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An *S*-transform approach to integration with respect to a fractional Brownian motion

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We give an elementary definition of the (Wick–)Itô integral with respect to a fractional Brownian motion using the expectation, the ordinary Lebesgue integral and the classical (simple) Wiener integral. Then we provide new and simple proofs of some basic properties of this integral, including the so-called fractional Itô isometry. We calculate the expectation of the fractional Itô integral under change of measure and prove a Girsanov theorem for the fractional Itô integral (not only for fractional Brownian motion). We then derive an Itô formula for functionals of a fractional Wiener integral. Finally, we compare our approach with other approaches that yield essentially the same integral.

Keywords: change of measure; fractional Brownian motion; fractional Girsanov theorem; fractional Itô integral; *S*-transform

1. Introduction

Since fractional Brownian motion with Hurst parameter $H \neq \frac{1}{2}$ is not a semimartingale, Itô's integration theory cannot be applied to this family of processes. Therefore different extensions have been proposed. They can be roughly divided into two groups by considering their change-of-variable formulae. An extension is said to be of Itô type if, analogously to the Itô integral for semimartingales, an additional term involving the second derivative occurs. It is said to be of Stratonovich type if the change-of-variable formula coincides with that from ordinary calculus.

After a preliminary section we give a motivation for a simple definition of an integral with respect to a fractional Brownian motion which turns out to be of Itô type. This definition is based on the *S*-transform, an important tool in white noise analysis, but carries over to an arbitrary probability space on which a two-sided Brownian motion lives. To this end we provide a new proof of the injectivity of the *S*-transform. Then we show that the new definition of the fractional Itô integral is an extension of the classical Itô integral to the case $H \neq \frac{1}{2}$ using the classical Girsanov theorem. Moreover, the relationship to the Wick product and the (Hitsuda–)Skorohod integral is explored and a simple proof of the so-called fractional Itô isometry is provided.

In Section 4 we examine the behaviour of the fractional Itô integral under change of measure $\mathrm{d}Q_f =: \mathrm{e}^{I(f)} : \mathrm{d}P$, where $: \mathrm{e}^{I(f)} :$ is the Wick exponential of a function $f \in L^2(\mathbb{R})$. We first calculate its expectation under the measure Q_f. Then we prove a Girsanov theorem for the fractional Itô integral. The essential new feature of this theorem is that it holds for the fractional Itô integral, and not just for fractional Brownian motion as do the Girsanov theorems due to Norros *et al*. (1999), Hu and Øksendal (2003) and Elliott and van der Hoek (2003). Since the fractional Itô integral is implictly based on the Wick product and the Wick product depends on the underlying probability measure, the extension of Girsanov's theorem from fractional Brownian motion to the fractional Itô integral is not at all obvious.

In Section 5 we first prove an Itô formula for functionals of fractional Wiener integrals (with non-constant coefficients in the case where $H > \frac{1}{2}$) – modifying the approach in Bender (2003a) – to show that our integral is of Itô type. We then extend the definition of a (Wick) geometric fractional Brownian motion from Hu and Øksendal (2003) to non-constant coefficients in the case where $H > \frac{1}{2}$. As a corollary, we obtain an Itô formula for geometric fractional Brownian motions. This in turn yields an analogue of the Doléans–Dade equation and thus justifies the name geometric fractional Brownian motion.

We conclude by comparing our S-transform approach with other approaches to Itô-type integration with respect to a fractional Brownian motion. It turns out that our definition is equivalent to the white noise definition (Hu and Øksendal 2003; Elliott and van der Hoek 2003; Bender 2003a) as long as we suppose the integrand and the integral to be $L^2(\Omega)$-valued. However, our definition is much simpler and does not make use of the complicated constructions from the white noise calculus. Compared to the Malliavin calculus approach (Alòs *et al*. 2001), our definition allows a wider class of integrands, particularly for small Hurst parameters. A different generalization of the Malliavin calculus approach has recently been developed independently by Cheridito and Nualart (2002).

2. Preliminaries

2.1. Construction of fractional Brownian motion

Definition 2.1. *A continuous stochastic process* $(B_t^H)_{t\in\mathbb{R}}$ *is called a* (two-sided) fractional Brownian motion with Hurst parameter *H, if the family* $(B_t^H)_{t\in\mathbb{R}}$ *is centred Gaussian with*

$$\mathrm{E}[B_t^H B_s^H] = \tfrac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \qquad t, s \in \mathbb{R}. \tag{1}$$

If $H = \frac{1}{2}$, $B^{1/2}$ *is said to be a* two-sided Brownian motion.

We recall a construction of a fractional Brownian motion starting from a Brownian motion. Let $(\Omega, \mathcal{F}, P)$ be a probability space that carries a two-sided Brownian motion B. For $a, b \in \mathbb{R}$ we define the indicator function

$$\mathbf{1}(a, b)(t) = \begin{cases} 1, & \text{if } a \leqslant t < b, \\ -1, & \text{if } b \leqslant t < a, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Furthermore, let

$$K_H := \Gamma\left(H + \frac{1}{2}\right)\left(\int_0^\infty \left((1+s)^{H-1/2} - s^{H-1/2}\right)\mathrm{d}s + \frac{1}{2H}\right)^{-1/2},$$

and define the operator

$$M_\pm^H f := \begin{cases} K_H D_\pm^{-(H-1/2)} f, & 0 < H < \frac{1}{2} \\ f, & H = \frac{1}{2} \\ K_H I_\pm^{H-1/2} f, & \frac{1}{2} < H < 1. \end{cases} \tag{3}$$

Here $I_\pm^\alpha$, $0 < \alpha < 1$, is the *fractional integral of Weyl's type*, defined by

$$(I_-^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^\infty f(t)(t-x)^{\alpha-1}\mathrm{d}t,$$

$$(I_+^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x f(t)(x-t)^{\alpha-1}\mathrm{d}t,$$

if the integrals exist for almost all $x \in \mathbb{R}$. $D_\pm^\alpha$, $0 < \alpha < 1$, is the *fractional derivative of Marchaud's type* given by ($\epsilon > 0$)

$$(D_{\pm,\epsilon}^\alpha f)(x) := \frac{\alpha}{\Gamma(1-\alpha)} \int_\epsilon^\infty \frac{f(x) - f(x \mp t)}{t^{1+\alpha}}\mathrm{d}t$$

and

$$(D_\pm^\alpha f) := \lim_{\epsilon \to 0+} (D_{\pm,\epsilon}^\alpha f),$$

if the limit exists in $L^p(\mathbb{R})$ for some $p > 1$. The notation $D_\pm^\alpha f \in L^p(\mathbb{R})$ indicates convergence in the $L^p(\mathbb{R})$ norm.

With these definitions we have:

Theorem 2.1. *For* $0 < H < 1$, *let the operators* $M_\pm^H$ *be defined by* (3). *Then* $M_-^H \mathbf{1}(0, t) \in L^2(\mathbb{R})$ *and a fractional Brownian motion* B^H *is given by a continuous version of the Wiener integral* $\int_\mathbb{R} (M_-^H \mathbf{1}(0, t))(s)\mathrm{d}B_s$.

Proof. Using elementary integration one can easily show that the representation of B^H is the well-known Mandelbrot–Van Ness representation (Mandelbrot and Van Ness 1968). More details can be found in Bender (2003a). □

2.2. The *S*-transform

The *S*-transform is an important tool in white noise analysis. Here we give a definition and state some results that do not depend on properties of the white noise space.

We first introduce some notation. $I^B(f)$ denotes the Wiener integral $\int_\mathbb{R} f(s)\mathrm{d}B_s$ for a function $f \in L^2(\mathbb{R})$. If there is no danger of confusion, we shall drop the superscript B.

$|f|_0$ is the usual L^2 norm, and the corresponding inner product is denoted by $(f, g)_0$.

Note that we interpret the functions in $L^2(\mathbb{R})$ and in $\mathcal{S}(\mathbb{R})$, the Schwartz space of smooth rapidly decreasing functions, as real-valued.

$\mathcal{G}$ is the σ-field generated by $\{I(f); f \in L^2(\mathbb{R})\}$, and we define $(L^2) := L^2(\Omega, \mathcal{G}, P)$. $\|\Phi\|_0$ denotes the (L^2) norm.

We can now define the S-transform:

Definition 2.2. *For $\Phi \in (L^2)$, the S-*transform *is defined by*

$$S\Phi(\eta) := \mathrm{E}\big[\Phi \cdot :\mathrm{e}^{I(\eta)}:\big], \qquad \eta \in \mathcal{S}(\mathbb{R}). \tag{4}$$

Here the Wick exponential *of $I(\eta)$ is given by* $:\mathrm{e}^{I(\eta)}: = \mathrm{e}^{I(\eta)-|\eta|_0^2/2}$.

The next theorem states that the S-transform is injective:

Theorem 2.2. *If $(S\Phi)(\eta) = (S\Psi)(\eta)$ for all $\eta \in \mathcal{S}(\mathbb{R})$, then $\Phi = \Psi$.*

This result is well known in the white noise setting. Here we give an elementary proof that relies neither on the Stone–Weierstrass theorem (and hence on topological properties of the probability space) nor on the chaos decomposition. It is inspired by Theorem 4.1 in Hida (1980).

Proof. Let ξ_n, $n \in \mathbb{N}$, be an orthonormal base for $L^2(\mathbb{R})$ in $\mathcal{S}(\mathbb{R})$, for example the Hermite functions. We define $\mathcal{G}_n$ as the σ-field generated by $\{I(\xi_k); k \leqslant n\}$.

By the linearity of the S-transform it is sufficient to prove

$$\forall \eta \in \mathcal{S}(\mathbb{R}), \qquad (S\Phi)(\eta) = 0 \Rightarrow \Phi = 0. \tag{5}$$

We set $\Phi_n := \mathrm{E}[\Phi|\mathcal{G}_n]$. Then there is a function ϕ_n such that $\phi_n \mathrm{e}^{-|\cdot|^2/4} \in L^2(\mathbb{R}^n)$ and $\Phi_n = \phi_n(I(\xi_1), \ldots, I(\xi_n))$. Using the orthonormality of (ξ_k) and the transformation theorem, we obtain, for $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$,

$$\begin{aligned} 0 &= \int_\Omega :\mathrm{e}^{I\left(\sum_{k=1}^n t_k f_k\right)}: \Phi \mathrm{d}P = \int_\Omega :\mathrm{e}^{I\left(\sum_{k=1}^n t_k f_k\right)}: \Phi_n \mathrm{d}P \\ &= \int_\Omega \exp\left\{\sum_{k=1}^n (t_k I(f_k) - \tfrac{1}{2}t_k^2)\right\} \phi_n(I(f_1), \ldots I(f_n)) \mathrm{d}P \\ &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-|t-u|^2/2} \phi_n(u) \mathrm{d}u \\ &= 2\mathrm{e}^{|t|^2/2} (2\pi)^{-n/2} \int_{\mathbb{R}^n} \mathrm{e}^{-|t-u|^2} \phi_n(2u) \mathrm{e}^{-|2u|^2/4} \mathrm{d}u. \end{aligned}$$

Consequently, the convolution of $\mathrm{e}^{-|x|^2}$ and $\psi_n(x) = \phi_n(2x)\mathrm{e}^{-|2x|^2/4} \in L^2(\mathbb{R}^n)$ vanishes. Applying the Fourier transform and taking into account that the Fourier transform of $\mathrm{e}^{-|x|^2}$ is strictly positive, we may conclude from the trivialization theorem for the convolution that the Fourier transform of ψ_n vanishes. Therefore, ψ_n equals zero and so do ϕ_n and thus Φ_n.

As (ξ_k) is an orthonormal basis of $L^2(\mathbb{R})$, $\mathcal{G}_n$ monotonically increases to $\mathcal{G}$ and hence $\Phi = 0$. □

One can also characterize (L^2) convergence in terms of the S-transform:

Theorem 2.3. *Let Φ_n be a sequence in (L^2) and $\Phi \in (L^2)$. Then the following assertions are equivalent:*

(i) Φ_n *(strongly) converges to* Φ *in* (L^2).
(ii) $\|\Phi_n\|_0 \to \|\Phi\|_0$ *and, for all* $\eta \in \mathcal{S}(\mathbb{R})$, $S(\Phi_n)(\eta) \to (S\Phi)(\eta)$.

Proof. The proof of (i) ⇒ (ii) is obvious.

As for (ii) ⇒ (i), by the preceding theorem the linear span of $\{: \mathrm{e}^{I(\eta)} : ; \eta \in \mathcal{S}(\mathbb{R})\}$ is dense in (L^2). Hence, by Yosida (1966, Theorem 3, p. 121) Φ_n weakly converges to Φ in (L^2). Then $\|\Phi_n\|_0 \to \|\Phi\|_0$ implies convergence even in the strong topology. □

The following lemma can easily be proved:

Lemma 2.4. *Let $f, g \in L^2(\mathbb{R})$. Then:*

$$\mathrm{E}[: \mathrm{e}^{I(f)} : \cdot : \mathrm{e}^{I(g)} :] = \mathrm{e}^{(f,g)_0}.$$

In particular,

$$(S : \mathrm{e}^{I(f)} :)(\eta) = \mathrm{e}^{(f,\eta)_0}.$$

Proof. If $g = 0$, we have

$$\mathrm{E}[: \mathrm{e}^{I(f)} :] = \frac{1}{\sqrt{2\pi}|f|_0} \int_{\mathbb{R}} \exp\left\{ u - \frac{1}{2}\left(|f|_0^2 + \frac{u^2}{|f|_0^2} \right) \right\} \mathrm{d}u = 1.$$

The general case can be reduced to this by

$$: \mathrm{e}^{I(f)} : \cdot : \mathrm{e}^{I(g)} : = \mathrm{e}^{(f,g)_0} : \mathrm{e}^{I(f+g)} : .$$

□

Lemma 2.4 and Theorem 2.3 imply:

Corollary 2.5. *Let f_n be a sequence that converges in $L^2(\mathbb{R})$ to f. Then $: \mathrm{e}^{I(f_n)} :$ converges to $: \mathrm{e}^{I(f)} :$ in (L^2).*

From Lemma 2.4 we know that $\mathrm{E}[: \mathrm{e}^{I(f)} :] = 1$ for $f \in L^2(\mathbb{R})$. Hence we can define a probability measure on $\mathcal{G}$ by

$$dQ_f =: \mathrm{e}^{I(f)} : \mathrm{d}P. \tag{6}$$

One can easily check that P and Q_f are equivalent. With the measures Q_η, $\eta \in \mathcal{S}(\mathbb{R})$, we can rewrite the S-transform as

$$(S\Phi)(\eta) = \mathrm{E}^{Q_\eta}[\Phi]. \tag{7}$$

2.3. Fractional integration by parts

Under appropriate conditions, the equation

$$\int_{\mathbb{R}} f(s)\left(M_-^H g\right)(s)\mathrm{d}s = \int_{\mathbb{R}} \left(M_+^H f\right)(s)g(s)\mathrm{d}s \tag{8}$$

holds. We refer to (8) as the fractional integration by parts rule. It is valid under the following conditions:

Theorem 2.6. (i) *Let* $0 < H < \frac{1}{2}$. *Then* (8) *holds if* $M_+^H f \in L^p(\mathbb{R})$, $M_-^H g \in L^r(\mathbb{R})$, $f \in L^s(\mathbb{R})$, $g \in L^t(\mathbb{R})$ *and* $p > 1$, $r > 1$, $1/p + 1/r = 3/2 - H$, $1 \leqslant s, t < \infty$.

(ii) *Let* $\frac{1}{2} < H < 1$. *Then* (8) *holds if* $f \in L^p(\mathbb{R})$, $g \in L^r(\mathbb{R})$ *and* $p > 1$, $r > 1$, $1/p + 1/r = 1/2 + H$.

Proof. In view of the definition of $M_\pm^H$ this is a simple reformulation of Corollary 2 (p. 129) and formula (5.16) in Samko *et al*. (1993). For (i) we note that by Theorem 6.2 in Samko *et al*. (1993) the conditions $1/s = 1/p - 1/2 + H$ and $1/t = 1/r - 1/2 + H$ are satisfied. □

From the proof of Lemma 2.6 in Bender (2003a), we know that $M_+^H \eta \in L^{1/(1-H)}(\mathbb{R})$ if $H < \frac{1}{2}$ and $\eta \in \mathcal{S}(\mathbb{R})$. Moreover, it is well known that $\mathcal{S}(\mathbb{R}) \subset L^p(\mathbb{R})$ for all $p \geqslant 1$. Hence, we have:

Corollary 2.7. *Let* $f \in \mathcal{S}(\mathbb{R})$. *Then:*

(i) (8) *holds if* $0 < H < \frac{1}{2}$, $M_-^H g \in L^2(\mathbb{R})$ *and* $g \in L^p(\mathbb{R})$ *for some* $1 \leqslant p < \infty$.
(ii) (8) *holds if* $\frac{1}{2} < H < 1$ *and* $g \in L^p(\mathbb{R})$ *for some* $1 < p < (H - \frac{1}{2})^{-1}$.

2.4. Pettis integral and stochastic fractional calculus

Throughout this paper we interpret a stochastic process as an (L^2)-valued function. Hence, the notion of Pettis integrability fits better than the pathwise integral:

Definition 2.3. *Let* $X : M \to (L^2)$ ($M \subset \mathbb{R}$ *a Borel set). Then* X *is said to be* Pettis integrable *if* $\mathrm{E}[X\Psi] \in L^1(M)$ *for any* $\Psi \in (L^2)$. *In that case there is a unique* $\Phi \in (L^2)$ *such that, for all* $\Psi \in (L^2)$,

$$\mathrm{E}[\Phi\Psi] = \int_M \mathrm{E}[X_t\Psi]\mathrm{d}t.$$

Φ *is called the* Pettis integral of X *and is denoted by* $\int_M X_t\,\mathrm{d}t$.

A proof of the existence and uniqueness of Φ can be found in Hille and Phillips (1957). Note that by this definition we have, for a Pettis integrable X,

$$\int_M \mathrm{E}[X_t\Psi]\mathrm{d}t = \mathrm{E}\left[\int_M X_t\Psi\mathrm{d}t\right] \tag{9}$$

for all $\Psi \in (L^2)$. In particular, the Pettis integral interchanges with the S-transform.

Before we state a useful criterion for Pettis integrability, we shall point out the relationship between the Pettis integral and the pathwise integral. Let $X : [a, b] \times \Omega \to \mathbb{R}$ be measurable and pathwise integrable such that the pathwise integral belongs to (L^2). If X is good enough to apply Fubini's theorem, we can interchange the integrals:

$$\mathrm{E}\left[\int_a^b X_t\,\mathrm{d}t \cdot \Psi\right] = \int_a^b \mathrm{E}[X_t\Psi]\mathrm{d}t,$$

where the integral on the left-hand side is the ordinary pathwise integral. Hence, the Pettis integral defined in Definition 2.3 coincides with the pathwise integral in that case.

The following criterion is based on the fact, that the Pettis integral is an extension of the Bochner integral:

Theorem 2.8. *Let* $X : \mathbb{R} \to (L^2)$ *such that* $(SX)(\eta)$ *is measurable for all* $\eta \in \mathcal{S}(\mathbb{R})$ *and* $\|X\|_0 \in L^1(\mathbb{R})$. *Then* X *is Pettis integrable and*

$$\left\|\int_{\mathbb{R}} X_t\,\mathrm{d}t\right\|_0 \leqslant \int_{\mathbb{R}} \|X_t\|_0\mathrm{d}t.$$

Proof. As the linear span of the set $\{: \mathrm{e}^{\langle\cdot,\eta\rangle} : ;\ \eta \in \mathcal{S}(\mathbb{R})\}$ is dense in (L^2) by Theorem 2.2, $\mathrm{E}[X_tF]$ is measurable for any $F \in (L^2)$. Hence, X_t is weakly measurable. The separability of (L^2) implies that X_t is strongly measurable (see Hille and Phillips, 1957, p. 73). Thus, X_t is Bochner integrable and the inequality holds by Theorem 3.7.4 in Hille and Phillips (1957). □

Let us now extend the operators $M_\pm^H$ to stochastic processes: For $X : \mathbb{R} \to (L^2)$ and $\frac{1}{2} < H < 1$, the *fractional integral of Weyl's type* is defined by

$$(I_-^{H-1/2}X)_x := \frac{1}{\Gamma(H-\frac{1}{2})}\int_x^\infty X_t(t-x)^{H-3/2}\mathrm{d}t,$$

$$(I_+^{H-1/2}X)_x := \frac{1}{\Gamma(H-\frac{1}{2})}\int_{-\infty}^x X_t(x-t)^{H-3/2}\mathrm{d}t,$$

if the integrals exist for almost all $x \in \mathbb{R}$ as Pettis integrals.

If $0 < H < \frac{1}{2}$ the *fractional derivative of Marchaud's type* is given by ($\epsilon > 0$)

$$\left(D_{\pm,\epsilon}^{-(H-1/2)}X\right)_x := \frac{-(H-\frac{1}{2})}{\Gamma(H+\frac{1}{2})}\int_\epsilon^\infty \frac{X_x - X_{x\mp t}}{t^{3/2-H}}\,\mathrm{d}t$$

and

$$\left(D_{\pm}^{-(H-1/2)}X\right) := \lim_{\epsilon\to 0+}\left(D_{\pm,\epsilon}^{-(H-1/2)}X\right),$$

if the above integrals exist for almost all $\epsilon > 0$ and $x \in \mathbb{R}$ as Pettis integrals and the limit exists in $L^p(\mathbb{R}, (L^2))$ for some $p > 1$. Again, $D_{\pm}^{-(H-1/2)}X \in L^p(\mathbb{R}, (L^2))$ indicates convergence in the $L^p(\mathbb{R}, (L^2))$ norm. The operators $M_{\pm}^H$ can then be defined as in (3).

It is important that the operators $M_{\pm}^H$ interchange with the S-transform:

Lemma 2.9. *Let $M_{\pm}^H X$ exist for some $X : \mathbb{R} \to (L^2)$. Then we have, for all $\Psi \in (L^2)$,*

$$\mathrm{E}\left[(M_{\pm}^H X)_t \Psi\right] = M_{\pm}^H(\mathrm{E}[X_t \Psi]).$$

In the case $H < \frac{1}{2}$ the convergence of the fractional derivative on the right-hand side is in the $L^p(\mathbb{R})$ sense, if $M_{\pm}^{-(H-1/2)}X \in L^p(\mathbb{R}, (L^2))$. In particular, the operators $M_{\pm}^H$ interchange with the S-transform.

Proof. The case $H \geqslant \frac{1}{2}$ is straightforward in view of (9).

Let $H < \frac{1}{2}$. Then by (9):

$$\begin{aligned}
&\int_{\mathbb{R}}\left|\mathrm{E}\left[(D_{\pm}^{-(H-1/2)}X)_x\Psi\right] - D_{\pm,\epsilon}^{-(H-1/2)}(\mathrm{E}[X_x\Psi])\right|^p \mathrm{d}x \\
&= \int_{\mathbb{R}}\left|\mathrm{E}\left[\Psi\left((D_{\pm}^{-(H-1/2)}X)_x - (D_{\pm,\epsilon}^{-(H-1/2)}X)_x\right)\right]\right|^p \mathrm{d}x \\
&\leqslant \mathrm{E}[\Psi^2]^{p/2}\cdot\int_{\mathbb{R}}\mathrm{E}\left[\left((D_{\pm}^{-(H-1/2)}X)_x - (D_{\pm,\epsilon}^{-(H-1/2)}X)_x\right)^2\right]^{p/2}\mathrm{d}x \\
&\to 0
\end{aligned}$$

as $\epsilon \to 0+$ by the definition of the fractional derivative of a stochastic process. In view of the definition of $M_{\pm}^H$ in the case where $H < \frac{1}{2}$ the proof is finished. □

We now prove a stochastic version of the Hardy–Littlewood theorem:

Theorem 2.10. *Let $H > \frac{1}{2}$. Then $M_{\pm}^H$ is a continuous operator from $L^p(\mathbb{R}, (L^2))$ into $L^q(\mathbb{R}, (L^2))$ if $1 < p < (H - \frac{1}{2})^{-1}$ and $q = p/(1 - p(H - 1/2))$.*

Proof. As $X \in L^p(\mathbb{R}, (L^2))$, $\|X\|_0 \in L^p(\mathbb{R})$. Thus,

$$\int_t^\infty \|X_s(s-t)^{H-3/2}\|_0 \mathrm{d}s = \int_t^\infty \|X_s\|_0 (s-t)^{H-3/2} \mathrm{d}s < \infty$$

for all $t \in \mathbb{R}$. Hence, by Theorem 2.8, $M_-^H X$ exists and

$$\int_{\mathbb{R}} \|(M_-^H X)_t\|_0^q \mathrm{d}t \leqslant \int_{\mathbb{R}} |(M_-^H \|X\|_0)(t)|^q \mathrm{d}t \leqslant C_H \left(\int_{\mathbb{R}} \|X_t\|_0^p \mathrm{d}t \right)^{q/p},$$

where the last inequality follows from the deterministic Hardy–Littlewood theorem (Samko *et al*., 1993, Theorem 5.3) applied to $\|X\|_0$. The proof for M_+^H is similar. □

We conclude this section with a straightforward consequence of Hölder's inequality:

Lemma 2.11. *Let* $X : \mathbb{R} \to (L^2)$ *such that* $X \in L^p(\mathbb{R}, (L^2))$, $p > 1$. *Then* $\mathrm{E}[X\Phi] \in L^p(\mathbb{R})$ *for all* $\Phi \in (L^2)$. *In particular,* $X \in L^p(\mathbb{R}, (L^2))$ *implies* $(SX)(\eta) \in L^p(\mathbb{R})$ *for all* $\eta \in \mathcal{S}(\mathbb{R})$.

3. The fractional Itô integral

3.1. The classical case from an *S*-transform point of view

Let $0 \leqslant a \leqslant b$ and $X : [a, b] \times \Omega \to \mathbb{R}$ a progressively measurable (with respect to the filtration $\mathcal{F}_t$ generated by the Brownian motion B_s, $0 \leqslant s \leqslant t$) process satisfying

$$\mathrm{E}\left[\int_a^b |X_t|^2 \mathrm{d}t \right] < \infty. \tag{10}$$

Then the classical Itô integral $\int_a^b X_t \, \mathrm{d}B_t$ with respect to the Brownian motion B exists. By the isometry property of the Itô integral it is an element of (L^2). We now calculate its S-transform.

Let Q_η, $\eta \in \mathcal{S}(\mathbb{R})$, be the measure defined by (6). Then by the classical Girsanov theorem $\tilde{B}_t := B_t - \int_0^t \eta(t)\mathrm{d}t$ is a two-sided Brownian motion under the measure Q_η. Moreover, by (10) and Hölder's inequality,

$$\mathrm{E}^{Q_\eta}\left[\left(\int_a^b |X_t|^2 \mathrm{d}t \right)^{1/2} \right] < \infty.$$

Consequently, $\int_a^s X_t \, \mathrm{d}\tilde{B}_t$, $a \leqslant s \leqslant b$, is a Q_η-martingale with zero expectation.

Using the above considerations, (7) and Fubini's theorem we obtain

$$
\begin{aligned}
S\left(\int_a^b X_t\,\mathrm{d}B_t\right)(\eta) &= \mathrm{E}^{Q_\eta}\left[\int_a^b X_t\,\mathrm{d}B_t\right] \\
&= \mathrm{E}^{Q_\eta}\left[\int_a^b X_t\,\mathrm{d}\tilde{B}_t + \int_a^b X_t\eta(t)\mathrm{d}t\right] \\
&= \int_a^b \mathrm{E}^{Q_\eta}[X_t]\eta(t)\mathrm{d}t \\
&= \int_a^b (SX_t)(\eta)\eta(t)\mathrm{d}t.
\end{aligned}
$$

If we replace X by a function $f \in L^2(\mathbb{R})$ we can repeat the argument and obtain for the Wiener integral,

$$S(I(f))(\eta) = \int_{\mathbb{R}} f(t)\eta(t)\mathrm{d}t.$$

In particular,

$$S(B_t)(\eta) = \int_0^t \eta(s)\mathrm{d}s.$$

Let us summarize the foregoing:

Theorem 3.1. (i) *Let $0 \leqslant a \leqslant b$ and $X : [a, b] \times \Omega \to \mathbb{R}$ a progressively measurable process satisfying* (10). *Then the Itô integral $\int_a^b X_t\,\mathrm{d}B_t$ is the unique element in (L^2) with S-transform given by*

$$\int_a^b (SX_t)(\eta)\eta(t)\mathrm{d}t.$$

(ii) *The Wiener integral $I(f)$, $f \in L^2(\mathbb{R})$, is the unique element in (L^2) with S-transform given by*

$$\int_{\mathbb{R}} f(t)\eta(t)\mathrm{d}t.$$

Using this result we can define an extension of the Itô integral in terms of the S-transform:

Definition 3.1. *Let $M \subset \mathbb{R}$ a Borel set, $X : M \to (L^2)$. Then X is said to be* Hitsuda–Skorohod integrable *if $(SX_\cdot)(\eta)\eta(\cdot) \in L^1(M)$ for any $\eta \in \mathcal{S}(\mathbb{R})$, and there is a $\Phi \in (L^2)$ such that, for all $\eta \in \mathcal{S}(\mathbb{R})$,*

$$S\Phi(\eta) = \int_M S(X_t)(\eta)\eta(t)\mathrm{d}t.$$

In that case Φ is uniquely determined by Theorem 2.2 and we denote it by $\int_M X_t \, \mathrm{d}B_t^{1/2}$.

Remark 3.1. In this terminology, Theorem 3.1(i) states that the Hitsuda–Skorohod integral is an extension of the Itô integral. This result is well known in the white noise setting. However, in this setting the proof is usually given either by calculating the chaos decomposition (Hida *et al*. 1993) or by approximating with step processes (Kuo 1996). Note that our proof is rather elementary.

Our definition of the Hitsuda–Skorohod integral makes use of the S-transform. It obviously coincides with the definition in Kuo (1996) in the white noise setting. However, there is another and more common approach to the Skorohod integral using the Malliavin calculus. As the fractional Itô integral is defined via the (Malliavin calculus) Skorohod integral in Decreusefond and Üstünel (1999) and Alòs *et al*. (2001), we briefly review this definition. For more information concerning the Malliavin calculus we refer to Nualart (1995).

For a smooth random variable of the form $F(I(\xi_1), \ldots, I(\xi_n))$ with $\xi_i \in L^2(\mathbb{R})$ and $F \in \mathcal{C}^\infty(\mathbb{R}^n)$, with polynomial growth, the Malliavin derivative is given by

$$D_t F = \sum_{i=1}^n \frac{\partial F}{\partial x_i}(I(\xi_1), \ldots, I(\xi_n))\xi_i(t).$$

($t \in M$, $M \subset \mathbb{R}$ a Borel set). Here DF is a closable operator from (L^2) to $L^2(\Omega, L^2(M))$. The domain of D is denoted by $\mathbb{D}^{1,2}$.

A stochastic process $X \in L^2(M, (L^2))$ is said to be *Skorohod integrable (with respect to B)* if $X: \Omega \times M \to \mathbb{R}$ is measurable and

$$\left| \mathrm{E}\left[\int_{\mathbb{R}} D_t F X_t \, \mathrm{d}t \right] \right| \leqslant c \|F\|_0$$

for all $F \in \mathbb{D}^{1,2}$ and a constant c depending on X. Then the Skorohod integral of X is the unique element $\Phi \in (L^2)$ such that, for all $F \in \mathbb{D}^{1,2}$,

$$\mathrm{E}[\Phi \cdot F] = \mathrm{E}\left[\int_M D_t F X_t \, \mathrm{d}t \right].$$

Now one can easily check that $: \mathrm{e}^{I(\eta)} : \in \mathbb{D}^{1,2}$ and

$$D_t : \mathrm{e}^{I(\eta)} : =: \mathrm{e}^{I(\eta)} : \cdot \eta(t)$$

for $\eta \in \mathcal{S}(\mathbb{R})$. Hence we have proved the first part of the following theorem:

Theorem 3.2. *The Hitsuda–Skorohod integral is an extension of the Skorohod integral. Conversely, if X is Hitsuda–Skorohod integrable, then X is in the domain of the Skorohod integral if and only if* $X \in L^2(M, (L^2))$.

Proof. The ‘only if’ part is trivial in view of the above definition. For the ‘if’ part suppose

$X \in L^2(M, (L^2))$ is Hitsuda–Skorohod integrable. Then there is a $Y \in (L^2)$ such that, for all $\eta \in \mathcal{S}(\mathbb{R})$,

$$\mathrm{E}[Y : \mathrm{e}^{I(\eta)} :] = (SY)(\eta) = \int_M (SX_s)(\eta)\eta(s)\mathrm{d}s = \int_M \mathrm{E}\big[X_s D_s : \mathrm{e}^{I(\eta)} :\big]\mathrm{d}s.$$

By linearity and the fact that the linear combinations of Wick exponentials form a dense set in $\mathbb{D}^{1,2}$, we may conclude that, for all $F \in \mathbb{D}^{1,2}$,

$$\mathrm{E}[Y \cdot F] = \int_{\mathbb{R}} \mathrm{E}[X_s \cdot D_s F]\mathrm{d}s,$$

which shows that X is in the domain of the Skorohod integral. □

3.2. Definition of the fractional Itô integral and basic properties

We shall now define an integral with respect to a fractional Brownian motion in analogy to the Hitsuda–Skorohod integral. To this end, note that

$$\eta(t) = \frac{\mathrm{d}}{\mathrm{d}t} S(B_t)(\eta)$$

and, hence,

$$\int_a^b S(X_t)(\eta)\eta(t)\mathrm{d}t = \int_a^b S(X_t)(\eta)\frac{\mathrm{d}}{\mathrm{d}t} S(B_t)(\eta)\mathrm{d}t.$$

Thus, we should define the fractional Itô integral as the unique random variable Φ (if it exists), that satisfies

$$(S\Phi)(\eta) = \int_a^b S(X_t)(\eta)\frac{\mathrm{d}}{\mathrm{d}t} S(B_t^H)(\eta)\mathrm{d}t.$$

From Theorem 3.1(ii) and the construction of B^H we see that

$$\frac{\mathrm{d}}{\mathrm{d}t} S(B_t^H)(\eta) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} M_-^H(\mathbf{1}(0, t))(s)\eta(s)\mathrm{d}s.$$

Applying the fractional integration by parts rule (Corollary 2.7) yields:

$$\frac{\mathrm{d}}{\mathrm{d}t} S(B_t^H)(\eta) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (M_+^H\eta)(s)\mathrm{d}s = (M_+^H\eta)(t).$$

Thus, we arrive at the following definition:

Definition 3.2. *Let X: $M \to (L^2)$ ($M \subset \mathbb{R}$ a Borel set). Then X is said to have a* fractional Itô integral (S-transform approach) *if $(SX_\cdot)(\eta)(M_+^H\eta)(\cdot) \in L^1(M)$ for any $\eta \in \mathcal{S}(\mathbb{R})$ and there is a $\Phi \in (L^2)$ such that, for all $\eta \in \mathcal{S}(\mathbb{R})$,*

$$S\Phi(\eta) = \int_M S(X_t)(\eta)(M_+^H\eta)(t)\mathrm{d}t.$$

In that case Φ is uniquely determined by Theorem 2.2 and we denote it by $\int_M X_t \, \mathrm{d}B_t^H$.

Remark 3.2. For the rest of this paper, except for Section 6, the phrase 'fractional Itô integral' always refers to the S-transform approach proposed in Definition 3.2.

The following properties of the fractional Itô integral are straightforward in view of the definition:

Proposition 3.3. (i) *For all* $a < b \in \mathbb{R}$,

$$B_b^H - B_a^H = \int_a^b \mathrm{d}B_t^H.$$

(ii) *Let* $X : M \to (L^2)$ *be fractional Itô integrable. Then*

$$\int_M X_t \, \mathrm{d}B_t^H = \int_{\mathbb{R}} \mathbf{1}_M(t) X_t \, \mathrm{d}B_t^H,$$

where $\mathbf{1}_M$ *denotes the indicator function of M.*

(iii) *Let* $X : M \to (L^2)$ *be fractional Itô integrable. Then*

$$\mathrm{E}\left[\int_M X_t \, \mathrm{d}B_t^H\right] = 0.$$

Note that (iii) holds since the expectation coincides with the S-transform at $\eta = 0$. Because of (ii) there is no loss of generality in proving the majority of results for $M = \mathbb{R}$ only.

Example 3.1. Let us calculate $\int_0^T B_t^H \, \mathrm{d}B_t^H$: Again, Q_n, $\eta \in \mathcal{S}(\mathbb{R})$, denotes the measure in (6). $\tilde{B}$ is the Brownian motion under Q_η given by the classical Girasanov theorem. Then we can apply integration by parts and the classical Girsanov theorem to obtain:

$$\begin{aligned}
2\int_0^T S(B_t^H)(\eta)(M_+^H \eta)(t)\mathrm{d}t &= 2\int_0^T \int_0^t (M_+^H \eta)(s)\mathrm{d}s (M_+^H \eta)(t)\mathrm{d}t \\
&= \left(\int_0^T (M_+^H \eta)(s)\mathrm{d}s\right)^2 \\
&= \mathrm{E}^{Q_\eta}\left[\left(\int_{\mathbb{R}} M_-^H \mathbf{1}(0,\, T)(s)\mathrm{d}\tilde{B}_s - \int_0^T (M_+^H \eta)(s)\mathrm{d}s\right)^2\right] - |M_-^H \mathbf{1}(0,\, T)|_0^2 \\
&= \mathrm{E}^{Q_\eta}\left[\left(\int_{\mathbb{R}} M_-^H \mathbf{1}(0,\, T)(s)\mathrm{d}B_s\right)^2\right] - T^{2H} = S\big((B_T^H)^2\big)(\eta) - T^{2H}
\end{aligned}$$

by Theorem 2.1 and (7). Hence,

$$\int_0^T B_t^H \, \mathrm{d}B_t^H = \tfrac{1}{2}\Big(\big(B_T^H\big)^2 - T^{2H} \Big).$$

Remark 3.3. The above argument can be generalized to obtain Itô formulae for functionals of a fractional Brownian motion or a geometric fractional Brownian motion; see Section 5.1 below or Bender (2003a) for the case of generalized functionals of B^H.

We now prove that the fractional Itô integral coincides with a Hitsuda–Skorohod integral under appropriate conditions:

Theorem 3.4. *Let* $0 < H < 1$. *Suppose that* $X \in L^p(\mathbb{R}, (L^2))$ *with* $1 < p < (H - \frac{1}{2})^{-1}$ *when* $H > 1/2$, *and that* $M_-^H X \in L^2(\mathbb{R}, (L^2))$ *and* $X \in L^p(\mathbb{R}, (L^2))$ *with* $1 \leqslant p < \infty$ *when* $H < \frac{1}{2}$. *Then*

$$\int_{\mathbb{R}} X_t \, \mathrm{d}B_t^H = \int_{\mathbb{R}} (M_-^H X)(t) \mathrm{d}B_t^{1/2}$$

in the usual sense, that is if one of the integrals exists then so does the other, and both coincide.

Proof. From Lemma 2.11 we know that $(SX)(\eta) \in L^p(\mathbb{R})$ for all $\eta \in \mathcal{S}(\mathbb{R})$ with the respective p from the assumption of the theorem. Moreover, $S(M_-^H X))(\eta) \in L^2(\mathbb{R})$, if $H < \frac{1}{2}$. Now we obtain from Lemma 2.9 and the fractional integration by parts rule (Corollary 2.7):

$$\int_{\mathbb{R}} S\big((M_-^H X)_t\big)(\eta) \cdot \eta(t) \mathrm{d}t = \int_{\mathbb{R}} M_-^H((SX_\cdot)(\eta))(t) \cdot \eta(t) \mathrm{d}t$$

$$= \int_{\mathbb{R}} (SX_t)(\eta) \cdot (M_+^H \eta)(t) \mathrm{d}t,$$

which (in view of Definition 3.2) proves the assertion. □

From this theorem we can easily deduce a criterion for fractional Itô integrability:

Corollary 3.5. *Let* $0 < H < 1$ *and* $X \in L^{1/H}(\mathbb{R}, (L^2))$ *and additionally* $M_-^H X \in L^2(\mathbb{R}, (L^2))$ *when* $H < \frac{1}{2}$. *Moreover, assume that the Malliavin derivative* $D_t[(M_-^H X)_s]$ *exists as an element of* (L^2) *for almost all* $(s, t) \in \mathbb{R}^2$, *and*

$$\int_{\mathbb{R}^2} \mathrm{E}\big[D_t[(M_-^H X)_s] \cdot D_s[(M_-^H X)_t]\big] \mathrm{d}(s, t) < \infty.$$

Then X *is* H*-fractional Itô integrable and*

$$\mathrm{E}\left[\left(\int_{\mathbb{R}} X_s \, \mathrm{d}B_s^H\right)^2\right] = \int_{\mathbb{R}^2} \mathrm{E}\big[D_t[(M_-^H X)_s] \cdot D_s[(M_-^H X)_t]\big] \mathrm{d}(s, t) + \int_{\mathbb{R}} \mathrm{E}[|(M_-^H X)_s|^2] \mathrm{d}s \quad (11)$$

Proof. The stochastic Hardy–Littlewood theorem (Theorem 2.10) yields $M_-^H X \in L^2(\mathbb{R}, (L^2))$

when $H > \frac{1}{2}$, too. Then, by Nualart (1995, pp. 38–39), $M_-^H X$ is Skorohod integrable and (11) holds with the left-hand side replaced by

$$\mathrm{E}\left[\left(\int_{\mathbb{R}} (M_-^H X)_s \,\mathrm{d}B_s^{1/2}\right)^2\right].$$

Hence, the assertion follows from Theorem 3.4. □

Remark 3.4. Note that (11) was first proved (in a slightly different formulation) by Elliott and van der Hoek (2003) in the white noise setting by a rather lengthy calculation involving the Wiener chaos expansion of the fractional Itô integral. In analogy to the classical isometry for the Itô integral, (11) is usually called the *fractional Itô isometry.* This name is misleading in the sense that in general (11) can be zero without X being zero. However, it becomes an isometry if restricted to an appropriate class of adapted processes (with respect to the filtration generated by the fractional Brownian motion starting at minus infinity); see Bender (2003b).

Let us now introduce the Wick product:

Definition 3.3. *Let* $\Phi, \Psi \in (L^2)$ *and assume that there is an element* $\Phi \diamond \Psi \in (L^2)$, *that satisfies* $S(\Phi \diamond \Psi)(\eta) = (S\Phi)(\eta)(S\Psi)(\eta)$ *for all* $\eta \in \mathcal{S}(\mathbb{R})$. *Then* $\Phi \diamond \Psi$ *is called the* Wick product *of* Φ *and* Ψ.

The next theorem explores the relationship between the fractional Itô integral and the Wick product:

Theorem 3.6. *Let* $X : \mathbb{R} \to (L^2)$ *and* $Y \in (L^2)$. *Then*

$$Y \diamond \int_{\mathbb{R}} X_s \mathrm{d}B_s^H = \int_{\mathbb{R}} Y \diamond X_s \mathrm{d}B_s^H$$

in the sense that if one side is well defined then so is the other, and both coincide.

The straightforward proof can be carried out by calculating the S-transform of both sides.

In particular, this theorem implies that, for good random variables Y,

$$Y \diamond (B_b^H - B_a^H) = \int_{\mathbb{R}} \mathbf{1}(a, b)(s) Y \mathrm{d}B_s^H \tag{12}$$

Together with the fractional Itô isometry (11), this shows that for sufficiently good processes X the fractional Itô integral is an (L^2) limit of Wick–Riemann sums.

Note that, in general, the Wick product does not coincide with the ordinary pathwise product. We provide an easy example:

Example 3.2. Let $a < b$ and $\eta \in \mathcal{S}(\mathbb{R})$. Again Q_η denotes the measure defined in (6) and $\tilde{B}$ is the Brownian motion under Q_η constructed by the classical Girsanov theorem. Then

$$
\begin{aligned}
S\big(B_a^H(B_b^H - B_a^H)\big)(\eta) &= S\bigg(\int_{\mathbb{R}}(M_-^H\mathbf{1}(0,a))(s)\mathrm{d}B_s \cdot \int_{\mathbb{R}}(M_-^H\mathbf{1}(a,b))(s)\mathrm{d}B_s\bigg)(\eta) \\
&= \mathrm{E}^{Q_\eta}\bigg[\bigg(\int_{\mathbb{R}}(M_-^H\mathbf{1}(0,a))(s)\mathrm{d}\tilde{B}_s + \int_{\mathbb{R}}(M_-^H\mathbf{1}(0,a))(s)\eta(s)\mathrm{d}s\bigg) \\
&\quad \times\bigg(\int_{\mathbb{R}}(M_-^H\mathbf{1}(a,b))(s)\mathrm{d}\tilde{B}_s + \int_{\mathbb{R}}(M_-^H\mathbf{1}(a,b))(s)\eta(s)\mathrm{d}s\bigg)\bigg] \\
&= \int_0^a (M_+^H\eta)(s)\mathrm{d}s \cdot \int_a^b (M_+^H\eta)(s)\mathrm{d}s + (M_-^H\mathbf{1}(0,a), M_-^H\mathbf{1}(a,b))_0 \\
&= S\big(B_a^H \diamond (B_b^H - B_a^H)\big)(\eta) + \tfrac{1}{2}\big(b^{2H} - a^{2H} - (b-a)^{2H}\big).
\end{aligned}
$$

Hence,

$$
B_a^H(B_b^H - B_a^H) = B_a^H \diamond (B_b^H - B_a^H) + \tfrac{1}{2}\big(b^{2H} - a^{2H} - (b-a)^{2H}\big). \tag{13}
$$

Remark 3.5. If $H = \frac{1}{2}$ the last term on the right-hand side of (13) is zero. This is not by chance, but a special case of the following more general reasoning. Assume that X is $\mathcal{F}_a$-measurable, where $\mathcal{F}_t$ is the filtration generated by the Brownian motion B. Then we can apply Theorem 3.1(i), to see that

$$
S(X(B_b - B_a))(\eta) = (SX)(\eta)\int_a^b \eta(t)\mathrm{d}t = (SX)(\eta)(S(B_b - B_a))(\eta).
$$

Hence,

$$
X(B_b - B_a) = X \diamond (B_b - B_a). \tag{14}
$$

This is why one does not need the Wick product in defining the classical Itô integral.

Using (12) we can rewrite (13) as

$$
B_a^H\int_a^b \mathrm{d}B_s^H = \int_a^b B_a^H \mathrm{d}B_s^H + \tfrac{1}{2}\big(b^{2H} - a^{2H} - (b-a)^{2H}\big).
$$

Thus, we arrive at the following unfortunate observation, which is different from the Itô integral in the Brownian motion case:

Proposition 3.7. *Let* $\mathcal{F}_t^H = \sigma\big(B_s^H; 0 \leqslant s \leqslant t\big)$ *and assume that* $X : [a, b] \to (L^2)$ *is* $\mathcal{F}_t^H$-*progressively measurable and* $Y \in (L^2)$ *is* $\mathcal{F}_a^H$-*measurable. Then, in general, the identity*

$$
Y \cdot \int_a^b X_s \mathrm{d}B_s^H = \int_a^b Y \cdot X_s \mathrm{d}B_s^H
$$

does not *hold if* $H \neq 1/2$.

The true relationship between the pathwise product and the fractional Itô integral is as follows (see Biagini *et al.* 2003):

Proposition 3.8. *Suppose X satisfies the assumptions of Corollary 3.5, and let $Y \in (L^2)$ such that, for almost all $t \in \mathbb{R}$, $D_t Y \in (L^2)$. Then*

$$\int_{\mathbb{R}} Y \cdot X_s \, \mathrm{d}B_s^H = Y \cdot \int_{\mathbb{R}} X_s \, \mathrm{d}B_s^H - \int_{\mathbb{R}} D_s Y \cdot (M_-^H X)_s \, \mathrm{d}s$$

in the sense that the integral of the left-hand side exists if the right-hand side belongs to (L^2).

The proof is a direct application of Theorem 3.4 above and formula (1.49) in Nualart (1995).

4. The fractional Itô integral under change of measure

4.1. Expectation

We now calculate the expectation of a fractional Itô integral under the measure Q_f, $f \in L^2(\mathbb{R})$, given by (6).

We begin with the rather simple case of Q_η, $\eta \in \mathcal{S}(\mathbb{R})$. To this end, let $X : \mathbb{R} \to (L^2)$ be H-fractional Itô integrable $(0 < H < 1)$. Then

$$\mathrm{E}^{Q_\eta}\left[\int_{\mathbb{R}} X_t \, \mathrm{d}B_t^H\right] = S\left(\int_{\mathbb{R}} X_t \, \mathrm{d}B_t^H\right)(\eta) = \int_{\mathbb{R}} \mathrm{E}^{Q_\eta}[X_t](M_+^H \eta)(t)\mathrm{d}t \tag{15}$$

by Definition 3.2 and (7).

Let us now consider the general case of $f \in L^2(\mathbb{R})$. We choose a sequence $(\eta_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R})$ such that η_n converges to f in $L^2(\mathbb{R})$. By Corollary 2.5 the left-hand side of (15) (with η replaced by η_n) converges to $\mathrm{E}^{Q_f}\left[\int_a^b X_t \, \mathrm{d}B_t^H\right]$. So we have to impose conditions that ensure the convergence of the right-hand side:

Theorem 4.1. *Let $0 < H < 1$, $f \in L^2(\mathbb{R})$ and Q_f be given by (6). Moreover, assume that $X : \mathbb{R} \to (L^2)$ is H-fractional Itô integrable and $X \in L^{1/H}(\mathbb{R}, (L^2))$. Additionally, suppose that $M_+^H f \in L^{1/(1-H)}(\mathbb{R})$ and $M_-^H X \in L^2(\mathbb{R}, (L^2))$ when $H < \frac{1}{2}$. Then*

$$\mathrm{E}^{Q_f}\left[\int_{\mathbb{R}} X_t \, \mathrm{d}B_t^H\right] = \int_{\mathbb{R}} \mathrm{E}^{Q_f}[X_t](M_+^H f)(t)\mathrm{d}t.$$

Proof. Note, first, that by Theorem 2.10, $M_-^H X \in L^2(\mathbb{R}, (L^2))$ when $H > \frac{1}{2}$, too. Let $(\eta_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R})$ be given such that η_n converges to f in $L^2(\mathbb{R})$. By the considerations at the beginning of this section it remains to prove that the right-hand side of (15) (with η replaced by η_n) converges to $\mathrm{E}^{Q_f}\left[\int_{\mathbb{R}} X_t (M_+^H \eta)(t)\mathrm{d}t\right]$. By the supposed integrability conditions, Lemmas 2.9 and 2.11, we may apply the fractional integration by parts rule in the form of Corollary 2.7 to obtain:

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} \mathrm{E}\big[X_t : \mathrm{e}^{I(\eta_n)} :\big](M_+^H\eta_n)(t) - \mathrm{E}\big[X_t : \mathrm{e}^{I(f)} :\big](M_+^H f)(t)\mathrm{d}t\right| \\
&\quad = \left|\int_{\mathbb{R}} \mathrm{E}\big[(M_-^H X)_t : \mathrm{e}^{I(\eta_n)} :\big]\eta_n(t) - \int_{\mathbb{R}} \mathrm{E}\big[(M_-^H X)_t : \mathrm{e}^{I(f)}:\big] f(t)\mathrm{d}t\right| \\
&\quad \leqslant \int_{\mathbb{R}} |\eta_n(t)| \mathrm{E}\big[\big|(M_-^H X)_t\big(: \mathrm{e}^{I(\eta_n)} : - : \mathrm{e}^{I(f)} :\big)\big|\big]\mathrm{d}t \\
&\qquad + \int_{\mathbb{R}} |\eta_n(t) - f(t)| \mathrm{E}\big[\big|(M_-^H X)_t \cdot : \mathrm{e}^{I(f)} :\big|\big]\mathrm{d}t \\
&\quad = I_1 + I_2.
\end{aligned}
$$

By Hölder's inequality,

$$
\begin{aligned}
I_2 &\leqslant |\eta_n - f|_0 \cdot \left(\int_{\mathbb{R}} \mathrm{E}\big|(M_-^H X)_t \cdot : \mathrm{e}^{I(f)} :\big|^2 \mathrm{d}t\right)^{1/2} \\
&\leqslant |\eta_n - f|_0 \cdot \left(\int_{\mathbb{R}} \mathrm{E}\left[\big|(M_-^H X)_t\big|^2\right]\mathrm{d}t\right)^{1/2} (\mathrm{E}[: \mathrm{e}^{I(f)} :^2])^{1/2} \\
&\to 0
\end{aligned}
$$

as $n \to \infty$. The same argument applied to I_1 yields

$$
\begin{aligned}
I_1 &\leqslant |\eta_n|_0 \cdot \left(\int_{\mathbb{R}} \mathrm{E}\left[\big|(M_-^H X)_t\big|^2\right]\mathrm{d}t\right)^{1/2} \\
&\quad \times \big(\mathrm{E}\big[|: \mathrm{e}^{I(\eta_n)} : - : \mathrm{e}^{I(f)} : |^2\big]\big)^{1/2} \\
&\to 0
\end{aligned}
$$

as $n \to \infty$ by Corollary 2.5. Hence,

$$
\lim_{n\to\infty} \int_{\mathbb{R}} \mathrm{E}^{Q_{\eta_n}}[X_t](M_+^H \eta_n)(t)\mathrm{d}t = \int_{\mathbb{R}} \mathrm{E}^{Q_f}[X_t](M_+^H f)(t)\mathrm{d}t,
$$

which proves the assertion. □

Remark 4.1. Note that under the assumptions of Theorem 4.1, for all $\Psi \in (L^2)$,

$$
\begin{aligned}
&\int_{\mathbb{R}} \left|\mathrm{E}\left[X_t (M_+^H f)(t)\Psi\right]\right| \mathrm{d}t \\
&\quad \leqslant \left(\int_{\mathbb{R}} |(M_+^H f)(t)|^{1/(1-H)}\mathrm{d}t\right)^{1-H} \left(\int_{\mathbb{R}} \mathrm{E}\big[|X_t|^2\big]^{1/(2H)}\mathrm{d}t\right)^{H} \cdot \mathrm{E}[|\Psi|^2]^{1/2} \\
&\quad < \infty.
\end{aligned}
$$

Thus, $\int_{\mathbb{R}} X_t(M_+^H f)(t)\mathrm{d}t$ exists as a Pettis integral and, by (9),

$$\mathrm{E}^{Q_f}\left[\int_{\mathbb{R}} X_t\,\mathrm{d}B_t^H\right] = \mathrm{E}^{Q_f}\left[\int_{\mathbb{R}} X_t(M_+^H f)(t)\mathrm{d}t\right]. \tag{16}$$

4.2. A Girsanov theorem

The classical Girsanov theorem states that a Brownian motion with drift becomes a Brownian motion without drift under some change of measure. In the fractional Brownian motion case similar results have been obtained by Norros *et al*. (1999), Hu and Øksendal (2003) and Elliott and van der Hoek (2003). We now explore how a fractional Itô integral behaves under change of measure. We thus generalize the results given above.

Let Q_f, $f \in L^2(\mathbb{R})$, again be the measure defined by (6). The probability space $(\Omega, \mathcal{G}, Q_f)$ carries a two-sided Brownian motion given by $\tilde{B}_t := B_t - \int_0^t f(t)\mathrm{d}t$ by virtue of the classical Girsanov theorem. Hence, all constructions from Section 2 carry over to this probability space. We shall use the notation S_{Q_f} for the S-transform with respect to this new probability space:

$$(S_{Q_f}X)(\eta) := \mathrm{E}^{Q_f}\left[: \mathrm{e}^{I^{\tilde{B}}(\eta)} : X\right].$$

S and I respectively denote the S-transform with respect to the space $(\Omega, \mathcal{G}, P)$ and the Wiener integral with respect to B, as before.

Later we shall need the following identity which can be verified directly for all $g \in L^2(\mathbb{R})$:

$$: \mathrm{e}^{I^{\tilde{B}}(g)} : \cdot : \mathrm{e}^{I(f)} : = : \mathrm{e}^{I(g+f)} : . \tag{17}$$

Our Girsanov theorem has the following form:

Theorem 4.2. *Let the assumptions of Theorem 4.1 hold with respect to the probability space* $(\Omega, \mathcal{G}, P)$*. Moreover, assume that*

$$\mathrm{E}^{Q_f}\left[\left|\int_{\mathbb{R}} X_t\,\mathrm{d}B_t^H - \int_{\mathbb{R}} X_t(M_+^H f)(t)\right|^2\right] < \infty. \tag{18}$$

Then the identity

$$\int_{\mathbb{R}} X_t\,\mathrm{d}\tilde{B}_t^H = \int_{\mathbb{R}} X_t\,\mathrm{d}B_t^H - \int_{\mathbb{R}} X_t(M_+^H f)(t)\mathrm{d}t$$

holds in $L^2(\Omega, \mathcal{G}, Q_f)$ *and consequently* Q_f*-almost surely and* P*-almost surely.*

Proof. We want to apply Theorem 4.1 to $f + \eta$, $\eta \in \mathcal{S}(\mathbb{R})$. Recall that, for $H < \frac{1}{2}$, $M_+^H \eta \in L^{1/(1-H)}(\mathbb{R})$ for all $\eta \in \mathcal{S}(\mathbb{R})$ by the proof of Lemma 2.6 in Bender (2003a). Hence, $M_+^H(f + \eta) \in L^{1/(1-H)}(\mathbb{R})$ if $H < \frac{1}{2}$. By (17) and Theorem 4.1 we have:

$$
\begin{aligned}
S_{Q_f}&\left(\int_{\mathbb{R}} X_t\,\mathrm{d}B_t^H - \int_{\mathbb{R}} X_t(M_+^H f)(t)\mathrm{d}t\right)(\eta)\\
&= \mathrm{E}^{Q_{f+\eta}}\left[\int_{\mathbb{R}} X_t\,\mathrm{d}B_t^H - \int_{\mathbb{R}} X_t(M_+^H f)(t)\mathrm{d}t\right]\\
&= \int_{\mathbb{R}} \mathrm{E}^{Q_{f+\eta}}[X_t] M_+^H(f+\eta)(t)\mathrm{d}t - \int_{\mathbb{R}} \mathrm{E}^{Q_{f+\eta}}[X_t](M_+^H f)(t)\mathrm{d}t\\
&= \int_{\mathbb{R}} \mathrm{E}^{Q_{f+\eta}}[X_t](M_+^H \eta)(t)\mathrm{d}t = \int_{\mathbb{R}} \big(S_{Q_f} X_t\big)(\eta)(M_+^H \eta)(t)\mathrm{d}t.
\end{aligned}
$$

To obtain the second identity, we also used the fact that $\int_{\mathbb{R}} X_t(M_+^H f)(t)\mathrm{d}t$ exists as a Pettis integral, which was proven in Remark 4.1. In view of Definition 3.2 applied to the space $(\Omega, \mathcal{G}, Q_f)$, the asserted identity is proved. Note that it also holds P-almost surely, as the measures P and Q_f are equivalent. □

Remark 4.2. (i) Notice that the fractional Girsanov theorem also holds for anticipating integrands.
(ii) Consider a special case where $X_t = \mathbf{1}(a, b)(t)Y$ with $Y \in (L^2)$ such that the conditions of Theorem 4.2 are valid. Then

$$
Y \diamond_{Q_f} (\tilde{B}_b^H - \tilde{B}_a^H) = Y \diamond_P (B_b^H - B_a^H) - Y \cdot \int_a^b (M_+^H f)(s)\mathrm{d}s.
$$

Note that a priori there occur different Wick products with respect to the different probability measures on both sides of the equation. It is therefore somewhat surprising and not at all obvious that the Girsanov theorem extends to the fractional Itô integral.
(iii) In Elliott and van der Hoek (2003) and in a preprint of Hu and Øksendal (2003) a Girsanov theorem is proved for the fractional Brownian motion but applied to the (white noise definition of the) fractional Itô integral. After we had informed Hu and Øksendal about this gap, they included an additional result, Lemma 3.21, in the final version of their paper. This states that (in our notation) in the spaces of Hida distributions $(S)_P^*$ and $(S)_{Q_f}^*$ corresponding to the measures P and Q_f respectively, the Wick products $\diamond_P$ and $\diamond_{Q_f}$ coincide. In a denseness argument in the proof they implicitly use the fact that both spaces of Hida distributions are equal as sets and carry equivalent topologies. Both assertions are not proved and seem not to be trivial.

4.3. Removal of a drift

Usually (and, in particular, in no-arbitrage arguments) the classical Girsanov theorem is applied to remove a given drift $\int_a^b X_t g(t)\mathrm{d}t$. The fractional version stated in Theorem 4.2 allows a drift of the form

$$\int_a^b X_t (M_+^H f)(t) \mathrm{d}t$$

to be killed. Hence, we need to explore whether the fractional equation

$$(M_+^H f) = g \quad \text{a.s.} \tag{19}$$

has a solution $f \in L^2(\mathbb{R})$ for given g.

We start with $H < \frac{1}{2}$. Note that in this case the Theorems 4.1 and 4.2 require that $(M_+^H f) \in L^{1/(1-H)}(\mathbb{R})$. So the best we can hope for is:

Theorem 4.3. *Let* $0 < H < \frac{1}{2}$ *and* $g \in L^{1/(1-H)}(\mathbb{R})$. *Then* (19) *has a solution* $f \in L^2(\mathbb{R})$.

Proof. Let $f(x) = K_H^{-1}(I_+^{1/2-H} g)(x)$, where K_H is the constant given by (3). Then by Theorem 6.1 in Samko *et al*. (1993) and (3) above,

$$M_+^H f = D_+^{1/2-H} \left(I_+^{1/2-H} g \right) = g.$$

Now the Hardy–Littlewood theorem (Samko *et al*., 1993, Theorem 5.3) implies that $f \in L^2(\mathbb{R})$. □

If $H > \frac{1}{2}$ we have:

Theorem 4.4. *Let* $\frac{1}{2} < H < 1$ *and assume that* $g = I_+^{H-1/2} \phi$ *for a* $\phi \in L^2(\mathbb{R})$. *Then* (19) *has a solution* $f \in L^2(\mathbb{R})$.

The proof is given in Samko *et al*. (1993, Theorem 30.6). Moreover, the solution f is explicitly calculated.

For our purposes it is sufficient to solve (19) on the interval $[a, b]$. Hence, the following corollary is useful, as its assumptions are easier to verify:

Corollary 4.5. *Let* $\frac{1}{2} < H < 1$, *and suppose that* $g = \mathbf{1}(a, b)\psi$, $\psi : [a, b] \to \mathbb{R}$, *is Hölder continuous with exponent* $\lambda > H - \frac{1}{2}$. *Then* (19) *has a solution* $f \in L^2(\mathbb{R})$.

Proof. Let us first extend ψ to a λ-Hölder continuous function $\tilde{\psi} : \mathbb{R} \to \mathbb{R}$ with compact support. Then: $g = \mathbf{1}(a, b)\tilde{\psi}$ and the assertion follows from Lemma 5.3 in Pipiras and Taqqu (2000), Theorem 6.7 in Samko *et al*. (1993) and Theorem 4.4 above. □

Thus, we have proved the following variant of Theorem 4.1:

Theorem 4.6. *Let* $0 < H < 1$ *and* $a < b \in \mathbb{R}$. *Moreover, assume that* $X : [a, b] \to (L^2)$ *is H-fractional Itô integrable and* $X \in L^{1/H}([a, b], (L^2))$. *Additionally, suppose that* $g \in L^{1/(1-H)}(\mathbb{R})$ *and* $M_-^H(\mathbf{1}(a, b)X) \in L^2(\mathbb{R}, (L^2))$ *if* $H < \frac{1}{2}$, *and that* g *is* λ-*Hölder continuous with* $\lambda > H - \frac{1}{2}$ *if* $H > \frac{1}{2}$. *Then there is an* $f \in L^2(\mathbb{R})$ *such that*

$$\mathrm{E}^{Q_f}\left[\int_a^b X_t \,\mathrm{d}B_t^H\right] = \int_a^b \mathrm{E}^{Q_f}[X_t] g(t)\mathrm{d}t.$$

Remark 4.3. We could also formulate a similar variant of Theorem 4.2. But notice that this variant would require the integrability condition (18) to hold with respect to the measure Q_f, where f is the solution to (19) on $[a, b]$.

5. Itô formula and geometric fractional Brownian motion

5.1. Itô formula for a fractional Wiener integral

We begin this section by proving an Itô formula for a class of indefinite fractional Wiener integrals using the S-transform approach in Bender (2003a). In this way we show that our definition of an integral with respect to a fractional Brownian motion deserves to be called of Itô type. In general, an *indefinite H-fractional Wiener integral* is to be understood as a process

$$X_t = \int_0^t \sigma(s)\mathrm{d}B_s^H, \qquad 0 \leqslant t \leqslant T,$$

provided σ is a deterministic function such that the above integral exists for all $0 \leqslant t \leqslant T$.

Proposition 5.1. *Let $T > 0$, and $\sigma : [0, T] \to \mathbb{R}$ be continuous (if $\frac{1}{2} \leqslant H < 1$), or λ-Hölder continuous with $\lambda > \frac{1}{2} - H$ (if $0 < H < \frac{1}{2}$). Then the indefinite H-fractional Wiener integral exists for all $0 \leqslant t \leqslant T$, and*

$$\int_0^t \sigma(s)\mathrm{d}B_s^H = I\big(M_-^H(\mathbf{1}(0, t)\sigma)\big).$$

Proof. The case where $H \geqslant \frac{1}{2}$ is straightforward in view of Theorems 3.4 and 3.1(ii). For $H < \frac{1}{2}$ we have to prove $M_-^H(\mathbf{1}(0, t)\sigma) \in L^2(\mathbb{R})$ for all $0 \leqslant t \leqslant T$ to apply Theorem 3.4: Similar to the argument in Corollary 4.5, we may conclude that for fixed t there is a function $\phi \in L^2(\mathbb{R})$ such that

$$\mathbf{1}(0, t)\sigma = I_-^{1/2-H}\phi.$$

Hence, $M_-^H(\mathbf{1}(0, t)\sigma) \in L^2(\mathbb{R})$ by the inversion formula of the fractional calculus (Samko *et al.*, 1993, Theorem 6.1). □

Let us now prepare the ground for the proof of our Itô formula:

Lemma 5.2. *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be continuous and $H > \frac{1}{2}$. Then*

$$|M_-^H(\mathbf{1}(0,\ t)\sigma)|_0^2 = 2H(2H-1)\int_0^t\int_0^\tau \sigma(s)\sigma(\tau)|s-\tau|^{2H-2}\,\mathrm{d}s\,\mathrm{d}\tau \tag{20}$$

In particular:

(i) *For all* $t \geqslant 0$,

$$|M_-^H(\mathbf{1}(0,\ t)\sigma)|_0^2 \leqslant \max_{s\in[0,t]}|\sigma(s)|^2 t^{2H}$$

(ii) $|M_-^H(\mathbf{1}(0,\ t)\sigma)|_0^2$ *is differentiable in* t *and, for all* $t \geqslant 0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|M_-^H(\mathbf{1}(0,\ t)\sigma)|_0^2 &= 2H(2H-1)\int_0^t \sigma(s)\sigma(t)|s-t|^{2H-2}\mathrm{d}s\\ &\leqslant 2H\max_{s\in[0,t]}|\sigma(s)|^2 t^{2H-1}.\end{aligned}$$

Proof. By an identity of Gripenberg and Norros (1996) we have

$$|M_-^H(\mathbf{1}(0,\ t)\sigma)|_0^2 = H(2H-1)\int_{\mathbb{R}}\int_{\mathbb{R}}\mathbf{1}(0,\ t)(s)\sigma(s)\mathbf{1}(0,\ t)(\tau)\sigma(\tau)|s-\tau|^{2H-2}\,\mathrm{d}s\,\mathrm{d}\tau.$$

Equation (20) easily follows. The other assertions are direct implications of (20). □

Remark 5.1 For $H < \frac{1}{2}$ there is a lack of a result similar to the above lemma. Hence, we can consider the case of constant σ only in this case. Then we have

$$|M_-^H(\mathbf{1}(0,\ t)\sigma)|_0^2 = \sigma^2 t^{2H}.$$

Note that the Itô formula below would hold for $H < \frac{1}{2}$ and a λ-Hölder continuous σ ($\lambda > \frac{1}{2} - H$), if estimates similar to the above lemma could be established under these assumptions for $t > 0$.

Theorem 5.3. *Let* $T > 0$ *and* X *be an indefinite H-fractional Wiener integral with continuous integrand* σ *when* $H \geqslant \frac{1}{2}$ *and constant integrand when* $H < \frac{1}{2}$*. Furthermore, assume that* $F \in \mathcal{C}^{1,2}([0,\ T]\times\mathbb{R})$ *and there are constants* $C \geqslant 0$ *and* $\lambda < (2T^H \cdot \max_{s\in[0,T]}|\sigma(s)|)^{-2}$ *such that, for all* $(t,\ x) \in [0,\ T]\times\mathbb{R}$,

$$\max\left\{|F(t,\ x)|,\ \left|\frac{\partial}{\partial t}F(t,\ x)\right|,\ \left|\frac{\partial}{\partial x}F(t,\ x)\right|,\ \left|\frac{\partial^2}{\partial x^2}F(t,\ x)\right|\right\} \leqslant Ce^{\lambda x^2}.$$

Then the following equality holds in (L^2):

$$\begin{aligned}\int_0^T \sigma(t)\frac{\partial}{\partial x}F(t,\ X_t)\mathrm{d}B_t^H &= F(T,\ X_T) - F(0,\ 0) - \int_0^T \frac{\partial}{\partial t}F(t,\ X_t)\mathrm{d}t\\ &\quad - \frac{1}{2}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}|M_-^H(\mathbf{1}(0,\ t)\sigma)|_0^2\frac{\partial^2}{\partial x^2}F(t,\ X_t)\mathrm{d}t.\end{aligned} \tag{21}$$

Proof. By Definition 3.2 we have to prove that the right-hand side is an element of (L^2) and has S-transform given by

$$\int_0^T (M_+^H\eta)(t)S\left(\sigma(t)\frac{\partial}{\partial x}F(t, X_t)\right)(\eta)\mathrm{d}t = \int_0^T (M_+^H\eta)(t)\sigma(t)S\left(\frac{\partial}{\partial x}F(t, X_t)\right)(\eta)\mathrm{d}t. \tag{22}$$

Notice first that, by the growth condition for $G = F$, $(\partial/\partial t)F$, $(\partial/\partial x)F$, $(\partial^2/\partial x^2)F$ and $t \in [0, T]$,

$$\|G(t, X_t)\|_0^2 \leqslant C^2(1 - 4\lambda|M_-^H(\mathbf{1}(0, t)\sigma)|_0^2)^{-1/2} \leqslant \text{const.} \tag{23}$$

using Lemma 5.2 or Remark 5.1 below. Consequently all terms on the right-hand side exist in (L^2) (the integrals as (L^2)-valued Pettis integrals). For the last integral this can be proved in the following way:

$$\begin{aligned}
&\int_0^T \left\| \frac{\mathrm{d}}{\mathrm{d}t}|M_-^H(\mathbf{1}(0, t)\sigma)|_0^2 \frac{\partial^2}{\partial x^2}F(t, X_t)\right\|_0 \mathrm{d}t \\
&\leqslant \int_0^T \left| \frac{\mathrm{d}}{\mathrm{d}t}|M_-^H(\mathbf{1}(0, t)\sigma)|_0^2\right| \left\| \frac{\partial^2}{\partial x^2}F(t, X_t)\right\|_0 \mathrm{d}t \\
&\leqslant \text{const.}\int_0^T t^{2H-1}\,\mathrm{d}t < \infty
\end{aligned}$$

using Lemma 5.2 and (23) and then applying Theorem 2.8.

We shall now calculate the S-transform of the right-hand side of (21): To this end, let

$$g(t, x) := \frac{1}{\sqrt{2\pi t}}\exp\left\{\frac{-x^2}{2t}\right\}$$

be the heat kernel. Applying the classical Girsanov theorem for the measure Q_η, $\eta \in \mathcal{S}(\mathbb{R})$, and fractional integration by parts (Corollary 2.7) we may conclude that X_t is a Gaussian random variable with mean $\int_0^t \sigma(s)(M_+^H\eta)(s)\mathrm{d}s$ and variance $|M_-^H(\mathbf{1}(0, t)\sigma)|_0^2$ under the measure Q_η.

Thus we obtain, for $0 < t \leqslant T$,

$$\begin{aligned}
S(F(t, X_t))(\eta) &= \mathrm{E}^{Q_\eta}[F(t, X_t)] \\
&= \int_{\mathbb{R}} F\left(t, u + \int_0^t (M_+^H\eta)(s)\sigma(s)\mathrm{d}s\right) g(|M_-^H(\mathbf{1}(0, t)\sigma)|_0^2, u)\mathrm{d}u
\end{aligned}$$

By the growth condition we may interchange integration and differentiation to obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}S(F(t, X_t))(\eta) &= S\left(\frac{\partial}{\partial t}F(t, X_t)\right)(\eta) + (M_+^H\eta)(t)\sigma(t)S\left(\frac{\partial}{\partial x}F(t, X_t)\right)(\eta) \\
&\quad + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|M_-^H(\mathbf{1}(0, t)\sigma)|_0^2 \cdot S\left(\frac{\partial^2}{\partial x^2}F(t, X_t)\right)(\eta)
\end{aligned}$$

for $0 < t \leqslant T$ in the same way as in Bender (2003a). Hence,

$$S(F(T, X_T) - F(0, 0))(\eta) = \lim_{\epsilon \to 0} S(F(T, X_T) - F(\epsilon, X_\epsilon))(\eta)$$
$$= \int_0^T S\left(\frac{\partial}{\partial t} F(t, X_t)\right)(\eta)\mathrm{d}t + \int_0^T (M_+^H \eta)(t)\sigma(t) S\left(\frac{\partial}{\partial x} F(t, X_t)\right)(\eta)\mathrm{d}t$$
$$+ \frac{1}{2}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t} |M_-^H(\mathbf{1}(0, t)\sigma)|_0^2 \cdot S\left(\frac{\partial^2}{\partial x^2} F(t, X_t)\right)(\eta)\mathrm{d}t.$$

Thus, the S-transform of the right-hand side is given by (22). □

Remark 5.2. In the case $\sigma = 1$ the above theorem generalizes the Itô formula from Alòs *et al*. (2001) to the case where $H < \frac{1}{4}$. A different proof has recently and independently been given by Biagini *et al*. (2003). The advantage of our proof is that we do not have to approximate the function F as in Biagini *et al*. (2003). Moreover, our result allows non-constant integrands when $H > \frac{1}{2}$. Note that Theorem 5.3 also contains the versions of Itô's formula in Bender (2003a) provided F is classically differentiable. However, the Itô formula in Bender (2003a) holds in the framework of tempered distributions and generalized functionals of a fractional Brownian motion.

5.2. Geometric fractional Brownian motion

We now introduce geometric fractional Brownian motions with deterministic (but not necessarily constant) coefficients:

Definition 5.1. *Let* $H \in (0, 1)$, $x_0 > 0$ *and* $\sigma, r: [0, \infty) \to \mathbb{R}$. *Then we call*

$$P_t := x_0 \exp\left\{\int_0^t r(s)\mathrm{d}s - \tfrac{1}{2}|M_-^H(\mathbf{1}(0, t)\sigma)|_0^2 + \int_0^t \sigma(s)\mathrm{d}B_s^H\right\}$$

a geometric fractional Brownian motion *with coefficients* H, x_0, σ, r, *provided the right-hand side exists as an element of* (L^2) *for all* $0 \leqslant t < \infty$.

The following Itô formula is a consequence of Theorem 5.3.

Theorem 5.4 *Let* $T > 0$, P *be a geometric fractional Brownian motion with continuous coefficients* r *and* σ, *and let* σ *be a constant in the case where* $H < \frac{1}{2}$. *Furthermore, assume that* $F \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$ *such that* F, $(\partial/\partial t)F(t, x)$, $(\partial/\partial x)F(t, x)$, $(\partial^2/\partial x^2)F(t, x)$ *are of polynomial growth. Then the following equality holds in* (L^2):

$$\int_0^T \sigma(t)P_t \frac{\partial}{\partial x} F(t, P_t)\mathrm{d}B_t^H = F(T, P_T) - F(0, x_0) - \int_0^T \frac{\partial}{\partial t} F(t, P_t)\mathrm{d}t - \int_0^T r(t)P_t \frac{\partial}{\partial x} F(t, P_t)\mathrm{d}t - \frac{1}{2}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}|M_-^H(\mathbf{1}(0, t)\sigma)|_0^2 P_t^2 \frac{\partial^2}{\partial x^2} F(t, P_t)\mathrm{d}t.$$

Remark 5.3. Note that the assumptions on r and σ obviously ensure the existence of P.

Proof. Apply Theorem 5.3 to $F(t, g(t, x))$ with

$$g(t, x) := x_0 \exp\left\{\int_0^t r(s)\mathrm{d}s - \tfrac{1}{2}|M_-^H(\mathbf{1}(0, t)\sigma)|_0^2 + x\right\}.$$

□

The special case $F(t, x) = x$ yields:

Corollary 5.5. *Let P be a geometric fractional Brownian motion as in Theorem 5.4. Then, for all $t \geqslant 0$,*

$$P_t = x_0 + \int_0^t \sigma(s)P_s \mathrm{d}B_s^H + \int_0^t r(s)P_s \mathrm{d}s.$$

This corollary is a fractional analogue of the Doléans–Dade identity. It justifies the name 'geometric fractional Brownian motion'.

6. Comparison with related approaches

We finally compare our definition of an Itô type integral with respect to a fractional Brownian motion with related approaches in the literature. However, to make these approaches comparable, we have to choose a common basic setting. In particular, we will always assume that fractional Brownian motion is related to the underlying Brownian motion as described in Section 2. Additionally, we interpret application of the operator $M_\pm^H$ in the sense of Pettis integration.

6.1. The white noise approach

The general idea of the white noise approach is as follows. There is an appropriate space of generalized random variables (Hida distributions) $(S)^*$ which contains (L^2) such that the mapping $B^H : \mathbb{R} \to (S)^*$ is continuously differentiable. Moreover, the Wick product can be extended to a continuous mapping, $\diamond : (S)^* \times (S)^* \to (S)^*$. Now suppose a *stochastic* *distribution process*, that is, a mapping $X : \mathbb{R} \to (S)^*$, is given. Then we can consider the stochastic distribution process $X \diamond (\mathrm{d}/\mathrm{d}t)B^H$.

$(S)^*$-valued integration can be defined as a Pettis-type integral with the help of the extension of the S-transform to $(S)^*$. To be precise, a stochastic distribution process X is *integrable* if $\int_{\mathbb{R}}(SX_t)(\eta)\mathrm{d}t$ exists for all $\eta \in \mathcal{S}(\mathbb{R})$ and is the S-transform of a Hida distribution Φ. We then call Φ the white noise integral of X. We thus arrive at the following definition:

Definition 6.1. *A stochastic distribution process X has a* fractional Itô integral (white noise approach) *if the process $X \diamond (\mathrm{d}/\mathrm{d}t)B^H$ is white noise integrable. We then define its fractional Itô integral (white noise approach) to be the white noise integral of $X \diamond (\mathrm{d}/\mathrm{d}t)B^H$.*

This definition was first proposed by Hu and Øksendal (2003) in the case $H > \frac{1}{2}$ and extendend in slightly different settings to arbitrary Hurst parameters in Elliott and van der Hoek (2003) and Bender (2003a).

We have:

Theorem 6.1. *Let $X : \mathbb{R} \to (L^2)$. Then the white noise approach to the fractional Itô integral extends the S-transform approach. Conversely, if the fractional Itô integral of X exists in the white noise approach and is itself a member of (L^2), then it also exists in the S-transform approach.*

The proof follows directly from the fact that $S((\mathrm{d}/\mathrm{d}t)B_t^H)(\eta) = (M_+^H\eta)(t)$, which is proved in Bender (2003a, Theorem 3.7).

Therefore the S-transform approach preserves the generality of the white noise approach as long as we suppose the integrand and the fractional Itô integral to be (L^2)-valued. However, following the S-tranform approach we do not need the complicated constructions from the white noise analysis, such as the space of Hida distributions and the extensions of the Wick product and the S-transform, for which we refer the reader to Hida *et al*. (1993) and Kuo (1996).

6.2. The Malliavin calculus approach

The Malliavin cacluclus approach to integration with respect to a fractional Brownian motion originates in Decreusefond and Üstünel (1999) and Privault (1998). Most related to our paper are the definitions in Alòs *et al*. (2001) and Cheridito and Nualart (2002). In our setting we may define:

Definition 6.2. *Suppose $X : \mathbb{R} \to (L^2)$ such that $M_-^H X$ is Skorohod integrable (with respect to the underlying Brownian motion). Then X is said to have a* fractional Itô integral (Malliavin calculus approach) *which is defined to be the Skorohod integral of $M_-^H X$.*

Theorems 3.2 and 3.4 imply:

Theorem 6.2. *Suppose* $X \in L^p(\mathbb{R}, (L^2))$ *for some* $1 \leqslant p < \infty$ *in the case where* $H < \frac{1}{2}$ *or for some* $1 < p < (H - \frac{1}{2})^{-1}$ *in the case where* $H > \frac{1}{2}$. *Then X has a fractional Itô integral in the Malliavin calculus approach if and only if X has a fractional Itô integral in the S-transform approach and* $M_-^H X \in L^2(\mathbb{R}, (L^2))$. *In that case both approaches yield the same integral.*

The condition $M_-^H X \in L^2(\mathbb{R}, (L^2))$ is particularly restrictive for small Hurst parameters. For the smaller the Hurst parameter the higher the order of the fractional Marchaud derivative and consequently the higher the required smoothness of X. For this reason the Malliavin calculus approach in Alòs *et al*. (2001) is only developed for $H > \frac{1}{4}$. For a further discussion of this smoothness problem for small Hurst parameters we refer to the recent paper by Cheridito and Nualart (2002), in particular Proposition 3.

Cheridito and Nualart (2002) also propose a different generalization of the Skorohod integral approach. Roughly speaking, they replace the Wick exponentials in our *S*-transform approach by Hermite polynomials. However, we believe that the use of the Wick exponentials has several advantages over the use of the Hermite polynomials. First, the definition of the fractional Itô integral with the aid of the Wick exponentials is analogous to the definition of the Hitsuda–Skorohod integral. Thus, the fractional Itô integral (*S*-transform approach) coincides with a well-known integral for $H = \frac{1}{2}$. Second, the relation to the classical Girsanov theorem, which crucially depends on the use of the Wick exponentials, yields simple proofs. The reader might like to compare our proof of the fractional Itô formula with the rather lenghty argument in Cheridito and Nualart (2002).

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