mathbb{E}\left[\int_0^t \mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t]^c}\big]\,\mathrm{d}W_\alpha|\mathbb{F}_{[s,t]^c}\right] = \int_0^s \mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t]^c}\big]\,\mathrm{d}W_\alpha\\ &= \mathbb{E}\left[\int_0^s \mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,s]^c}\big]\,\mathrm{d}W_\alpha|\mathbb{F}_{[s,t]^c}\right] = \mathbb{E}\big(Y_s|\mathbb{F}_{[s,t]^c}\big).\end{aligned}$$

Concerning (d), we have, by (3),

$$\begin{aligned}\mathbb{E}(Y_{t_{i+1}} - Y_{t_i})^2 &= \mathbb{E}\left(\int_0^{t_{i+1}} \mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t_{i+1}]^c}\big]\,\mathrm{d}W_\alpha\right)^2\\ &\quad - 2\mathbb{E}\left(\int_0^{t_{i+1}} \mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t_{i+1}]^c}\big]\,\mathrm{d}W_\alpha\right)\left(\int_0^{t_i} \mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t_i]^c}\big]\,\mathrm{d}W_\alpha\right) + \left(\int_0^{t_i} \mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t_i]^c}\big]\,\mathrm{d}W_\alpha\right)^2\\ &= \mathbb{E}\int_{t_i}^{t_{i+1}} \mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t_{i+1}]^c}\big]^2\,\mathrm{d}\alpha + \mathbb{E}\int_0^{t_i}\big(\mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t_i]^c}\big] - \mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t_{i+1}]^c}\big]\big)^2\,\mathrm{d}\alpha.\end{aligned} \tag{8}$$

Another application of (6) and the properties of the Malliavin derivative yields, for $\alpha < t_i$,

$$\begin{aligned}\mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t_i]^c}\big] - \mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t_{i+1}]^c}\big] &= \mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t_i]^c}\big] - \mathbb{E}\big[\mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t_i]^c}\big]|\mathbb{F}_{[\alpha,t_{i+1}]^c}\big]\\ &= \int_{t_i}^{t_{i+1}} \mathbb{E}\big[\mathbb{E}\big[D_\beta v_\alpha|\mathbb{F}_{[\alpha,t_i]^c}\big]|\mathbb{F}_{[\beta,t_{i+1}]^c}\big]\,\mathrm{d}W_\beta,\end{aligned}$$

and therefore

$$\mathbb{E}\big(\mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t_i]^c}\big]-\mathbb{E}\big[v_\alpha|\mathbb{F}_{[\alpha,t_{i+1}]^c}\big]\big)^2 = \mathbb{E}\int_{t_i}^{t_{i+1}}\big(\mathbb{E}\big[\mathbb{E}\big[D_\beta v_\alpha|\mathbb{F}_{[\alpha,t_i]^c}\big]|\mathbb{F}_{[\beta,t_{i+1}]^c}\big]\big)^2\,\mathrm{d}\beta$$

$$\leqslant \mathbb{E}\int_{t_i}^{t_{i+1}}(D_\beta v_\alpha)^2\,\mathrm{d}\beta. \tag{9}$$

Finally, by (8) and (9), we obtain that $\sup_\Delta \mathbb{E}\sum_{i=0}^{n-1}(Y_{t_{i+1}}-Y_{t_i})^2$ is bounded by

$$\sup_\Delta \mathbb{E}\sum_{i=0}^{n-1}\mathbb{E}\int_{t_i}^{t_{i+1}} v_\alpha^2\,\mathrm{d}\alpha+\sup_\Delta \mathbb{E}\sum_{i=0}^{n-1}\mathbb{E}\int_{t_i}^{t_{i+1}}\int_0^1 (D_\beta v_\alpha)^2\,\mathrm{d}\alpha\,\mathrm{d}\beta \leqslant \|v\|_{1,2}^2<\infty.$$

The next step is to find an expression for u in terms of v. In order to do that, we consider the limits

$$L^2(\Omega)-\lim_{\varepsilon\to 0} D_t Y_{t+\varepsilon} \text{ and } L^2(\Omega)-\lim_{\varepsilon\to 0} D_t Y_{t-\varepsilon},$$

and we have that

$$u_t+\delta\big(1_{[0,t]}D_t u\big)=v_t \text{ and } \delta\big(1_{[0,t]}D_t u\big)=\delta\big(1_{[0,t]}\mathbb{E}\big[D_t v\cdot|\mathcal{F}_{(\cdot,t]^c}\big]\big). \tag{10}$$

Therefore (7) holds and, as in the proof of Propostition 1, $u\in\mathbb{L}^{k-2,p}$.

Uniqueness. Suppose that there exists another process $u'\in\mathbb{L}^{k-2,p}$ satisfying $Y_t=\int_0^t u'_s\,\mathrm{d}W_s$. Then the difference $z=u-u'$ satisfies $\int_0^t z_s\,\mathrm{d}W_s=0$ for every t and therefore its quadratic variation, which equals $\int_0^t z_s^2\,\mathrm{d}s$, is zero. This completes the proof.

□

Remark 1. If the process v in Proposition 2 is adapted, then $u=v$. Otherwise this is not true, although averaging over the future time dependence should intuitively act in a smoothing manner over the integrand.

Remark 2. Formula (7) can be understood as the effect of the symmetrization of the kernels appearing in the chaotic expression of the Skorohod integral. Recall that if the process u admits the Wiener–Itô chaos expansion $u_t=\sum_n I_n(f_n(\cdot,t))$, where I_n is the multiple Wiener–Itô integral of order n and ‘ $\cdot$ ’ denotes n variables, then $\delta(u)=\sum_n I_n(\tilde{f}_n)$, with $\tilde{f}_n$ denoting the symmetrization of f_n in $n+1$ variables.

Remark 3. Relation (7) can be directly verified in the case where $u=DF$, with $F\in\mathbb{D}^{1,2}$. We will use the notation $\mathbb{F}_{t^c}$ for $\mathbb{F}_{[0,t]^c}$. We note that, by (6), $r_\alpha=D_\alpha F-\mathbb{E}[D_\alpha F|\mathbb{F}_{\alpha^c}]$, and therefore

$$\int_0^t(u_\alpha-r_\alpha)\,\mathrm{d}W_\alpha = \int_0^t\mathbb{E}[D_\alpha F|\mathbb{F}_{\alpha^c}]\mathrm{d}W_\alpha=\int_0^T\mathbb{E}[D_\alpha F|\mathbb{F}_{\alpha^c}]\mathrm{d}W_\alpha-\int_t^T\mathbb{E}[D_\alpha F|\mathbb{F}_{\alpha^c}]\mathrm{d}W_\alpha$$

$$= F-\mathbb{E}[F|\mathbb{F}_{t^c}]=\int_0^t\mathbb{E}\big[D_\alpha F|\mathbb{F}_{[\alpha,t]^c}\big]\mathrm{d}W_\alpha.$$

Remark 4. If $v \in \mathbb{L}^{1,p}$ with $p > 2$, then we can prove that the process Y, $Y_t = \int_0^t \mathbb{E}\left[v_s | \mathbb{F}_{[s,t]^c}\right] \mathrm{d}W_s$, admits a continuous modification. Indeed, using (6) and the Kolmogorov criterion,

$$\begin{aligned} Y_t - Y_s &= \int_s^t \mathbb{E}\left[v_\alpha | \mathbb{F}_{[\alpha,t]^c}\right] \mathrm{d}W_\alpha + \int_0^s \left(\mathbb{E}\left[v_\alpha | \mathbb{F}_{[\alpha,t]^c}\right] - \mathbb{E}\left[v_\alpha | \mathbb{F}_{[\alpha,s]^c}\right]\right) \mathrm{d}W_\alpha \\ &= \int_s^t \mathbb{E}\left[v_\alpha | \mathbb{F}_{[\alpha,t]^c}\right] \mathrm{d}W_\alpha - \int_0^s \int_s^t \mathbb{E}\left[\mathbb{E}\left[D_\beta u_\alpha | \mathbb{F}_{[\alpha,s]^c} | \mathbb{F}_{[\beta,t]^c}\right]\right] \mathrm{d}W_\beta \, \mathrm{d}W_\alpha \end{aligned}$$

and therefore, if C_p is a generic constant depending only on p and $\|u\|_{1,p}$,

$$\mathbb{E}|Y_t - Y_s|^p \leqslant C_p(t-s)^{p/2-1}\left(\int_s^t \mathbb{E}|u_\alpha|^p \,\mathrm{d}\alpha + \int_0^s \int_s^t \mathbb{E}\left|D_\beta u_\alpha\right|^p \mathrm{d}\beta \, \mathrm{d}\alpha\right) \leqslant C_p(t-s)^{p/2-1}.$$

As a consequence the processes X and Y from Propositions 1 and 2 coincide as stochastic processes.

Our main result is a consequence of Propositions 1 and 2 and of Remarks 1–4.

Theorem 1. *For every $p > 2$, the sets of stochastic processes $\mathcal{M}^{\infty,p}$ and $\mathcal{N}^{\infty,p}$ coincide.*

Since the Itô–Skorohod integrals are also indefinite Skorohod integrals, the Skorohod stochastic calculus developed in Nualart and Pardoux (1988) or Nualart (1995) can be applied to Y. We now state an Itô formula in the Skorohod sense for Itô–Skorohod integral processes.

Proposition 3. *Let $F \in C^2(\mathbb{R})$ and denote, for every $t \in T$, $r_t = \delta(1_{[0,t]}\mathbb{E}[D_t u_\cdot | \mathcal{F}_{(\cdot,t]^c}])$. Assume that $u - r \in \mathbb{L}^{2,4}$, and let $Y_t = \int_0^t \mathbb{E}\left[u_\alpha | \mathbb{F}_{[\alpha,t]^c}\right] \mathrm{d}W_\alpha$. Then the following Itô formula holds for the process Y_t:*

$$F(Y_t) = F(0) + \int_0^t F'(Y_\alpha)(u_\alpha - r_\alpha)\,\mathrm{d}W_\alpha + \frac{1}{2}\int_0^t F''(Y_\alpha)(u_\alpha^2 - r_\alpha^2)\,\mathrm{d}\alpha. \tag{11}$$

Proof. By (7), Y_t can be written as $Y_t = \int_0^t v_\alpha \,\mathrm{d}W_\alpha$ with $v_\alpha = u_\alpha - r_\alpha \in \mathbb{L}^{2,4}$. We can write Itô's formula for Skorohod integrals (see Nualart 1995) as

$$F(Y_t) = F(0) + \int_0^t F'(Y_\alpha) v_\alpha \,\mathrm{d}W_\alpha + \frac{1}{2}\int_0^t F''(Y_\alpha) v_\alpha^2 \,\mathrm{d}\alpha + \int_0^t F''(Y_\alpha) v_\alpha \delta(1_{(0,\alpha)} D_\alpha v)\,\mathrm{d}\alpha$$

and since, by (10), $\delta\left(1_{[0,t]} D_t v\right) = r_t$ for every $t \in T$, we obtain

$$\begin{aligned} F(Y_t) &= F(0) + \int_0^t F'(Y_\alpha)(u_\alpha - r_\alpha)\,\mathrm{d}W_\alpha + \int_0^t F''(Y_\alpha) u_\alpha v_\alpha \,\mathrm{d}\alpha - \frac{1}{2}\int_0^t F''(Y_\alpha) v_\alpha^2 \\ &= F(0) + \int_0^t F'(Y_\alpha)(u_\alpha - r_\alpha)\,\mathrm{d}W_\alpha + \frac{1}{2}\int_0^t F''(Y_\alpha)(u_\alpha^2 - r_\alpha^2)\,\mathrm{d}\alpha. \end{aligned}$$

□

Remark 5. If the integrand u is adapted to the Brownian filtration, then $D_\alpha u_s = 0$ for $\alpha > s$ and therefore $r = 0$ on $T \times \Omega$. We thus retrieve the classical Itô formula. Note that the quadratic variation of Y is $\int_0^t (u_\alpha - r_\alpha)^2 \, \mathrm{d}\alpha$ if $u - r \in \mathbb{L}^{1,2}$.

3.2. Itô-type stochastic calculus for anticipating integrals

We have just shown that the Skorohod stochastic calculus can be used for a process $Y \in \mathcal{N}^{k,p}$ if the integrand u belongs to a large enough Sobolev space. We are now interested in the converse direction. That is, since a process in $\mathcal{N}^{k,p}$ seems to be 'nicer' than a Skorohod integral process, can we find interesting properties for a Skorohod integral process using the properties of the processes in $\mathcal{N}^{k,p}$?

First, note that the integral $Y_t = \int_0^t \mathbb{E}\left[u_\alpha | \mathbb{F}_{[\alpha,t]^c}\right] \mathrm{d}W_\alpha$ exists even for $u \in L^2(T \times \Omega)$ and has similarities with a classical Itô integral. Observe that this integral is an 'isometry' in the sense that

$$\mathbb{E}\left(\int_0^t \mathbb{E}\left[u_\alpha | \mathbb{F}_{[\alpha,t]^c}\right] \mathrm{d}W_\alpha\right)^2 = \mathbb{E}\int_0^t \left(\mathbb{E}\left[u_\alpha | \mathbb{F}_{[\alpha,t]^c}\right]\right)^2 \mathrm{d}\alpha.$$

The following lemma will be the basic tool for developing an 'Itô'-type stochastic calculus for anticipating processes.

Lemma 1. *For every $\lambda \leqslant t$ and $u \in L^2(T \times \Omega)$, let us define Y_t^λ by*

$$Y_t^\lambda = \int_0^\lambda \mathbb{E}\left[u_\alpha | \mathbb{F}_{[\alpha,t]^c}\right] \mathrm{d}W_\alpha. \tag{12}$$

Then for fixed $t \in T$, the process $\left(Y_t^\lambda\right)_{\lambda \leqslant t}$ is an $\mathcal{F}_{[\lambda,t]^c}$ martingale and we have

$$\lim_{\lambda \to t, \lambda \leqslant t} Y_t^\lambda = Y_t \text{ almost surely and in } L^2. \tag{13}$$

Proof. It easy to see that we can express Y_t^λ as

$$\begin{aligned} Y_t^\lambda = &\left[\mathbb{E}\left(\left(\int_0^t \mathbb{E}\left[u_\alpha | \mathbb{F}_{[\alpha,t]^c}\right] \mathrm{d}W_\alpha\right) | \mathbb{F}_{[\lambda,t]^c}\right]\right] = \mathbb{E}\left(Y_t | \mathbb{F}_{[\lambda,t]^c}\right) \\ = &\left[\mathbb{E}\left(\left(\int_0^\lambda \mathbb{E}\left[u_\alpha | \mathbb{F}_{[\alpha,\lambda]^c}\right] \mathrm{d}W_\alpha\right) | \mathbb{F}_{[\lambda,t]^c}\right]\right] = \mathbb{E}\left(Y_\lambda | \mathbb{F}_{[\lambda,t]^c}\right). \end{aligned}$$

The key observation is that, for every $t \in T$, the process $\left(Y_t^\lambda\right)_{\lambda \leqslant t}$ is an $\mathcal{F}_{(\lambda,t]^c}$ martingale and $L^2(\Omega) - \lim_{\lambda \to t, \lambda \leqslant t} Y_t^\lambda = Y_t$. Indeed,

$$\lim_{\lambda \to t, \lambda \leqslant t} \| Y_t^\lambda - Y_t \|_{L^2(\Omega)}^2 = \lim_{\lambda \to t, \lambda \leqslant t} \mathbb{E}\left(\int_\lambda^t \mathbb{E}\left[u_\alpha | \mathbb{F}_{[\alpha,t]^c}\right] \mathrm{d}W_\alpha\right) = \lim_{\lambda \to t, \lambda \leqslant t} \mathbb{E}\int_\lambda^t \left(\mathbb{E}\left[u_\alpha | \mathbb{F}_{[\alpha,t]^c}\right]\right)^2 \mathrm{d}\alpha = 0.$$

Since

$$\sup_{\lambda\leqslant t}\mathbb{E}|Y_t^\lambda|^2=\sup_{\lambda\leqslant t}\mathbb{E}\int_0^\lambda\left(\mathbb{E}\left[u_\alpha|\mathbb{F}_{[\alpha,t]^c}\right]\right)^2\mathrm{d}\alpha\leqslant\mathbb{E}\int_0^\lambda u_\alpha^2\,\mathrm{d}\alpha\leqslant\|u\|^2_{L^2(T\times\Omega)},$$

the martingale $\left(Y_t^\lambda\right)_{\lambda\leqslant t}$ is uniformly integrable and, by the martingale convergence theorem, we obtain that the prior limit exists almost surely. □

Property (13) will be the key to developing an alternative stochastic calculus for anticipating integral processes. In general, we will use the classical Itô theory for Y_t^λ and take the limits almost surely as λ goes to t.

A different Itô formula can be derived for Itô–Skorohod integral processes using the martingale stochastic calculus.

Proposition 4. *Let $f\in C^2(\mathbb{R})$, $u\in L^2(T\times\Omega)$ and $Y_t=\int_0^t\mathbb{E}\left[u_\alpha|\mathbb{F}_{[\alpha,t]^c}\right]\mathrm{d}W_\alpha$. Then*

$$f(Y_t)=f(0)+\int_0^t f'(Y_t^\beta)\mathbb{E}\left[u_\beta|\mathbb{F}_{[\beta,t]^c}\right]\mathrm{d}W_\beta+\frac{1}{2}\int_0^t f''(Y_t^\beta)\left(\mathbb{E}\left[u_\beta|\mathbb{F}_{[\beta,t]^c}\right]\right)^2\mathrm{d}\beta. \tag{14}$$

Proof. Let us write Itô's formula for $(Y_t^\lambda)_{\lambda\leqslant t}$ which is an $\mathcal{F}_{(\lambda,t]^c}$-martingale

$$f(Y_t^\lambda)=f(0)+\int_0^\lambda f'(Y_t^\beta)\mathbb{E}\left[u_\beta|\mathbb{F}_{[\beta,t]^c}\right]\mathrm{d}W_\beta+\frac{1}{2}\int_0^\lambda f''(Y_t^\beta)\left(\mathbb{E}\left[u_\beta|\mathbb{F}_{[\beta,t]^c}\right]\right)^2\mathrm{d}\beta \tag{15}$$

Note that

$$M_t^\lambda=\int_0^\lambda f'(Y_t^\beta)\mathbb{E}\left[u_\beta|\mathbb{F}_{[\beta,t]^c}\right]\mathrm{d}W_\beta=\mathbb{E}\left[\int_0^t f'(Y_t^\beta)\mathbb{E}\left[u_\beta|\mathbb{F}_{[\beta,t]^c}\right]\mathrm{d}W_\beta|\mathbb{F}_{[\lambda,t]^c}\right]$$

is again an $\mathcal{F}_{(\lambda,t]^c}$-martingale converging almost surely (see the proof of Lemma 1) to

$$\int_0^t f'(Y_t^\beta)\mathbb{E}\left[u_\beta|\mathbb{F}_{[\beta,t]^c}\right]\mathrm{d}W_\beta.$$

Letting $\lambda\to t$, we obtain almost surely

$$f(Y_t)=f(0)+\int_0^t f'(Y_t^\beta)\mathbb{E}\left[u_\beta|\mathbb{F}_{[\beta,t]^c}\right]\mathrm{d}W_\beta+\frac{1}{2}\int_0^t f''(Y_t^\beta)\left(\mathbb{E}\left[u_\beta|\mathbb{F}_{[\beta,t]^c}\right]\right)^2\mathrm{d}\beta.$$

□

Since a Skorohod integral process can be written as

$$X_t=\int_0^t u_\alpha\,\mathrm{d}W_\alpha=\int_0^t\mathbb{E}\left[(u_\alpha+r_\alpha|\mathbb{F}_{[\alpha,t]^c}\right]\mathrm{d}W_\alpha$$

if $r_\alpha=\delta\left(1_{[0,\alpha]}D_\alpha u\right)$, an Itô formula for X can be written as

$$f(X_t)=f(0)+\int_0^t f'(X_t^\beta)\mathbb{E}\left[(u_\beta+r_\beta)|\mathbb{F}_{[\beta,t]^c}\right]\mathrm{d}W_\beta+\frac{1}{2}\int_0^t f''(X_t^\beta)\left(\mathbb{E}\left[(u_\beta+r_\beta)|\mathbb{F}_{[\beta,t]^c}\right]\right)^2\mathrm{d}\beta \tag{16}$$

where $X_t^\beta = \int_0^\beta \mathbb{E}[(u_s + r_s)|\mathbb{F}_{[s,t]^c}]\mathrm{d}W_s$.

We can also generalize the Tanaka formula and introduce an extension of the martingale local time. By the Tanaka formula for Y_t^λ, we obtain

$$|Y_t^\lambda - a| = |a| + \int_0^\lambda \operatorname{sgn}(Y_t^\beta)\mathbb{E}\left[u_\beta|\mathbb{F}_{[\beta,t]^c}\right]\mathrm{d}W_\beta + L_t^\lambda(a), \tag{17}$$

where the local time $(L_t^\lambda(a))_{\lambda\leqslant t}$ satisfies

$$L_t^\lambda(a) = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^\lambda 1_{(a-\varepsilon,a+\varepsilon)}(Y_t^\beta)\left(\mathbb{E}\left[u_\beta|\mathbb{F}_{[\beta,t]^c}\right]\right)^2\mathrm{d}\beta \tag{18}$$

and the occupation time formula: for every Borel function Φ,

$$\int_0^\lambda \Phi(Y_t^\beta)\left(\mathbb{E}\left[u_\beta|\mathbb{F}_{[\beta,t]^c}\right]\right)^2\mathrm{d}\beta = \int_{-\infty}^{\infty}\Phi(a)L_t^\lambda(a)\,\mathrm{d}a. \tag{19}$$

Taking the limit as $\lambda \to t$ in (17), it is clear that the following limits exist almost surely and, by the dominated convergence theorem, in $L^2(\Omega)$:

$$|Y_t^\lambda - a| \to |Y_t - a| \quad \text{and} \quad \int_0^\lambda \operatorname{sgn}(Y_t^\beta)\,\mathrm{d}W_\beta \to \int_0^t \operatorname{sgn}(Y_t^\beta)\,\mathrm{d}W_\beta.$$

Proposition 5. *For every $t \in T$, let $(L_t^\lambda(a))_{a\in\mathbb{R},\lambda\leqslant t}$ be the local time of the martingale $(Y_t^\lambda)_{\lambda\leqslant t}$ given by* (12). *Then, for every a real and $t \in T$, the following limit exists in L^2 and almost surely:*

$$L_t(a) := \lim_{\lambda\to t,\lambda<t} L_t^\lambda(a).$$

Definition 1. *The process $(L_t(a))_{a\in\mathbb{R},t\in T}$ given by Proposition* 4 *will be called* the generalized local time *of the Itô–Skorohod process Y.*

Obviously, by (17), (18) and (19), we can prove the following result.

Proposition 6. *Let $(L_t(a))_{a\in\mathbb{R},t\in T}$ be the generalized local time of the Itô–Skorohod integral process $Y_t = \int_0^t \mathbb{E}[u_\alpha|\mathbb{F}_{[\alpha,t]^c}]\mathrm{d}W_\alpha$ with $u \in L^2(T\times\Omega)$, and let Φ be a Borel function. Then we have the Tanaka formula,*

$$|Y_t - a| = |a| + \int_0^t \operatorname{sgn}(Y_t^\beta - a)\mathbb{E}\left[u_\beta|\mathbb{F}_{[\beta,t]^c}\right]\mathrm{d}W_\beta + L_t(a),$$

the occupation time formula,

$$\int_0^t \Phi(Y_t^\beta)\left(\mathbb{E}\left[u_\beta|\mathbb{F}_{[\beta,t]^c}\right]\right)^2\mathrm{d}\beta = \int_{-\infty}^{\infty}\Phi(a)L_t(a)\,\mathrm{d}a,$$

and

$$L_t(a) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_0^t 1_{(a-\varepsilon,a+\varepsilon)}(Y_t^\beta)\big(\mathbb{E}\big[u_\beta|\mathbb{F}_{[\beta,t]^c}\big]\big)^2 \,\mathrm{d}\beta.$$

Using the same arguments as before, if $u \in \mathbb{L}^{2,2}$ and $X_t = \int_0^t u_\alpha \,\mathrm{d}W_\alpha$, we can define a generalized local time for X as

$$L_t^X(a) = \lim_{\lambda\to t} L_t^{X,\lambda}(a) \text{ almost surely in } L^2$$

$L_t^{X,\lambda}(a)$ being the local time of the martingale $X_t^\lambda = \int_0^\lambda \mathbb{E}(u_\beta + \delta(1_{[0,\beta]}D_\beta u))|\mathcal{F}_{[\beta,t]^c})\,\mathrm{d}W_\beta$, and the results of Proposition 5 will hold with $u_t + \delta(1_{[0,t]}D_t u)$ instead of u_t.

Remark 6. In Imkeller and Nualart (1994) a local time $l_t^X(a)$ of the anticipating process $X_t = \int_0^t u_\alpha \,\mathrm{d}W_\alpha$ was introduced as the density of the occupation measure $\mu(B) = \int_0^T 1_B(X_s)\,\mathrm{d}s$, where B is a subset of the real line. Its existence is proved using non-trivial conditions on the integrand. This local time satisfies the Tanaka formula

$$(X_t - a)^+ = (-a) + \int_0^t 1_{(a,\infty)}(X_s)u_s \,\mathrm{d}W_s + \tilde{l}_t^X(a),$$

where $\tilde{l}_t^X(a) = \int_0^t u_s\big(\frac{1}{2}u_s + r_s\big)\, l(\mathrm{d}s,\, a)$ and $r_s = \delta\big(1_{[0,s]}D_s u\big)$. The processes L^X and $\tilde{l}^X$ do not coincide in general, they coincide only if the integrand is adapted, but they always have the same expectation.

4. Some consequences

The aim of this section is to present some applications of the stochastic calculus introduced in this paper. We do not claim to give an exhaustive list of possible consequences. We believe that the correspondence between Skorohod and Itô–Skorohod integrals could open the door to further exploitation. We have chosen here only two immediate facts. The first concerns Burkholder inequalities for Skorohod integrals. The upper bound is a version of Meyer's inequality, and the lower bound seems to be new and interesting in itself. The second consists of an Itô formula for the product of a martingale and a backward martingale; in a particular case we find rather surprising identities for the functionals of the Brownian motion.

4.1. Burkholder inequalities for Skorohod integrals

Proposition 7. *Let $u \in L^2(T\times\Omega)$, and let Y be the Itô–Skorohod integral process of u. Then, for every $t \in T$ and p real, there exist two constants, $c_1(p) > 0$ and $c_2(p) > 0$, such that*

$$c_1(p)\mathbb{E}\bigg(\int_0^t \big(\mathbb{E}\big[u_\alpha|\mathbb{F}_{[\alpha,t]^c}\big]\big)^2 \,\mathrm{d}\alpha\bigg)^p \leqslant \mathbb{E}|Y_t|^{2p} \leqslant c_2(p)\mathbb{E}\bigg(\int_0^t \big(\mathbb{E}\big[u_\alpha|\mathbb{F}_{[\alpha,t]^c}\big]\big)^2 \,\mathrm{d}\alpha\bigg)^p. \qquad (23)$$

Proof. For the upper bound, we can write, by the classical Burkholder inequality,

$$\mathbb{E}|Y_t|^{2p} \leqslant \mathbb{E}\left(\sup_{\lambda \leqslant t}|Y_t^{\lambda}|^{2p}\right) \leqslant c_2(p)\mathbb{E}\left(\int_0^t \left(\mathbb{E}\left[u_\alpha|\mathbb{F}_{[\alpha,t]^c}\right]\right)^2 \mathrm{d}\alpha\right)^p.$$

Concerning the lower bound, since $\mathbb{E}|Y_t|^{2p} = \mathbb{E}(\lim_{\lambda\to t,\lambda\leqslant t}|Y_t^{\lambda}|^{2p})$ and since $\mathbb{E}|Y_t^{\lambda}|^{2p} \leqslant \mathbb{E}\|u\|_{L^2(T)}^p$, for every $\lambda < t$, by the dominated convergence theorem,

$$\begin{aligned}\mathbb{E}|Y_t|^{2p} &= \lim_{\lambda\to t,\lambda\leqslant t}\mathbb{E}|Y_\lambda|\mathbb{F}_{[\lambda,t]^c}|^p \geqslant c_1(p)\lim_{\lambda\to t,\lambda\leqslant t}\mathbb{E}\left(\int_0^{\lambda}\mathbb{E}\left(u_\alpha|\mathbb{F}_{(\alpha,t]^c}\right)^2 \mathrm{d}\alpha\right)^p \\ &= c_1(p)\mathbb{E}\left(\int_0^t \mathbb{E}\left(u_\alpha|\mathbb{F}_{[\alpha,t]^c}\right)^2 \mathrm{d}\alpha\right)^p.\end{aligned}$$

□

We can write Burkholder inequalities for an anticipating integral process $X_t = \int_0^t u_\alpha \,\mathrm{d}W_\alpha$ where $u \in \mathbb{L}^{1,2}$. That is, for every t, there exist $C_1(p)$, $C_2(p) > 0$ such that

$$C_1(p)\mathbb{E}\left(\int_0^t \left(\mathbb{E}\left[u_\alpha + r_\alpha|\mathbb{F}_{[\alpha,t]^c}\right]\right)^2 \mathrm{d}\alpha\right)^p \leqslant \mathbb{E}\left|X_t\right|^{2p} \leqslant C_2(p)\mathbb{E}\left(\int_0^t \left(\mathbb{E}\left[u_\alpha + r_\alpha|\mathbb{F}_{[\alpha,t]^c}\right]\right)^2 \mathrm{d}\alpha\right)^p$$

with $r_\alpha = \delta(1_{[0,\alpha]}D_\alpha u)$. As far as we know, the lower bound is new.

4.2. Itô formula for the product of a martingale and a backward martingale

Let $M = (M_t)_{t\in T}$ be a Brownian martingale and $N = (N_t)_{t\in T}$ a backward martingale (basically, N_t is adapted to $\mathbb{F}_{t^c}$ for every t and $\mathbb{E}[N_s|\mathbb{F}_{t^c}] = M_t$ for every $s < t$, see Revuz and Yor (1994). Let $a = (a_t)_{t\in T}$ be an adapted square-integrable process such that $M_t = \int_0^t a_s \,\mathrm{d}W_s$. Then the process MN can be expressed as

$$M_tN_t = N_t\int_0^t a_s \,\mathrm{d}W_s = \int_0^t a_sN_t \,\mathrm{d}W_s = \int_0^t a_s\mathbb{E}[N_s|\mathbb{F}_{t^c}]\mathrm{d}W_s = \int_0^t \mathbb{E}\left[a_sN_s|\mathbb{F}_{[s,t]^c}\right]\mathrm{d}W_s.$$

Therefore the process MN belongs to the class of Itô–Skorohod integral processes and, using Itô's formula,

$$\begin{aligned}f(M_tN_t) = {} & f(0) + \int_0^t f'\left(\mathbb{E}\left[M_sN_s|\mathbb{F}_{[s,t]^c}\right]\right)E\left[a_sN_s|\mathbb{F}_{[s,t]^c}\right]\mathrm{d}W_s \\ & + \frac{1}{2}\int_0^t f''\left(\mathbb{E}\left[M_sN_s|\mathbb{F}_{[s,t]^c}\right]\right)\left(E\left[a_sN_s|\mathbb{F}_{[s,t]^c}\right]\right)^2 \mathrm{d}s \\ = {} & f(0) + \int_0^t f'(M_sN_t)a_sN_t \,\mathrm{d}W_s + \frac{1}{2}\int_0^t f''(M_sN_t)a_s^2N_t^2 \,\mathrm{d}s\end{aligned}$$

Considering the particular case $M_t = W_t$ and $N_t = W_1 - W_t$, we obtain

$$f(W_t(W_1 - W_t)) = (W_1 - W_t)\int_0^t f'(W_s(W_1 - W_t))\mathrm{d}W_s + \frac{1}{2}(W_1 - W_t)^2\int_0^t f''(W_s(W_1 - W_t))\mathrm{d}s.$$

For different (non-polynomial) choices of f, the last formula gives surprising identities between the functionals of the Wiener process.

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