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# Monotonicity of the difference between median and mean of gamma distributions and of a related Ramanujan sequence

SVEN ERICK ALM

*Department of Mathematics, Uppsala University, P.O. Box 480, SE-751 06 Uppsala, Sweden. E-mail: sea@math.uu.se*

For $n \geqslant 0$, let $\lambda_n$ be the median of the $\Gamma(n+1, 1)$ distribution. We prove that the sequence $\{\alpha_n = \lambda_n - n\}$ decreases from $\log 2$ to $\frac{2}{3}$ as $n$ increases from 0 to $\infty$. The difference, $1 - \alpha_n$, between the mean and the median thus increases from $1 - \log 2$ to $\frac{1}{3}$. This result also proves a conjecture by Chen and Rubin about the Poisson distributions: if $Y_\mu \sim \text{Poisson}(\mu)$, and $\lambda_n$ the largest $\mu$ such that $P(Y_\mu \leqslant n) = \frac{1}{2}$, then $\lambda_n - n$ is decreasing in $n$. The sequence $\{\alpha_n\}$ is related to a sequence $\{\theta_n\}$, introduced by Ramanujan, which is known to be decreasing and of the form $\theta_n = \frac{1}{3} + 4/(135(n + k_n))$, where $\frac{2}{21} < k_n \leqslant \frac{8}{45}$. We also show that the sequence $\{k_n\}$ is decreasing.

*Keywords:* gamma distribution; mean; median; Poisson distribution; Ramanujan

## 1. Introduction

Let $Y_\mu \sim \text{Poisson}(\mu)$, and $\lambda_n$ be the largest $\mu$ such that $P(Y_\mu \leqslant n) = \frac{1}{2}$. Using the well-known relation between the Poisson and gamma distributions, we obtain

$$\tfrac{1}{2} = P(Y_{\lambda_n} \leqslant n) = P(X_{n+1} > \lambda_n),$$

where $X_{n+1} \sim \Gamma(n+1, 1)$, so that $\lambda_n$ is the median of the $\Gamma(n+1, 1)$ distribution.

Chen and Rubin (1986) proved, in our notation, that

$$n + \tfrac{2}{3} < \lambda_n < n + 1, \tag{1}$$

and conjectured that

$$\alpha_n = \lambda_n - n$$

is decreasing in $n$. By (1),

$$\tfrac{2}{3} < \alpha_n < 1.$$

This result was sharpened by Choi (1994) to

$$\tfrac{2}{3} < \alpha_n \leqslant \log 2.$$

Choi also gave the asymptotic expansion

$$\alpha_n = \frac{2}{3} + \frac{8}{405n} - \frac{64}{5103n^2} + \frac{2^7 \cdot 23}{3^9 \cdot 25n^3} + O\left(\frac{1}{n^4}\right),$$

which gives

$$\Delta\alpha_n = \alpha_n - \alpha_{n+1} = \frac{8}{405n^2} - \frac{1144}{25\,515n^3} + O\left(\frac{1}{n^4}\right),$$

so that $\{\alpha_n\}$ is decreasing for sufficiently large $n$. In the next section, we will show that the sequence $\{\alpha_n\}$ is in fact decreasing for all $n \geqslant 0$, with $\alpha_0 = \log 2$ and $\alpha_\infty = \frac{2}{3}$. This proves Conjecture 2 of Chen and Rubin (1986).

The analysis of $\{\alpha_n\}$ (or $\{\lambda_n\}$) is closely related to the problem, set by Ramanujan (1911), of showing that

$$\frac{1}{2}\mathrm{e}^n = 1 + \frac{n}{1!} + \frac{n^2}{2!} + \ldots + \theta_n \frac{n^n}{n!}, \qquad \text{where } \theta_n \text{ lies between } \frac{1}{2} \text{ and } \frac{1}{3}, \tag{2}$$

which, in his edition of Ramanujan's notebooks, Berndt (1989) refers to as 'ultimately famous'.

A solution was outlined by Ramanujan (1912). Complete proofs were given by Szegő (1928), who also proved that the sequence $\{\theta_n\}$ is decreasing, and Watson (1929). In his first letter to Hardy dated 16 January, 1913, Ramanujan (1962) further claimed that

$$\theta_n = \frac{1}{3} + \frac{4}{135(n + k_n)}, \qquad \text{where } k_n \text{ lies between } \frac{8}{45} \text{ and } \frac{2}{21}. \tag{3}$$

This was proved by Flajolet *et al.* (1995). We will use this result in the next section to prove that the sequence $\{\alpha_n\}$ is decreasing, and in Section 3 we also prove that the sequence $\{k_n\}$ decreases for all $n \geqslant 0$, from $k_0 = \frac{8}{45}$ to $k_\infty = \frac{2}{21}$.

***Remark 1.*** Ramanujan's claim (3) is given as Exercise 1.2.11.3.13 by Knuth (1968)!

***Remark 2.*** Our interest in the sequence $\{\lambda_n\}$ came from a statistical problem in the analysis of safety at Swedish nuclear power plants, where, in order to estimate the mean $\mu$ of a Poisson distribution, we needed to create an upper 50% confidence limit for $\mu$ given the observation $n$. This confidence limit is $\lambda_n = n + \alpha_n$. As $\alpha_n > \frac{2}{3}$, and $\alpha_5 \approx 0.67$ (see Table 1), the estimate $n + 0.67$ is conservative for $n > 5$ if we can show that the sequence $\{\alpha_n\}$ is decreasing.

## 2. Monotonicity of $\{\alpha_n\}$

The values of $\alpha_n$ can easily be computed for small $n$. For $n \leqslant 10$ they are given in Table 1.

**Theorem 1.** *The sequence* $\{\alpha_n\}_0^\infty$ *is decreasing in* $n$ *for all* $n \geqslant 0$.

**Table 1.** Values of $\alpha_n$ and $\Delta\alpha_n$ for small $n$

| $n$ | $\alpha_n$ | $\Delta\alpha_n$ |
|---|---|---|
| 0 | 0.693 147 | 0.014 800 |
| 1 | 0.678 347 | 0.004 287 |
| 2 | 0.674 060 | 0.002 000 |
| 3 | 0.672 061 | 0.001 152 |
| 4 | 0.670 909 | 0.000 748 |
| 5 | 0.670 161 | 0.000 524 |
| 6 | 0.669 637 | 0.000 388 |
| 7 | 0.669 249 | 0.000 298 |
| 8 | 0.668 951 | 0.000 237 |
| 9 | 0.668 715 | 0.000 192 |
| 10 | 0.668 522 | 0.000 159 |

The proof of the theorem consists of a number of steps. The first step is to establish a relation between $\alpha_n$ and Ramanujan's $\theta_n$.

**Lemma 2.**

$$1-\theta_n=\int_0^{\alpha_n}\mathrm{e}^{-x}\cdot\left(1+\frac{x}{n}\right)^n\mathrm{d}x.$$

***Proof.*** As in Knuth (1968), let

$$I_1=\int_n^{\infty}\mathrm{e}^{-t}\cdot t^n\,\mathrm{d}t$$

and

$$I_2=\int_n^{n+\alpha_n}\mathrm{e}^{-t}\cdot t^n\,\mathrm{d}t.$$

Then, by (2),

$$\frac{I_1}{n!}=P(X_{n+1}>n)=P(Y_n\leqslant n)=\frac{1}{2}+(1-\theta_n)\frac{n^n}{n!}\mathrm{e}^{-n}$$

and

$$\frac{I_2}{n!}=\frac{1}{2}-P(X_{n+1}<n)=\frac{1}{2}-\left(1-\frac{I_1}{n!}\right)=(1-\theta_n)\frac{n^n}{n!}\mathrm{e}^{-n}.$$

By substituting $t=x+n$ in $I_2$, we obtain

$$I_2=\mathrm{e}^{-n}\int_0^{\alpha_n}\mathrm{e}^{-x}(n+x)^n\,\mathrm{d}x=\mathrm{e}^{-n}\cdot n^n\int_0^{\alpha_n}\mathrm{e}^{-x}\left(1+\frac{x}{n}\right)^n\mathrm{d}x,$$

which proves the lemma. □

The second step is to constructively estimate the integral in Lemma 2. Let $\gamma_n = 1 - \theta_n$. The sequence $\{\gamma_n\}$ is then increasing for all $n \geqslant 0$. By (2), we have an explicit expression for $\theta_n$, and hence for $\gamma_n$. Later, we will need $\gamma_n$ for some small values of $n$, so the first few are given in Table 2.

**Lemma 3.** *For* $n \geqslant 3$, *with* $\gamma_n = 1 - \theta_n$,

$$\gamma_n < \alpha_n - \frac{\alpha_n^3}{6n} + \frac{\alpha_n^4}{12n^2} + \frac{\alpha_n^5}{40n^2} - \frac{0.0022}{n^3},$$

$$\gamma_n > \alpha_n - \frac{\alpha_n^3}{6n} + \frac{\alpha_n^4}{12n^2} + \frac{\alpha_n^5}{40n^2} - \frac{0.0114}{n^3}.$$

***Proof.*** For $0 < x < 1$,

$$e^{-x}\left(1+\frac{x}{n}\right)^n \begin{cases} = \exp\left(-x + n\log\left(1+\frac{x}{n}\right)\right) = \exp\left(-\frac{x^2}{2n} + \frac{x^3}{3n^2} - \frac{x^4}{4n^3} + \dots\right), \\ < \exp\left(-\left(\frac{x^2}{2n} - \frac{x^3}{3n^2}\right)\right), \\ > \exp\left(-\left(\frac{x^2}{2n} - \frac{x^3}{3n^2} + \frac{x^4}{4n^3}\right)\right). \end{cases}$$

**Table 2.** Values of $\gamma_n$ for small $n$

| $n$ | $\gamma_n = 1 - \theta_n$ | Decimal |
|---|---|---|
| 0 | $\frac{1}{2}$ | 0.500 000 |
| 1 | $\frac{4-e}{2}$ | 0.640 859 |
| 2 | $\frac{10-e^2}{4}$ | 0.652 736 |
| 3 | $\frac{26-e^3}{9}$ | 0.657 163 |
| 4 | $\frac{206-3e^4}{64}$ | 0.659 462 |
| 5 | $\frac{2194-12e^5}{625}$ | 0.660 867 |

Now, for $0 < x < 1$,

$$1 - x + \frac{x^2}{2} - \frac{x^3}{6} < \mathrm{e}^{-x} < 1 - x + \frac{x^2}{2},$$

so that

$$\begin{aligned} \mathrm{e}^{-x}\left(1+\frac{x}{n}\right)^n &< \exp\left(-\left(\frac{x^2}{2n} - \frac{x^3}{3n^2}\right)\right) \\ &< 1 - \left(\frac{x^2}{2n} - \frac{x^3}{3n^2}\right) + \frac{1}{2}\left(\frac{x^2}{2n} - \frac{x^3}{3n^2}\right) \\ &= 1 - \frac{x^2}{2n} + \frac{x^3}{3n^2} + \frac{x^4}{8n^2} - \frac{x^5}{6n^3} + \frac{x^6}{18n^4}. \end{aligned} \tag{4}$$

Integrating (4), using $\alpha_n > \frac{2}{3}$ and $n \geqslant 3$, gives

$$\begin{aligned} \gamma_n &< \alpha_n - \frac{\alpha_n^3}{6n} + \frac{\alpha_n^4}{12n^2} + \frac{\alpha_n^5}{40n^2} - \frac{\alpha_n^6}{36n^3} + \frac{\alpha_n^7}{126n^4} \\ &< \alpha_n - \frac{\alpha_n^3}{6n} + \frac{\alpha_n^4}{12n^2} + \frac{\alpha_n^5}{40n^2} - \frac{0.0022}{n^3}, \end{aligned}$$

which proves the first part of the lemma.

Further,

$$\begin{aligned} \mathrm{e}^{-x}\left(1+\frac{x}{n}\right)^n &> \exp\left(-\left(\frac{x^2}{2n} - \frac{x^3}{3n^2} + \frac{x^4}{4n^3}\right)\right) \\ &> 1 - \left(\frac{x^2}{2n} - \frac{x^3}{3n^2} + \frac{x^4}{4n^3}\right) + \frac{1}{2}\left(\frac{x^2}{2n} - \frac{x^3}{3n^2} + \frac{x^4}{4n^3}\right)^2 - \frac{1}{6}\left(\frac{x^2}{2n} - \frac{x^3}{3n^2} + \frac{x^4}{4n^3}\right)^3. \end{aligned} \tag{5}$$

Integration of (5) and straightforward, but tedious, estimations, using $\alpha_n \leqslant \log 2$ and $n \geqslant 3$, give

$$\begin{aligned} \gamma_n &> \alpha_n - \frac{\alpha_n^3}{6n} + \frac{\alpha_n^4}{12n^2} + \frac{\alpha_n^5}{40n^2} \\ &\quad - \frac{\alpha_n^5}{20n^3} - \frac{\alpha_n^6}{36n^3} + \frac{\alpha_n^7}{7}\left(\frac{17}{72n^4} - \frac{1}{48n^3}\right) + \frac{\alpha_n^8}{8}\left(\frac{1}{24n^4} - \frac{1}{12n^5}\right) \\ &\quad + \frac{\alpha_n^9}{9}\left(\frac{1}{32n^6} - \frac{17}{288n^5}\right) + \frac{31\alpha_n^{10}}{6480n^6} - \frac{17\alpha_n^{11}}{6336n^7} + \frac{\alpha_n^{12}}{1152n^8} - \frac{\alpha_n^{13}}{4992n^9} \\ &= \alpha_n - \frac{\alpha_n^3}{6n} + \frac{\alpha_n^4}{12n^2} + \frac{\alpha_n^5}{40n^2} - \frac{0.0114}{n^3}, \end{aligned}$$

which proves the second part of the lemma. □

The third step is to invert Lemma 3, that is, to give upper and lower bounds for $\alpha_n$ expressed in $\gamma_n$.

**Lemma 4.** *For* $n \geqslant 3$,

$$\alpha_n < \gamma_n + \frac{\gamma_n^3}{6n} - \frac{\gamma_n^4}{12n^2} + \frac{7\gamma_n^5}{120n^2} + \frac{0.0068}{n^3},$$

$$\alpha_n > \gamma_n + \frac{\gamma_n^3}{6n} - \frac{\gamma_n^4}{12n^2} + \frac{7\gamma_n^5}{120n^2} - \frac{0.0079}{n^3}.$$

***Proof.*** Lemma 3, with $C_1 = 0.0022$ and $C_2 = 0.0114$, gives

$$\alpha_n < \gamma_n + \frac{\alpha_n^3}{6n} - \frac{\alpha_n^4}{12n^2} - \frac{\alpha_n^5}{40n^2} + \frac{C_2}{n^3}, \tag{6}$$

$$\alpha_n > \gamma_n + \frac{\alpha_n^3}{6n} - \frac{\alpha_n^4}{12n^2} - \frac{\alpha_n^5}{40n^2} + \frac{C_1}{n^3}. \tag{7}$$

On the right-hand side of these expressions, we need to replace the powers of $\alpha_n$ by expressions in $\gamma_n$. It is sufficient to do so with estimates of order $C/n^2$ and $C/n$. From (6) and (7) we immediately obtain

$$\alpha_n < \gamma_n + \frac{\alpha_n^3}{6n} - \frac{1}{n^2} \cdot \left( \frac{\left(\frac{2}{3}\right)^4}{12} + \frac{\left(\frac{2}{3}\right)^5}{40} - \frac{C_2}{3} \right) = \gamma_n + \frac{\alpha_n^3}{6n} - \frac{C_3}{n^2}, \tag{8}$$

$$\alpha_n > \gamma_n + \frac{\alpha_n^3}{6n} - \frac{1}{n^2} \cdot \left( \frac{(\log 2)^4}{12} + \frac{(\log 2)^5}{40} \right) = \gamma_n + \frac{\alpha_n^3}{6n} - \frac{C_4}{n^2}, \tag{9}$$

where $C_3 > 0$, and

$$\alpha_n < \gamma_n + \frac{(\log 2)^3}{6n} = \gamma_n + \frac{C_5}{n}, \tag{10}$$

$$\alpha_n > \gamma_n + \frac{\left(\frac{2}{3}\right)^3}{6n} - \frac{C_4}{3n} = \gamma_n + \frac{C_6}{n}. \tag{11}$$

This gives, with repeated use of (8) and recalling that $\alpha_n > \frac{2}{3}$ and $\gamma_n \geqslant \gamma_3$ for $n \geqslant 3$,

$$\begin{aligned}
\alpha_n^3 &< \alpha_n^2 \left( \gamma_n + \frac{\alpha_n^3}{6n} - \frac{C_3}{n^2} \right) < \alpha_n \gamma_n \left( \gamma_n + \frac{\alpha_n^3}{6n} - \frac{C_3}{n^2} \right) + \frac{\alpha_n^5}{6n} - \frac{C_3 \alpha_n^2}{n^2} \\
&< \gamma_n^3 + \frac{\alpha_n^3(\gamma_n^2 + \gamma_n \alpha_n + \alpha_n^2)}{6n} - \frac{C_3(\gamma_3^2 + \gamma_3 \cdot \frac{2}{3} + \left(\frac{2}{3}\right)^2)}{n^2} \\
&< \gamma_n^3 + \frac{\alpha_n^3(\gamma_n^2 + \gamma_n \alpha_n + \alpha_n^2)}{6n} - \frac{C_7}{n^2},
\end{aligned} \tag{12}$$

and, in the same way, using (9) and recalling that $\alpha_n \leqslant \log 2$ and $\gamma_n < \frac{2}{3}$,

$$\alpha_n^3 > \gamma_n^3 + \frac{\alpha_n^3(\gamma_n^2 + \gamma_n\alpha_n + \alpha_n^2)}{6n} - \frac{C_4\left(\left(\frac{2}{3}\right)^2 + \frac{2}{3}\log 2 + (\log 2)^2\right)}{n^2}$$
$$> \gamma_n^3 + \frac{\alpha_n^3(\gamma_n^2 + \gamma_n\alpha_n + \alpha_n^2)}{6n} - \frac{C_8}{n^2}. \tag{13}$$

Using (10) and (11), we obtain estimates of $\alpha_n^3$ of order $C/n$,

$$\alpha_n^3 < \alpha_n^2\left(\gamma_n + \frac{C_5}{n}\right) < \alpha_n\gamma_n\left(\gamma_n + \frac{C_5}{n}\right) + \frac{C_5\alpha_n^2}{n}$$
$$< \gamma_n^3 + \frac{C_5\left(\left(\frac{2}{3}\right)^2 + \frac{2}{3}\log 2 + (\log 2)^2\right)}{n} = \gamma_n^3 + \frac{C_9}{n}, \tag{14}$$

$$\alpha_n^3 > \gamma_n^3 + \frac{C_6(\gamma_3^2 + \gamma_3 \cdot \frac{2}{3} + \left(\frac{2}{3}\right)^2)}{n} = \gamma_n^3 + \frac{C_{10}}{n}, \tag{15}$$

and, with the same method estimates of $\alpha_n^4$,

$$\alpha_n^4 < \gamma_n^4 + \frac{C_5(\gamma_n^3 + \gamma_n^2\alpha_n + \gamma_n\alpha_n^2 + \alpha_n^3)}{n}$$
$$< \gamma_n^4 + \frac{C_5\left(\left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^2\log 2 + \frac{2}{3}(\log 2)^2 + (\log 2)^3\right)}{n} = \gamma_n^4 + \frac{C_{11}}{n}, \tag{16}$$

$$\alpha_n^4 > \gamma_n^4 + \frac{C_6\left(\gamma_3^3 + \gamma_3^2 \cdot \frac{2}{3} + \gamma_3\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3\right)}{n} = \gamma_n^4 + \frac{C_{12}}{n}, \tag{17}$$

and of $\alpha_n^5$,

$$\alpha_n^5 < \gamma_n^5 + \frac{C_5\left(\left(\frac{2}{3}\right)^4 + \left(\frac{2}{3}\right)^3\log 2 + \left(\frac{2}{3}\right)^2(\log 2)^2 + \frac{2}{3}(\log 2)^3 + (\log 2)^4\right)}{n} = \gamma_n^5 + \frac{C_{13}}{n}, \tag{18}$$

$$\alpha_n^5 > \gamma_n^5 + \frac{C_6\left(\gamma_3^4 + \gamma_3^3 \cdot \frac{2}{3} + \gamma_3^2\left(\frac{2}{3}\right)^2 + \gamma_3\left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4\right)}{n} = \gamma_n^5 + \frac{C_{14}}{n}. \tag{19}$$

Combining (12) with (14), (16) and (18) gives

$$\alpha_n^3 < \gamma_n^3 + \frac{\gamma_n^2}{6n}\left(\gamma_n^3 + \frac{C_9}{n}\right) + \frac{\gamma_n}{6n}\left(\gamma_n^4 + \frac{C_{11}}{n}\right) + \frac{1}{6n}\left(\gamma_n^5 + \frac{C_{13}}{n}\right) - \frac{C_7}{n^2}$$
$$< \gamma_n^3 + \frac{\gamma_n^5}{2n} + \frac{1}{6n^2}\left(C_9\left(\frac{2}{3}\right)^2 + C_{11} \cdot \frac{2}{3} + C_{13} - 6 \cdot C_7\right) = \gamma_n^3 + \frac{\gamma_n^5}{2n} + \frac{C_{15}}{n^2}, \tag{20}$$

and, using (13) with (15), (17) and (19),

$$\alpha_n^3 > \gamma_n^3 + \frac{\gamma_n^2}{6n}\left(\gamma_n^3 + \frac{C_{10}}{n}\right) + \frac{\gamma_n}{6n}\left(\gamma_n^4 + \frac{C_{12}}{n}\right) + \frac{1}{6n}\left(\gamma_n^5 + \frac{C_{14}}{n}\right) - \frac{C_8}{n^2}$$

$$> \gamma_n^3 + \frac{\gamma_n^5}{2n} + \frac{1}{6n^2}\left(C_{10}\cdot\gamma_3^2 + C_{12}\cdot\gamma_3 + C_{14} - 6\cdot C_8\right) = \gamma_n^3 + \frac{\gamma_n^5}{2n} + \frac{C_{16}}{n^2}. \qquad (21)$$

Further, inserting (16)–(21) into (6) and (7), we obtain

$$\alpha_n < \gamma_n + \frac{\gamma_n^3 + \gamma_n^5/2n + C_{15}/n^2}{6n} - \frac{\gamma_n^4 + C_{12}/n}{12n^2} - \frac{\gamma_n^5 + C_{14}/n}{40n^2} + \frac{C_2}{n^3}$$

$$< \gamma_n + \frac{\gamma_n^3}{6n} - \frac{\gamma_n^4}{12n^2} + \frac{7\gamma_n^5}{120n^2} + \frac{1}{n^3}\left(\frac{C_{15}}{6} - \frac{C_{12}}{12} - \frac{C_{14}}{40} + C_2\right),$$

and

$$\alpha_n > \gamma_n + \frac{\gamma_n^3 + \gamma_n^5/2n + C_{16}/n^2}{6n} - \frac{\gamma_n^4 + C_{11}/n}{12n^2} - \frac{\gamma_n^5 + C_{13}/n}{40n^2} + \frac{C_1}{n^3}$$

$$> \gamma_n + \frac{\gamma_n^3}{6n} - \frac{\gamma_n^4}{12n^2} + \frac{7\gamma_n^5}{120n^2} + \frac{1}{n^3}\left(\frac{C_{16}}{6} - \frac{C_{11}}{12} - \frac{C_{13}}{40} + C_1\right).$$

Finally, computing

$$\frac{C_{15}}{6} - \frac{C_{12}}{12} - \frac{C_{14}}{40} + C_2 < 0.0068$$

and

$$\frac{C_{16}}{6} - \frac{C_{11}}{12} - \frac{C_{13}}{40} + C_1 > -0.0079$$

finishes the proof of the lemma. □

In order to estimate $\Delta\alpha_n = \alpha_n - \alpha_{n+1}$, using Lemma 4, we first need to estimate $\Delta\gamma_n = \gamma_n - \gamma_{n+1}$.

**Lemma 5.**

$$0 > \Delta\gamma_n = \gamma_n - \gamma_{n+1} > -\frac{1364}{42\,525}\cdot\frac{1}{n(n+1)}.$$

***Proof.*** $\gamma_n - \gamma_{n+1} = (1-\theta_n) - (1-\theta_{n+1}) = \theta_{n+1} - \theta_n < 0$, and, by (3),

$$\theta_{n+1} - \theta_n = \frac{1}{3} + \frac{4}{135(n+1+k_{n+1})} - \frac{1}{3} - \frac{4}{135(n+k_n)}$$

$$= \frac{4}{135}\left(\frac{1}{n+1+k_{n+1}} - \frac{1}{n+k_n}\right)$$

$$> \frac{4}{135}\left(\frac{1}{n+1+\frac{8}{45}} - \frac{1}{n+\frac{2}{21}}\right) = -\frac{4}{135}\cdot\frac{341}{315}\cdot\frac{1}{\left(n+\frac{53}{45}\right)\left(n+\frac{2}{21}\right)}$$

$$> -\frac{1364}{42\,525}\cdot\frac{1}{n(n+1)}.$$

□

***Proof of Theorem 1.*** Let $C_1 = 0.0068$ and $C_2 = 0.0079$ denote the constants of Lemma 4 and $C_\gamma = \frac{1364}{42\,525}$ denote the constant of Lemma 5. Then, by Lemma 4,

$$\alpha_n - \alpha_{n+1} > \gamma_n + \frac{\gamma_n^3}{6n} - \frac{\gamma_n^4}{12n^2} + \frac{7\gamma_n^5}{120n^2} - \frac{C_2}{n^3}$$

$$- \left(\gamma_{n+1} + \frac{\gamma_{n+1}^3}{6(n+1)} - \frac{\gamma_{n+1}^4}{12(n+1)^2} + \frac{7\gamma_{n+1}^5}{120(n+1)^2} + \frac{C_1}{(n+1)^3}\right)$$

$$= \gamma_n - \gamma_{n+1} + \frac{\gamma_n^3 - \gamma_{n+1}^3}{6n} + \frac{\gamma_{n+1}^3}{6n(n+1)} + \frac{7}{120}\cdot\frac{\gamma_n^5 - \gamma_{n+1}^5}{n^2}$$

$$+ \frac{7}{120}\cdot\gamma_{n+1}^5\cdot\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) - \frac{\gamma_n^4 - \gamma_{n+1}^4}{12n^2}$$

$$- \frac{\gamma_{n+1}^4}{12}\cdot\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) - \frac{C_2}{n^3} - \frac{C_1}{(n+1)^3}. \tag{22}$$

As $\Delta\gamma_n = \gamma_n - \gamma_{n+1} < 0$, we obtain, for $n \geqslant 3$, using $\gamma_3 \leqslant \gamma_n < \gamma_{n+1} < \frac{2}{3}$,

$$\gamma_n^3 - \gamma_{n+1}^3 = \Delta\gamma_n \cdot (\gamma_n^2 + \gamma_n\gamma_{n+1} + \gamma_{n+1}^2) > \Delta\gamma_n \cdot 3 \cdot \left(\tfrac{2}{3}\right)^2 = \tfrac{4}{3}\cdot\Delta\gamma_n,$$

$$\gamma_n^4 - \gamma_{n+1}^4 = \Delta\gamma_n \cdot (\gamma_n^3 + \gamma_n^2\gamma_{n+1} + \gamma_n\gamma_{n+1}^2 + \gamma_{n+1}^3) < \Delta\gamma_n \cdot 4 \cdot \gamma_3^3,$$

$$\gamma_n^5 - \gamma_{n+1}^5 = \Delta\gamma_n \cdot (\gamma_n^4 + \gamma_n^3\gamma_{n+1} + \gamma_n^2\gamma_{n+1}^2 + \gamma_n\gamma_{n+1}^3 + \gamma_{n+1}^4) > \Delta\gamma_n \cdot 5 \cdot \left(\tfrac{2}{3}\right)^4 = \tfrac{80}{81}\Delta\gamma_n.$$

Further, for $n \geqslant 3$,

$$\gamma_4 \leqslant \gamma_{n+1} < \frac{2}{3}, \qquad \frac{3}{4} \leqslant \frac{n}{n+1} < 1,$$

$$\frac{2}{n(n+1)^2} < \frac{1}{n^2} - \frac{1}{(n+1)^2} < \frac{2}{n^2(n+1)}.$$

Inserting these estimates into (22), and using Lemma 5 and the fact that $\gamma_3^3/3 > 14/243$, we obtain

$$\begin{aligned}
\alpha_n - \alpha_{n+1} &> \Delta\gamma_n + \frac{\frac{4}{3}\Delta\gamma_n}{6n} + \frac{\gamma_4^3}{6n(n+1)} + \frac{7}{120}\cdot\frac{\frac{80}{81}\Delta\gamma_n}{n^2} + \frac{7}{120}\cdot\gamma_4^5\cdot\frac{2}{n(n+1)^2} \\
&\quad - \frac{4\gamma_3^3\Delta\gamma_n}{12n^2} - \frac{\left(\frac{2}{3}\right)^4}{12}\cdot\frac{2}{n^2(n+1)} - \frac{C_2}{n^3} - \frac{C_1}{(n+1)^3} \\
&> \frac{1}{n(n+1)}\cdot\left(\frac{\gamma_4^3}{6} - C_\gamma\right) + \frac{1}{n^2(n+1)}\cdot\left(-\frac{2}{9}\cdot C_\gamma - \frac{8}{243}\right) \\
&\quad + \frac{1}{n(n+1)^2}\cdot\frac{7}{60}\cdot\gamma_4^5 - \frac{C_2}{n^3} - \frac{C_1}{(n+1)^3} + \frac{C_\gamma}{n^3(n+1)}\left(\frac{\gamma_3^3}{3} - \frac{14}{243}\right) \\
&> \frac{1}{n(n+1)}\cdot\left(\frac{\gamma_4^3}{6} - C_\gamma\right) - \frac{1}{n^3}\left(\frac{2}{9}\cdot C_\gamma + \frac{8}{243} - \left(\frac{3}{4}\right)^2\cdot\frac{7}{60}\cdot\gamma_4^5 + C_1 + C_2\right) \\
&\quad + \frac{C_\gamma}{n^3(n+1)}\left(\frac{\gamma_3^3}{3} - \frac{14}{243}\right) \\
&> \frac{0.0157}{n(n+1)} - \frac{0.0466}{n^3} + \frac{0.0369}{n^3(n+1)} > 0 \qquad \text{if } n > 3.17,
\end{aligned}$$

so that $\{\alpha_n\}$ is decreasing for $n > 3$. Checking in Table 1 that $\{\alpha_n\}$ is decreasing also for $n \leqslant 3$ finishes the proof. □

## 3. Monotonicity of $\{k_n\}$

**Theorem 6.** *The Ramanujan sequence* $\{k_n\}$ *of* (3) *is decreasing for all* $n \geqslant 0$.

To prove this theorem we will use the technique of Flajolet *et al.* (1995) in their proof of (3), but we need to improve some of their estimates.

First, we need an asymptotic expansion for $\theta_n$. Marsaglia (1986) provides a method which gives an arbitrary number of terms in the expansion, the first being

$$\theta_n = \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} - \frac{16}{8505n^3} + \frac{8992}{12\,629\,925n^4} + O\left(\frac{1}{n^5}\right). \tag{23}$$

Solving for $k_n$ in (3) gives

$$k_n = \frac{4}{135(\theta_n - \frac{1}{3})} - n, \tag{24}$$

which, after inserting (23), gives the expansion

$$k_n = \frac{2}{21} + \frac{32}{441n} - \frac{50\,752}{4\,584\,195n^2} + O\left(\frac{1}{n^3}\right),$$

which shows that $\{k_n\}$ is decreasing for sufficiently large $n$, as the difference

$$\Delta k_n = k_n - k_{n+1} = \frac{32}{441n^2} + O\left(\frac{1}{n^3}\right) \tag{25}$$

obviously is positive for $n > n_0$, for some sufficiently large $n_0$.

In order to specify $n_0$, we need constructive bounds in (23) of the type

$$\theta_n < \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} - \frac{16}{8505n^3} + \frac{8992}{12\,629\,925n^4} + \frac{C_1}{n^5},$$

$$\theta_n > \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} - \frac{16}{8505n^3} + \frac{8992}{12\,629\,925n^4} + \frac{C_2}{n^5},$$

which give corresponding bounds for $k_n$:

$$k_n < \frac{2}{21} + \frac{32}{441n} - \frac{50\,752}{4\,584\,195n^2} + \frac{D_1}{n^3},$$

$$k_n > \frac{2}{21} + \frac{32}{441n} - \frac{50\,752}{4\,584\,195n^2} + \frac{D_2}{n^3}.$$

Then, $\Delta k_n = k_n - k_{n+1} > A_1/n^2 - A_2/n^3 > 0$ if $n > n_0 = A_2/A_1$. Checking that $\Delta k_n > 0$ for $n \leqslant n_0$ can then be done numerically, provided that $n_0$ is not too large.

Flajolet *et al.* (1995) give constructive bounds for the quantity

$$D(n) = 2 \cdot \theta_n,$$

introduced by Knuth (1968) as an example of asymptotic expansions, namely

$$D_{10}(n) - \Delta_{10}(n) \leqslant D(n) \leqslant D_{10}(n) + \Delta_{10}(n), \tag{26}$$

where

$$D_{10}(n) = \sum_{k=0}^{9} \frac{d_k}{n^k} = \frac{2}{3} + \frac{8}{135n} - \frac{16}{2835n^2} - \frac{32}{8505n^3} + \frac{17\,984}{12\,629\,925n^4} + \frac{13\,159\,709}{9\,699\,782\,400n^5}$$
$$- \frac{977\,069}{1\,039\,262\,400n^6} - \frac{36\,669\,961}{28\,291\,032\,000n^7} + \frac{117\,191}{56\,582\,064n^8} - \frac{479}{561\,330n^9}, \tag{27}$$

and the remainder $\Delta_{10}(n)$ is estimated by

$$\Delta_{10}(n) < F_1 \cdot n^{3/2} \cdot 2^{-n/2} + \frac{F_2}{n^5}, \tag{28}$$

where

$$F_1 = 13.06, \qquad F_2 = 56.593\,98.$$

Both constants, $F_1$ and $F_2$, depend on the coefficients $c_k$ in the expansion

$$\log\left(\frac{z^2}{2(1-(1+z)\mathrm{e}^{-z})}\right) = \sum_{k=1}^{\infty} c_k \cdot z^k. \tag{29}$$

***Remark 3.*** There is a misprint in Flajolet *et al.*'s (1995, p. 109) asymptotic expansion of $D_{10}(n)$, where the numerator of the term

$$\frac{17\,984}{12\,629\,925 n^4}$$

is given as 1794.

The estimate of $\Delta_{10}(n)$ used in Flajolet *et al.* (1995) is

$$\Delta_{10}(n) < \frac{10^{-7}}{n^3} + \frac{57}{n^5}, \qquad \text{for } n \geqslant 116, \tag{30}$$

which is insufficient for our needs, as we need an estimate of order

$$\Delta_{10}(n) < \frac{C}{n^5} \qquad \text{for } n \geqslant n_0.$$

This can, however, be obtained by replacing their estimate of the first term in (28) by

$$13.06 \cdot n^{3/2} \cdot 2^{-n/2} < \frac{K_0}{n^5}, \qquad \text{for } n \geqslant 116,$$

with $K_0 = 13.06 \cdot 116^{13/2} \cdot 2^{-58} < 0.001\,189$. The numerator 57 in (30) is actually $F_2 = 56.593\,98$, so that

$$\Delta_{10}(n) < \frac{56.595\,169}{n^5}. \tag{31}$$

Unfortunately, performing the analysis outlined above only shows that $\{k_n\}$ is decreasing for $n > n_1 > 26\,324$, so we need to improve the bound in (31). We will do this by a more careful estimation of the remainder term, $\Delta_{10}(n)$ of (28).

***Remark 4.*** With sufficient computing power, it may be possible to check by brute force, for example using Maple with sufficient precision, that the first 26 324 values of $k_n$ are indeed decreasing. □

As both terms on the right-hand side of (28) depend on $|c_k|$ from (29), it is natural to try to improve the estimate given in Lemma 4 of Flajolet *et al.* (1995):

$$|c_k| < \frac{10.967\,148\,33}{\pi^k}, \qquad \text{for all } k \geqslant 1.$$

This can be achieved by a slight modification of their proof, and by noting that we only need an estimate for $k > 10$.

**Lemma 7.** *For $k > 10$, we have*

$$|c_k| < \frac{0.4593}{\left(\frac{6}{5}\pi\right)^k}.$$

***Proof.*** Recall that $c_k$ are defined, in (29), as the coefficients in the expansion of $\log f(z)$, where

$$f(z) = \frac{z^2}{2(1-(1+z)\mathrm{e}^{-z})},$$

and thus, by Cauchy's formula, can be written

$$c_k = \frac{1}{2\pi \mathrm{i}} \oint_{\mathcal{A}} \frac{\log f(z)}{z^{k+1}}\, \mathrm{d}z,$$

where $\mathcal{A}$ is a contour encircling the origin, and chosen so that $\log f(z)$ is well defined on it.

In Flajolet *et al*. (1995), $\mathcal{A}$ is chosen as $\mathcal{D}$, the boundary of the square $|\mathrm{Re}(z)| \leqslant \pi$, $|\mathrm{Im}(z)| \leqslant \pi$, that is with side $2\pi$. We will use the slightly larger square $|\mathrm{Re}(z)| \leqslant 6\pi/5$, $|\mathrm{Im}(z)| \leqslant 6\pi/5$, with side $12\pi/5$. Figure 1, and the argument principle, show that there are no poles or zeros of $f(z)$ on $\mathcal{A}$. We will estimate $\log f(z)$ separately on the four sides of the square. Let, for $-1 \leqslant \tau \leqslant 1$,

$$\mathcal{A}_1 : z = \tfrac{6}{5}\pi(1 + \mathrm{i}\tau),$$

$$\mathcal{A}_2 : z = \tfrac{6}{5}\pi(\tau + \mathrm{i}),$$

$$\mathcal{A}_3 : z = \tfrac{6}{5}\pi(-1 + \mathrm{i}\tau),$$

$$\mathcal{A}_4 : z = \tfrac{6}{5}\pi(\tau - \mathrm{i}),$$

and let

$$a_i = \max|\log f(z)| \text{ on } \mathcal{A}_i.$$

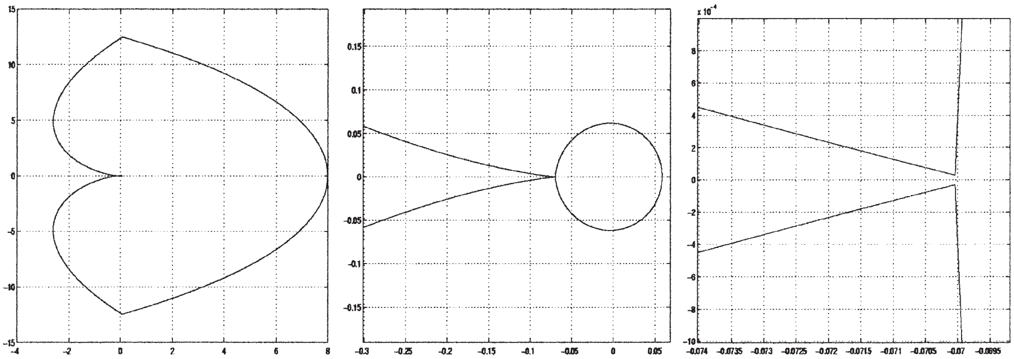


**Figure 1.** A plot of $f(z)$ on $\mathcal{A}$ and zoomed close to the origin

As $|z| \geqslant \frac{6}{5}\pi$ on $\mathcal{A}$, we obtain, for $k > 10$,

$$\oint_{\mathcal{A}_1} \frac{|\log f(z)|}{|z|^{k+1}}\,\mathrm{d}z \leqslant \frac{a_1}{\left(\frac{6}{5}\pi\right)^{k+1}} \int_{-1}^{1} \frac{d\tau}{|1+\mathrm{i}\tau|^{k+1}} = \frac{2a_1}{\left(\frac{6}{5}\pi\right)^{k+1}} \int_{0}^{1} \frac{d\tau}{(1+\tau^2)^{(k+1)/2}}$$

$$< \frac{2a_1}{\left(\frac{6}{5}\pi\right)^{k+1}} \int_{0}^{1} \frac{d\tau}{(1+\tau^2)^6} = \frac{2a_1}{\left(\frac{6}{5}\pi\right)^{k+1}} \left(\frac{63\pi}{1024} + \frac{61}{320}\right)$$

$$= \frac{a_1}{\left(\frac{6}{5}\pi\right)^{k+1}} \left(\frac{63\pi}{512} + \frac{61}{160}\right),$$

and, in the same way,

$$\oint_{\mathcal{A}_i} \frac{|\log f(z)|}{|z|^{k+1}}\,\mathrm{d}z < \frac{a_i}{\left(\frac{6}{5}\pi\right)^{k+1}} \left(\frac{63\pi}{512} + \frac{61}{160}\right), \qquad \text{for } k > 10,$$

so that

$$\oint_{\mathcal{A}} \frac{|\log f(z)|}{|z|^{k+1}}\,\mathrm{d}z < \frac{a_1+a_2+a_3+a_4}{\left(\frac{6}{5}\pi\right)^{k+1}} \left(\frac{63\pi}{512} + \frac{61}{160}\right), \qquad \text{for } k > 10. \tag{32}$$

As confirmed by Figure 2, $|\log f(z)|$ has its maxima in the corners of $\mathcal{A}$. This gives

$$a_1 = \left|\log f\left(\tfrac{6}{5}\pi(1+\mathrm{i})\right)\right| \leqslant 2.969\,411\,47,$$

$$a_2 = \left|\log f\left(\tfrac{6}{5}\pi(-1+\mathrm{i})\right)\right| \leqslant 4.115\,288\,07,$$

$$a_3 = \left|\log f\left(\tfrac{6}{5}\pi(-1+\mathrm{i})\right)\right| = a_2,$$

$$a_4 = \left|\log f\left(\tfrac{6}{5}\pi(-1-\mathrm{i})\right)\right| = a_2.$$

From Figure 2, we also see that, by splitting the integral into eight parts, instead of four, we can improve the estimate of (32) to

$$\oint_{\mathcal{A}} \frac{|\log f(z)|}{|z|^{k+1}}\,\mathrm{d}z < \frac{4a_1+4a_2}{\left(\frac{6}{5}\pi\right)^{k+1}} \left(\frac{63\pi}{1024} + \frac{61}{320}\right) = \frac{a_1+a_2}{\left(\frac{6}{5}\pi\right)^{k+1}} \left(\frac{63\pi}{256} + \frac{61}{80}\right), \qquad \text{for } k > 10.$$

Thus,

$$|c_k| \leqslant \frac{1}{2\pi} \frac{a_1+a_2}{\left(\frac{6}{5}\pi\right)^{k+1}} \left(\frac{63\pi}{256} + \frac{61}{80}\right) < \frac{0.4593}{\left(\frac{6}{5}\pi\right)^{k}}, \qquad \text{for } k > 10,$$

which proves the lemma. □

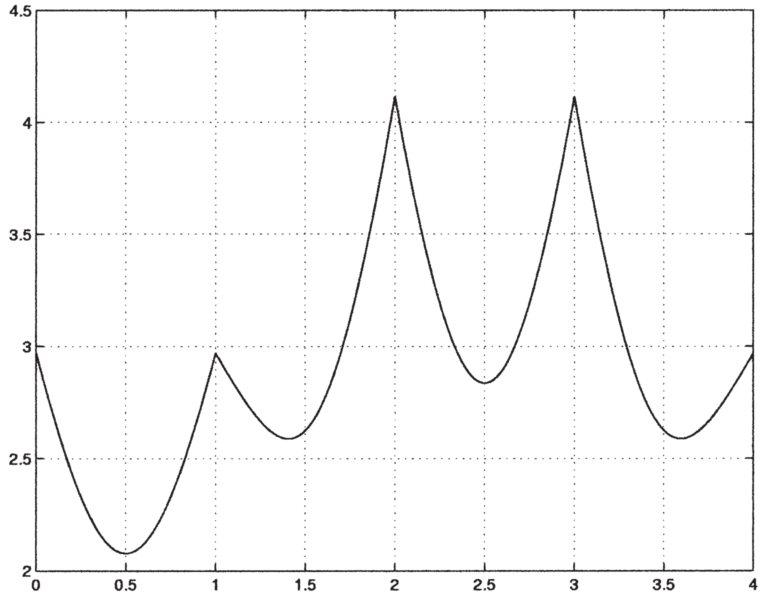


**Figure 2.** A plot of $|\log f(z)|$ on $\mathcal{A}$, in the order $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$

***Remark 5.*** There seems to be a mistake in Figure 1 of Flajolet *et al*. (1995), as it does not have winding number 0, as claimed, and does not resemble our plot of $f(z)$ on $\mathcal{D}$ (see Figure 3).

Using the estimate of Lemma 7 instead of the one given in Lemma 4 of Flajolet *et al.* (1995) gives a much improved estimate of the remainder $\Delta_{10}(n)$.

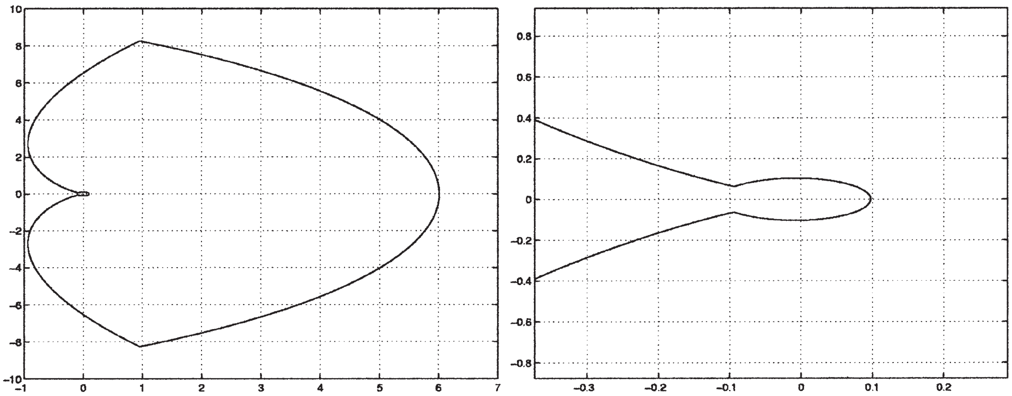

**Figure 3.** A plot of $f(z)$ on $\mathcal{D}$ from Flajolet *et al*. (1995) and zoomed at the origin

**Lemma 8.**

$$\Delta_{10}(n) < 0.0474 n^{3/2}\, 2^{-n/2} + \frac{0.295\,96}{n^5}.$$

***Proof.*** The lemma is obtained by a straightforward modification of Lemma 5 of Flajolet *et al*. (1995), and of the estimate of $\mu_{10}$ of their Lemma 6, by simply replacing the estimate of $|c_k|$. □

It is sufficient to bound $\Delta_{10}(n)$ by $C/n^4$, provided that the constant $C$ is sufficiently small; less than the coefficient $d_4$ of $1/n^4$ in (27).

**Lemma 9.** *For* $n \geqslant 208$,

$$\Delta_{10}(n) \leqslant \frac{C_\Delta}{n^4},$$

*where* $C_\Delta = 0.001\,422\,9$.

***Proof.*** $n^{11/2} \cdot 2^{-n/2}$ is decreasing for $n \geqslant 16$, so that, for $n \geqslant n_0 \geqslant 16$,

$$\Delta_{10}(n) \leqslant \frac{1}{n^4}\left(0.0474 n_0^{11/2}\, 2^{-n_0/2} + \frac{0.295\,96}{n_0}\right).$$

Choosing $n_0 = 208$ gives the lemma. □

The next lemma gives the necessary upper and lower bounds for $\theta_n$.

**Lemma 10.** *For* $n \geqslant 208$,

$$\theta_n \leqslant \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} - \frac{16}{8505n^3} + \frac{C_1}{n^4},$$

$$\theta_n \geqslant \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} - \frac{16}{8505n^3} + \frac{C_2}{n^4}$$

$$\geqslant \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} - \frac{C_3}{n^3},$$

*with* $C_1 = 0.001\,427$, $C_2 = 0.000\,000\,5$ *and* $C_3 = \frac{16}{8505}$.

***Proof.*** By (26) and Lemma 9,

$$2\,\theta_n = D(n) < D_{10}(n) + \Delta_{10}(n) < \sum_{k=0}^{9} \frac{d_k}{n^k} + \frac{C_\Delta}{n^4}.$$

Here, $d_6 < 0$, $d_9 < 0$ and $nd_7 + d_8 < 0$, so that

$$D(n) < \sum_{k=0}^{3} \frac{d_k}{n^k} + \frac{1}{n^4}\left(d_4 + C_\Delta + \frac{d_5}{n}\right).$$

Choosing $C_1 > (d_4 + C_\Delta + d_5/208)/2$ proves the first inequality.
Similarly,

$$D(n) > D_{10}(n) - \Delta_{10}(n) > \sum_{k=0}^{9} \frac{d_k}{n^k} - \frac{C_\Delta}{n^4}.$$

Here, $n\, d_8 + d_9 > 0$ and $n^2\, d_5 + n\, d_6 + d_7 > 0$, so that

$$D(n) > \sum_{k=0}^{3} \frac{d_k}{n^k} + \frac{1}{n^4}(d_4 - C_\Delta).$$

Choosing $C_2 < (d_4 - C_\Delta)/2$ proves the second inequality, and the third follows immediately as $C_2 > 0$. □

***Proof of Theorem 6.*** First, assume that $n \geqslant 208$. Using (24), we obtain

$$\begin{aligned}\Delta k_n = k_n - k_{n+1} &= \frac{4}{135\left(\theta_n - \frac{1}{3}\right)} - n - \left(\frac{4}{135\left(\theta_{n+1} - \frac{1}{3}\right)} - (n+1)\right) \\ &= 1 + \frac{4}{135\left(\theta_n - \frac{1}{3}\right)} - \frac{4}{135\left(\theta_{n+1} - \frac{1}{3}\right)} = 1 + \frac{4}{135}\,\frac{\theta_{n+1} - \theta_n}{\left(\theta_n - \frac{1}{3}\right)\left(\theta_{n+1} - \frac{1}{3}\right)} \\ &= 1 - \frac{4}{135}\,\frac{\theta_n - \theta_{n+1}}{\left(\theta_n - \frac{1}{3}\right)\left(\theta_{n+1} - \frac{1}{3}\right)}. \end{aligned} \tag{33}$$

Using Lemma 10, we obtain

$$\begin{aligned}\theta_n - \theta_{n+1} &< \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} - \frac{16}{8505n^3} + \frac{C_1}{n^4} \\ &\quad - \left(\frac{1}{3} + \frac{4}{135\,(n+1)} - \frac{8}{2835\,(n+1)^2} - \frac{16}{8505\,(n+1)^3} + \frac{C_2}{(n+1)^4}\right) \\ &= \frac{4}{135n(n+1)} - \frac{8}{2835}\,\frac{2n+1}{n^2\,(n+1)^2} - \frac{16}{8505}\,\frac{3n^2+3n+1}{n^3\,(n+1)^3} + \frac{C_1}{n^4} - \frac{C_2}{(n+1)^4} \\ &< \frac{4}{135n(n+1)} - \frac{16}{2835}\,\frac{1}{n^2\,(n+1)} + \frac{8-16}{2835}\,\frac{1}{n^2\,(n+1)^2} + \frac{C_1}{n^4} - \frac{C_2}{(n+1)^4} \\ &< \frac{4}{135n(n+1)} - \frac{16}{2835}\,\frac{1}{n^2\,(n+1)} - \frac{C_\theta}{n^3(n+1)}, \end{aligned} \tag{34}$$

where

$$C_\theta = \frac{208}{209}\frac{8}{2835} - \frac{209}{208}\,C_1 + \left(\frac{208}{209}\right)^3 \cdot C_2 > 0.$$

Further,

$$\begin{aligned}
&\left(\theta_n - \frac{1}{3}\right)\cdot\left(\theta_{n+1} - \frac{1}{3}\right)\\
&> \left(\frac{4}{135n} - \frac{8}{2835n^2} - \frac{C_3}{n^3}\right)\cdot\left(\frac{4}{135\,(n+1)} - \frac{8}{2835\,(n+1)^2} - \frac{C_3}{(n+1)^3}\right)\\
&= \left(\frac{4}{135}\right)^2 \frac{1}{n(n+1)}\left(1 - \frac{2}{21n} - \frac{135\,C_3}{4n^2}\right)\cdot\left(1 - \frac{2}{21(n+1)} - \frac{135\,C_3}{4(n+1)^2}\right)\\
&= \left(\frac{4}{135}\right)^2 \frac{1}{n(n+1)}(1 - g(n)),
\end{aligned}$$

where, recalling that $C_3 = \frac{16}{8505} > 0$,

$$\begin{aligned}
g(n) = {}& \frac{2}{21n} + \frac{2}{21(n+1)} + \frac{135\,C_3}{4n^2} - \frac{4}{441n(n+1)} + \frac{135\,C_3}{4(n+1)^2}\\
&- \frac{45\,C_3}{14n^2(n+1)} - \frac{45\,C_3}{14n(n+1)^2} - \left(\frac{135}{4}\right)^2 \frac{C_3^2}{n^2(n+1)^2}\\
< {}& \frac{4}{21n} - \frac{2}{21n(n+1)} + \frac{4}{63n^2} - \frac{4}{441n(n+1)} + \frac{4}{63(n+1)^2}\\
= {}& \frac{4}{21n} + \frac{4}{63}\left(\frac{1}{n} - \frac{1}{n+1}\right)^2 + \frac{8}{63n(n+1)} - \frac{46}{441n(n+1)}\\
= {}& \frac{4}{21n} + \frac{4}{63n^2(n+1)^2} + \frac{10}{441n(n+1)} < \frac{4}{21n} + \frac{C_g}{n^2},
\end{aligned}$$

where

$$C_g = \frac{4}{63\cdot(209)^2} + \frac{10}{441}.$$

Using the identity

$$\frac{1}{1-x} = 1 + x + \frac{x^2}{1-x},$$

we obtain

$$\frac{1}{1-g(n)} < 1 + \frac{4}{21n} + \frac{C_g}{n^2} + \frac{(4/21n + C_g/n^2)^2}{1 - 4/21n - C_g/n^2}$$

$$< 1 + \frac{4}{21n} + \frac{1}{n^2}\left(C_g + \frac{(4/21 + C_g/n)^2}{1 - 4/21n - C_g/n^2}\right)$$

$$< 1 + \frac{4}{21n} + \frac{C_0}{n^2},$$

where

$$C_0 = C_g + \frac{(4/21 + C_g/208)^2}{1 - 4/(21 \cdot 208) - C_g/208^2}.$$

Thus,

$$\frac{1}{\left(\theta_n - \frac{1}{3}\right) \cdot \left(\theta_{n+1} - \frac{1}{3}\right)} < \left(\frac{135}{4}\right)^2 \cdot n(n+1) \cdot \left(1 + \frac{4}{21n} + \frac{C_0}{n^2}\right), \tag{35}$$

so that, inserting (34) and (35) into (33),

$$\Delta k_n > 1 - \frac{4}{135}\left(\frac{4}{135}\,\frac{1}{n(n+1)} - \frac{16}{2835}\,\frac{1}{n^2(n+1)} - \frac{C_\theta}{n^3(n+1)}\right)$$

$$\cdot \left(\frac{135}{4}\right)^2 \cdot n(n+1) \cdot \left(1 + \frac{4}{21n} + \frac{C_0}{n^2}\right)$$

$$= 1 - \left(1 - \frac{4}{21n} - \frac{135\,C_\theta}{4n^2}\right) \cdot \left(1 + \frac{4}{21n} + \frac{C_0}{n^2}\right)$$

$$= 1 - \left(1 + \frac{4}{21n} + \frac{C_0}{n^2} - \frac{4}{21n} - \frac{16}{441n^2} - \frac{4\,C_0}{21n^3} - \frac{135\,C_\theta}{4n^2} - \frac{45\,C_\theta}{7n^3} - \frac{135\,C_\theta\,C_0}{4n^4}\right)$$

$$= \frac{1}{n^2}\left(\frac{16}{441} - C_0 + \frac{135\,C_\theta}{4}\right) + \frac{1}{n^3}\left(\frac{4\,C_0}{21} + \frac{45\,C_\theta}{7}\right) + \frac{1}{n^4} \cdot \frac{135\,C_\theta\,C_0}{4}$$

$$> \frac{A_2}{n^2},$$

where

$$A_2 = \frac{16}{441} - C_0 + \frac{135\,C_\theta}{4} > 0.0236 > 0,$$

so that $\Delta k_n > 0$ for all $n \geqslant 208$.

It only remains to verify that $\Delta k_n > 0$ also for $n < 208$. The first few $k_n$ and $\Delta k_n$

**Table 3.** Values of $k_n$ and $\Delta k_n$ for small $n$

| $n$ | $k_n$ | $\Delta k_n$ |
|---|---|---|
| 0 | 0.177 778 | 0.029 680 |
| 1 | 0.148 098 | 0.021 166 |
| 2 | 0.126 932 | 0.009 370 |
| 3 | 0.117 562 | 0.005 163 |
| 4 | 0.112 399 | 0.003 245 |
| 5 | 0.109 155 | 0.002 221 |
| 6 | 0.106 933 | 0.001 613 |
| 7 | 0.105 320 | 0.001 224 |
| 8 | 0.104 096 | 0.000 960 |
| 9 | 0.103 136 | 0.000 773 |
| 10 | 0.102 363 | 0.000 635 |

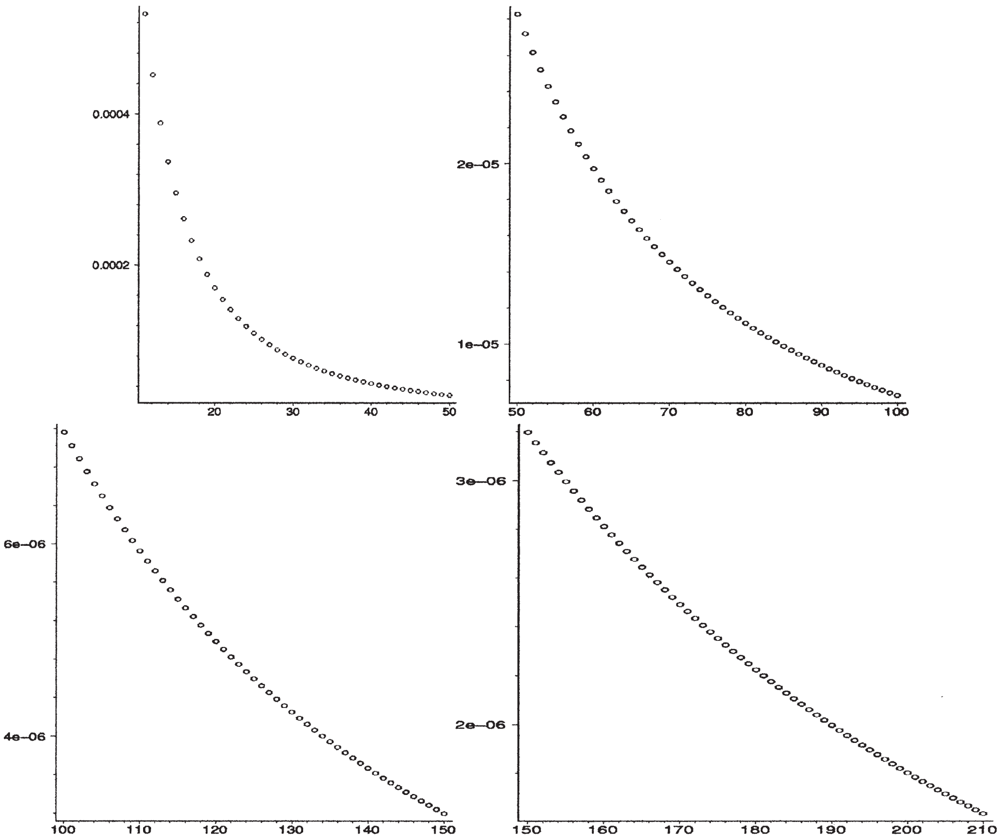


**Figure 4.** A plot of $\Delta k_n$ for $10 < n \leqslant 210$

($n \leqslant 10$) are given in Table 3. For $10 < n \leqslant 210$, we see from the plot in Figure 4 that $\Delta k_n > 0$, which finishes the proof. □

***Remark 6.*** Computations were performed with Maple and Matlab. $k_n$ and $\Delta k_n$ were computed by Maple with 100 digits' precision. Figures 1–3 were produced by Matlab, where the zoom option was most useful for Figures 1 and 3, whereas Figure 4 was produced with Maple.

***Remark 7.*** Table 3, Figure 4 and (25) indicate that also the sequence $\{\Delta k_n\}$ is decreasing for all $n$.

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