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Which negative multinomial distributions are infinitely divisible?

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We define a negative multinomial distribution on $\mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers, by its probability generating function which will be of the form $(A(a_1, \ldots, a_n)/A(a_1 z_2, \ldots, a_n z_n))^\lambda$, where

$$A(\mathbf{z}) = \sum_{T \subset \{1,2,\ldots,n\}} a_T \prod_{i \in T} z_i,$$

$a_\varnothing \neq 0$, and λ is a positive number. Finding couples (A, λ) for which we obtain a probability generating function is a difficult problem. We establish necessary and sufficient conditions on the coefficients of A for which we obtain a probability generating function for any positive number λ. In consequence, we obtain all infinitely divisible negative multinomial distributions on $\mathbb{N}_0^n$.

Keywords: discrete exponential families; infinitely divisible distribution; negative multinomial distribution; probability generating function

1. Negative multinomial distributions

In order to generalize a given standard distribution on the set $\mathbb{N}_0$ of non-negative integers to $\mathbb{N}_0^n$, one could simply replace its generating function $f(z)$ by $f(a_1 z_1 + \ldots + a_n z_n)$. Thus, for instance, the Bernoulli distribution $q\delta_0 + p\delta_1$ becomes $a_0 + a_1\delta_{e_1} + \ldots + a_n\delta_{e_n}$, where δ_a denotes the probability measure concentrated at a, $(e_1, \ldots e_n)$ is the standard basis for $\mathbb{R}^n$, $a_j > 0$, and $\sum_{j=0}^n a_j = 1$. Similarly, the negative binomial distribution $\sum_{n=0}^\infty (n!)^{-1}\lambda(\lambda+1)\ldots(\lambda+n-1)p^\lambda q^n \delta_n$, whose probability generating function is $f(z) = p^\lambda(1-qz)^{-\lambda}$, becomes $\sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n} p_{\boldsymbol{\alpha}} \delta_{\boldsymbol{\alpha}}$, whose generating function is

$$\sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n} p_{\boldsymbol{\alpha}} z_1^{\alpha_1} \ldots z_n^{\alpha_n} = \left(\frac{p}{1 - q(p_1 z_1 + \ldots + p_n z_n)} \right)^\lambda.$$

Some authors (see Johnson *et al*. 1997, p. 93) call this distribution a negative multinomial distribution.

However, it seems natural to have a wider generalization, by replacing $q + pz$ in the Bernoulli case or $1 - qz$ in the negative binomial case by an affine polynomial P, that is, a polynomial which is affine with respect to each z_j, $j = 1, \ldots, n$, or for which $\partial^2 P/\partial z_j^2 = 0$ for all $j = 1, \ldots, n$. For instance, for $n = 2$, P has the form

$P(z_1, z_2) = a_0 + a_1 z_1 + a_2 z_2 + a_{12} z_1 z_2$. Actually, some authors (see references in Bar-Lev *et al.* 1994; Doss 1979; Griffiths and Milne 1987) define the multinomial distribution as a distribution on $\mathbb{N}_0^n$ whose probability generating function is P^N, where P is an affine polynomial and $N \in \mathbb{N}$, the set of positive integers.

Similarly, for the negative binomial case we shall say that the probability distribution $\sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n} p_{\boldsymbol{\alpha}} \delta_{\boldsymbol{\alpha}}$ on $\mathbb{N}_0^n$ is a *negative multinomial distribution* if there exists an affine polynomial $P(z_1, \ldots, z_n)$ and $\lambda > 0$ such that

$$\sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n} p_{\boldsymbol{\alpha}} z_1^{\alpha_1} \ldots z_n^{\alpha_n} = \big(P(z_1, \ldots, z_n)\big)^{-\lambda}. \tag{1.1}$$

Of course, not all affine polynomials will give rise to a valid probability generating function. First of all, the number $P(1, \ldots, 1)$ must equal 1. Furthermore, the coefficients of P must be such that $(z_1, \ldots, z_n) \mapsto P^{-\lambda}$ is analytic at $(0, \ldots, 0)$, which implies that $P(0, \ldots, 0) \neq 0$. Finally, the Taylor expansion given by (1.1) must have non-negative coefficients. These negative multinomial distributions occur naturally in the classification of natural exponential families in $\mathbb{R}^n$ (see Bar-Lev *et al.* 1994).

However, finding exactly which pairs (P, λ) are compatible is an unsolved problem. Before giving details, let us make some observations. If $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, then we denote

$$\mathbf{z}^{\boldsymbol{\alpha}} = \prod_{i=1}^{n} z_i^{\alpha_i} = z_1^{\alpha_1} \ldots z_n^{\alpha_n}. \tag{1.2}$$

Let A be any affine polynomial such that $A(0, \ldots, 0) = 1$, and suppose that the Taylor expansion

$$\big(A(z_1, \ldots, z_n)\big)^{-\lambda} = \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n} c_{\boldsymbol{\alpha}}(\lambda) \mathbf{z}^{\boldsymbol{\alpha}} \tag{1.3}$$

has non-negative coefficients $c_{\boldsymbol{\alpha}}(\lambda)$. Let $a_1, \ldots, a_n$ be positive numbers such that $\sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n} c_{\boldsymbol{\alpha}}(\lambda) a_1^{\alpha_1} \ldots a_n^{\alpha_n} < +\infty$. With such a sequence $\mathbf{a} = (a_1, \ldots, a_n)$ we associate the negative multinomial distribution $\sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n} p_{\boldsymbol{\alpha}} \delta_{\boldsymbol{\alpha}}$ defined by

$$\sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n} p_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}} = \left(\frac{A(a_1, \ldots, a_n)}{A(a_1 z_1, \ldots, a_n z_n)}\right)^{\lambda}$$

(thus $P(z_1, \ldots, z_n) = A(a_1 z_1, \ldots, a_n z_n)/A(a_1, \ldots, a_n)$ in the notation of (1.1)).

Therefore the problem of finding all negative multinomial distributions, that is, of finding all acceptable (P, λ), is equivalent to the problem of finding all (A, λ) such that $A(0, \ldots, 0) = 1$ and the $c_{\boldsymbol{\alpha}}(\lambda)$ defined by (1.3) are non-negative. This is a venerable and difficult problem. For instance, if

$$A_\rho(x, y, z) = 1 + \frac{1-\rho}{\rho}(x + y + z) + \frac{\rho - 2}{\rho}(xy + yz + zx) + \frac{3-\rho}{\rho} xyz, \qquad \rho > 0,$$

Szegö (1933) has shown that $1/A_3$ has positive coefficients and Askey and Gasper (1977) generalize Szegö's result by showing that $A_\rho^{-\lambda}$ has positive coefficients for $\lambda \geqslant \frac{1}{2}$ and $\rho \geqslant 3$. As we shall see, there exist $0 < \lambda < \frac{1}{2}$ such that the coefficients of $A_\rho^{-\lambda}$ are not all positive. Askey and Gasper (1977) show that even finding acceptable (A, λ) is a formidable problem. Therefore the present paper solves the more modest problem of finding the affine polynomials A such that *for all* $\lambda > 0$, $A^{-\lambda}$ has non-negative coefficients. This is equivalent to finding all negative multinomial distributions on $\mathbb{N}_0^n$ which are infinitely divisible. The following simple proposition shows that the problem is much simpler.

Proposition 1. *Let $A(z_1, \ldots, z_n)$ be an affine polynomial such that $A(\mathbf{0}) = 1$. Let $P = 1 - A$ and consider the following Taylor expansions:*

$$(1-P)^{-\lambda} = \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n} c_{\boldsymbol{\alpha}}(\lambda) \mathbf{z}^{\boldsymbol{\alpha}} \quad \textit{and} \quad \log \frac{1}{1-P} = \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}} \mathrm{d}_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}}.$$

Then $c_{\boldsymbol{\alpha}}(\lambda) \geqslant 0$ for all $\lambda > 0$, for all $\boldsymbol{\alpha} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}$ if and only if $d_{\boldsymbol{\alpha}} \geqslant 0$ for all $\boldsymbol{\alpha} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}$.

Proof. The 'if' part is clear since $(1-P)^{-\lambda} = \exp\left(\lambda \log(1-P)^{-1}\right)$, that is,

$$\sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n} c_{\boldsymbol{\alpha}}(\lambda) \mathbf{z}^{\boldsymbol{\alpha}} = \exp\left(\lambda \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}} d_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}}\right).$$

The converse is easy since $c_{\boldsymbol{\alpha}}(\lambda) = d_{\boldsymbol{\alpha}}\lambda + o(\lambda)$ for $\lambda \to 0$. Indeed, since

$$\exp\left(\lambda \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}} d_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}}\right) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(\lambda \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}} d_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}}\right)^n,$$

the coefficient of λ in $c_{\boldsymbol{\alpha}}(\lambda)$ is obtained for $n = 1$ and is $d_{\boldsymbol{\alpha}}$. Thus $c_{\boldsymbol{\alpha}}(\lambda) \geqslant 0$ for all $\lambda > 0$ implies $d_{\boldsymbol{\alpha}} \geqslant 0$. □

Let us write $P(\mathbf{z}) = \sum_{T \in \mathfrak{B}_n^*} a_T \mathbf{z}^T$, where $\mathfrak{B}_n^*$ is the set of non-empty subsets of $\{1, \ldots, n\}$ and $\mathbf{z}^T = \prod_{t \in T} z_t$. The aim of the present paper is to find necessary and sufficient conditions such that the $d_{\boldsymbol{\alpha}}$ in Proposition 1 are non-negative for all $\boldsymbol{\alpha}$ in $\mathbb{N}_0^n \setminus \{\mathbf{0}\}$; in Theorem 2 we show that this infinite set of inequalities $\{d_{\boldsymbol{\alpha}} \geqslant 0; \boldsymbol{\alpha} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}\}$ is equivalent to a finite one. More specifically, we shall define $2^n - 1$ polynomials b_T in the coefficients of A and we shall prove (Theorem 2) that $d_{\boldsymbol{\alpha}} \geqslant 0$ for all $\boldsymbol{\alpha}$ in $\mathbb{N}_0^n \setminus \{\mathbf{0}\}$ if and only if the b_T are non-negative.

Griffiths and Milne (1987) find all infinitely divisible negative multinomial distributions whose probability generating function is $|\mathbf{I}_n - \mathbf{Q}||\mathbf{I}_n - \mathbf{QZ}|^{-1}$, where $\mathbf{Z} = \mathrm{diag}(z_1, \ldots, z_n)$, $\mathbf{I}_n$ is the $n \times n$ identity matrix and $\mathbf{Q}$ is a $n \times n$ real matrix. Clearly $|\mathbf{I}_n - \mathbf{QZ}|$ is an affine polynomial. Since these distributions depend on up to n^2 parameters and since our distributions depend on up to 2^n parameters, our distributions are more general. Since Griffiths and Milne obtain a necessary and sufficient condition (Theorem 2 in their paper) as we do, it will be interesting to compare their result to ours.

Section 2 expresses the $c_{\boldsymbol{\alpha}}(\lambda)$ and $d_{\boldsymbol{\alpha}}$ in terms of the a_T. Section 3 introduces the b_T, which are polynomials in the a_T, and proves the basic result (Theorem 1) that the $d_{\boldsymbol{\alpha}}$ are polynomials in the b_T with non-negative coefficients. From this point, it is easy to conclude (Theorem 2) that the distribution is infinitely divisible if and only if the $2^n - 1$ inequalities $b_T \geqslant 0$ are satisfied. Section 4 applies the above result when $n = 2$ and 3. Section 5 applies Theorem 2 of Section 3 to the particular case studied by Griffiths and Milne, namely $P(\mathbf{z}) = 1 - |\mathbf{I}_n - \mathbf{QZ}|$.

2. Computation of the $c_{\boldsymbol{\alpha}}(\lambda)$ and the $d_{\boldsymbol{\alpha}}$

First we introduce some notation taken from Comtet (1974). Given a positive integer n, we denote $[n] = \{1, 2, \ldots, n\}$. If $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, then we denote $\boldsymbol{\alpha}! = \alpha_1! \ldots \alpha_n!$, $|\boldsymbol{\alpha}| = \alpha_1 + \ldots + \alpha_n$ and $a_{\boldsymbol{\alpha}} = a_{\alpha_1,\ldots,\alpha_n}$. For any k in $\mathbb{N}$ and λ in the set $\mathbb{R}_+$ of positive real numbers, we define the following symbols: $(\lambda)_k = \lambda(\lambda-1)\ldots(\lambda-k+1)$ and $(\lambda)_0 = 1$; $\langle\lambda\rangle_k = \lambda(\lambda+1)\ldots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda)$, where Γ is Euler's gamma function and $\langle\lambda\rangle_0 = 1$. In particular, for $\lambda = n$ in $\mathbb{N}$, for all k in $\mathbb{N}$, we have

$$(n)_k = \prod_{i=1}^{k}(n-i+1) = \frac{n!}{(n-k)!};$$

$$\langle n\rangle_k = \prod_{i=1}^{k}(n+i-1) = \frac{(n+k-1)!}{(n-1)!}.$$

If S is a set, Π_S^k denotes the set of all partitions of S into k *non-empty* subsets of S. We call these k-partitions, and $\Pi_S = \bigcup_{k\geqslant 1}\Pi_S^k$ is the set of all partitions of S. If $S = [n]$, we write $\Pi_{[n]}^k = \Pi_n^k$, and $\Pi_n = \bigcup_{k=1}^n \Pi_n^k$ is the set of all partitions of $[n]$. Let $\mathfrak{B}_n = \mathfrak{B}([n])$ be the family of all subsets of $[n]$ and $\mathfrak{B}_n^*$ the family of non-empty subsets of $[n]$. For simplicity, if n is fixed and if there is no ambiguity, we denote these families by $\mathfrak{B}$ and $\mathfrak{B}^*$, respectively. For T in $\mathfrak{B}_n$, we simplify the notation (1.2) by writing $\mathbf{z}^T = \prod_{t\in T} z_t$ instead of $\mathbf{z}^{\mathbf{1}_T}$ where $\mathbf{1}_T = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i = 1$ if $i \in T$ and $\alpha_i = 0$ if $i \notin T$. For a mapping $a : \mathfrak{B}^* \to \mathbb{R}$, we shall use the notation $a : \mathfrak{B}^* \to \mathbb{R}$, $T \mapsto a_T$. The set of all mappings $k : \mathfrak{B}^* \to \mathbb{N}_0$ is denoted by K. For $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$ fixed, we denote $Z_{\mathbf{z}} : \mathfrak{B}^* \to \mathbb{R}$, $T \mapsto \mathbf{z}^T$, and we write $Z = Z_{\mathbf{z}}$ for the sake of simplicity. If k is in K, we denote $|k| = \sum_{T\in\mathfrak{B}^*} k_T$, $a^k = \prod_{T\in\mathfrak{B}^*} a_T^{k_T}$ and $k! = \prod_{T\in\mathfrak{B}^*} k_T!$. Notice that for $\mathbf{z}$ in $\mathbb{R}^n$ and k in K we have

$$Z^k = \prod_{T\in\mathfrak{B}^*}(\mathbf{z}^T)^{k_T} = \prod_{T\in\mathfrak{B}^*}(\mathbf{z}^{\mathbf{1}_T})^{k_T} = \prod_{T\in\mathfrak{B}^*}\mathbf{z}^{k_T\mathbf{1}_T} = \mathbf{z}^{\sum_{T\in\mathfrak{B}^*} k_T\mathbf{1}_T}. \tag{2.1}$$

For $\boldsymbol{\alpha}$ in $\mathbb{N}_0^n$, we denote

$$K_{\boldsymbol{\alpha}} = \left\{k \in K : \sum_{T\in\mathfrak{B}^*} k_T\mathbf{1}_T = \boldsymbol{\alpha}\right\} = \left\{k \in K : \sum_{T\in\mathfrak{B}^*} k_T\mathbf{1}_T(i) = \alpha_i, \forall i \in [n]\right\}. \tag{2.2}$$

For $\mathcal{T} = \{T_1, \ldots, T_k\}$ in Π_S, we denote

$$a_{\mathcal{T}} = \prod_{i=1}^{k} a_{T_i}. \tag{2.3}$$

We conclude this section with the following simple proposition.

Proposition 2. *Let* $P(\mathbf{z}) = \sum_{T \in \mathfrak{B}_n^*} a_T \mathbf{z}^T$. *For the coefficients in the Taylor expansions*

$$(1 - P(\mathbf{z}))^{-\lambda} = \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n} \mathrm{c}_{\boldsymbol{\alpha}}(\lambda) \mathbf{z}^{\boldsymbol{\alpha}} \tag{2.4}$$

and

$$\log \frac{1}{1 - P(\mathbf{z})} = \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n} \mathrm{d}_{\boldsymbol{\alpha}} \mathbf{z}^{\boldsymbol{\alpha}}, \tag{2.5}$$

it follows that

$$c_{\boldsymbol{\alpha}}(\lambda) = \sum_{k \in K_{\boldsymbol{\alpha}}} \langle \lambda \rangle_{|k|} \frac{a^k}{k!} \tag{2.6}$$

and

$$d_{\boldsymbol{\alpha}} = \sum_{k \in K_{\boldsymbol{\alpha}}} (|k| - 1)! \frac{a^k}{k!}. \tag{2.7}$$

Proof. From (2.4) we can write

$$(1 - P(\mathbf{z}))^{-\lambda} = \sum_{N=0}^{+\infty} \frac{\lambda(\lambda + 1) \ldots (\lambda + N - 1)}{N!} \left(\sum_{T \in \mathfrak{B}^*} a_T \mathbf{z}^T \right)^N.$$

Using the multinomial identity, we obtain

$$\begin{aligned}
(1 - P(\mathbf{z}))^{-\lambda} &= \sum_{N=0}^{+\infty} \langle \lambda \rangle_N \left(\sum_{|k|=N} \frac{a^k}{k!} \prod_{T \in \mathfrak{B}^*} (\mathbf{z}^T)^{k_T} \right) \\
&= \sum_{N=0}^{+\infty} \langle \lambda \rangle_N \left(\sum_{|k|=N} \frac{a^k}{k!} \mathbf{z}^{\sum_{T \in \mathfrak{B}^*} k_T \mathbf{1}_T} \right) = \sum_{k \in K} \langle \lambda \rangle_{|k|} \frac{a^k}{k!} \mathbf{z}^{\sum_{T \in \mathfrak{B}^*} \mathrm{k}_T \mathbf{1}_T} \\
&= \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^n} \left(\sum_{k \in K_{\boldsymbol{\alpha}}} \langle \lambda \rangle_{|k|} \frac{a^k}{k!} \right) \mathbf{z}^{\boldsymbol{\alpha}}.
\end{aligned}$$

Similarly,

$$\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n\setminus\{\mathbf{0}\}} \mathrm{d}_{\boldsymbol{\alpha}}\mathbf{z}^{\boldsymbol{\alpha}} = \sum_{N=1}^{+\infty}\frac{1}{N}\left(\sum_{T\in\mathfrak{B}^*} a_T\mathbf{z}^T\right)^N = \sum_{N=1}^{+\infty}\frac{N!}{N}\left(\sum_{|k|=N}\prod_{T\in\mathfrak{B}^*}\frac{(a_T\mathbf{z}^T)^{k_T}}{k_T!}\right)$$

$$= \sum_{N=1}^{+\infty}(N-1)!\left(\sum_{|k|=N}\frac{a^k}{k!}\mathbf{z}^{\sum_{T\in\mathfrak{B}^*}k_T\mathbf{1}_T}\right) = \sum_{k\in K}(|k|-1)!\frac{a^k}{k!}\mathbf{z}^{\sum_{T\in\mathfrak{B}^*}k_T\mathbf{1}_T}$$

$$= \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n\setminus\{\mathbf{0}\}}\left(\sum_{k\in K_{\boldsymbol{\alpha}}}(|k|-1)!\frac{a^k}{k!}\right)\mathbf{z}^{\boldsymbol{\alpha}}.$$

□

3. The polynomials b_T

We introduce in this section the important polynomials b_T mentioned in the Introduction.

Proposition 3. *Let S be in $\mathfrak{B}_n^*$. If $\boldsymbol{\alpha} = \mathbf{1}_S$, denote by b_S the number $d_{\mathbf{1}_S}$ defined by* (2.5). *Then*

$$b_S = \sum_{l=1}^{|S|}(l-1)!\sum_{\mathcal{T}\in\Pi_S^l} a_{\mathcal{T}}. \tag{3.1}$$

Proof. According to (2.2),

$$K_{\mathbf{1}_S} = \left\{k\in K : \sum_{T\in\mathfrak{B}^*} k_T\mathbf{1}_T = \mathbf{1}_S\right\} = \left\{k\in K : \sum_{T\in\mathfrak{B}^*} k_T\mathbf{1}_T(i) = \mathbf{1}_S(i),\ \forall i\in[n]\right\}.$$

Thus k_T is equal to 1 or 0. If there are a number l of k_T which are equal to 1 then $|k| = l$, $k! = 1$ and $a^k = \prod_{T\in\mathfrak{B}^*} a_T^{k_T} = \prod_{T\in\mathfrak{B}^*:k_T=1} a_T$. Now $\sum_{T\in\mathfrak{B}^*} k_T\mathbf{1}_T(i) = \mathbf{1}_S(i)$ for all i in $[n]$ implies that the T such that k_T is equal to 1 are disjoint, otherwise $\sum_{T\in\mathfrak{B}^*} k_T\mathbf{1}_T(i) \neq \mathbf{1}_S(i)$. Thus

$$a^k = \prod_{T\in\mathfrak{B}^*:k_T=1} a_T = a_{\mathcal{T}},$$

where $\mathcal{T}$ is the l-partition of S formed by the T of $\mathfrak{B}^*$ such that $k_T = 1$. Thus for $\boldsymbol{\alpha} = \mathbf{1}_S$,

$$d_{\boldsymbol{\alpha}} = \sum_{k\in K_{\boldsymbol{\alpha}}}(|k|-1)!\frac{a^k}{k!} = \sum_{l=1}^{|S|}(l-1)!\sum_{\mathcal{T}\in\Pi_S^l} a_{\mathcal{T}} = b_S,$$

which is therefore the coefficient of z^S in the Taylor expansion of $\log(1-P(\mathbf{z}))^{-1}$. □

The following result is fundamental.

Theorem 1. *For n in $\mathbb{N}$, let $P_n(\mathbf{z}) = \sum_{T\in\mathfrak{B}_n^*} a_T\mathbf{z}^T$. Then the coefficient $d_{\boldsymbol{\alpha}}^n$ of $\mathbf{z}^{\boldsymbol{\alpha}}$ in the Taylor expansion of* $\log(1 - P_n(\mathbf{z}))^{-1}$ *is a polynomial in the* $2^n - 1$ *variables* b_T, $T \in \mathfrak{B}_n^*$, *with non-negative coefficients.*

If there is no ambiguity we omit the index n in $d_{\boldsymbol{\alpha}}^n$.

Proof. We proceed by induction on n. Since $d_{\boldsymbol{\alpha}}^n$ is the coefficient of $\mathbf{z}^{\boldsymbol{\alpha}}$ in the Taylor expansion of $\log(1 - P_n(\mathbf{z}))^{-1}$, we have seen in Proposition 3 that $d_{\mathbf{1}_T}^n = b_T$, for T in $\mathfrak{B}_n^*$. Moreover,

$$d_{\boldsymbol{\alpha}}^n = \frac{1}{\boldsymbol{\alpha}!}\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial \mathbf{z}^{\boldsymbol{\alpha}}} \log \frac{1}{1 - P_n(\mathbf{z})}(0, \ldots, 0).$$

For $n = 1$, $\boldsymbol{\alpha} = \alpha_1$ and

$$\log\left(1 - a_{\{1\}}z_1\right)^{-1} = \sum_{j=1}^{\infty} \frac{a_{\{1\}}^j}{j} z_1^j.$$

As $b_{\{1\}} = a_{\{1\}}$, we have

$$d_j^n = \frac{a_{\{1\}}^j}{j} = \frac{b_{\{1\}}^j}{j},$$

which is a polynomial in the variable $b_{\{1\}}$ with non-negative coefficients.

Let n be a fixed positive integer and suppose that, for all $\boldsymbol{\alpha}$ in $\mathbb{N}_0^n\setminus\{\mathbf{0}\}$, $d_{\boldsymbol{\alpha}}^n$ is a polynomial in the variables b_T, $T \in \mathfrak{B}_n^*$, with non-negative coefficients. We now embark on the proof of the proposition for $n + 1$. Let

$$\begin{aligned} P_{n+1}\left((\mathbf{z}, z_{n+1})\right) &= \sum_{T\in\mathfrak{B}_{n+1}^*} a_T\left(\mathbf{z}, z_{n+1}\right)^T \\ &= \sum_{T\in\mathfrak{B}_n^*} a_T\mathbf{z}^T + z_{n+1} \sum_{T\in\mathfrak{B}_N} a_{T\cup\{n+1\}}\mathbf{z}^T \\ &= P_n(\mathbf{z}) + z_{n+1}Q_n(\mathbf{z}), \end{aligned}$$

where $Q_n(\mathbf{z}) = Q_n(z_1, \ldots, z_n)$ is an affine polynomial with respect to each z_j, $j = 1, \ldots, n$. In particular, $P_{n+1}\left((\mathbf{z}, 0)\right) = P_n(\mathbf{z})$. We show first that for $\boldsymbol{\alpha}' = (\boldsymbol{\alpha}, \alpha_{n+1}) = (\alpha_1, \ldots, \alpha_n, \alpha_{n+1})$ in $\mathbb{N}_0^{n+1}\setminus\{\mathbf{0}\}$, the coefficient $d_{\boldsymbol{\alpha}'}^{n+1}$ of $(\mathbf{z}, z_{n+1})^{\boldsymbol{\alpha}'}$ in the Taylor expansion of $\log\left(1 - P_{n+1}\left((\mathbf{z}, z_{n+1})\right)\right)^{-1}$ is a polynomial in the variables b_T, $T \in \mathfrak{B}_{n+1}^*$, with non-negative coefficients.

Step 1. We assume first that $\alpha_{n+1} = 0$. In this case,

$$\begin{aligned} d^{n+1}_{\boldsymbol{\alpha}'} &= d^{n+1}_{(\boldsymbol{\alpha},0)} \\ &= \frac{1}{(\boldsymbol{\alpha},0)!}\frac{\partial^{|(\boldsymbol{\alpha},0)|}}{\partial(\mathbf{z}, z_{n+1})^{(\boldsymbol{\alpha},0)}}\log\frac{1}{1-P_{n+1}((\mathbf{z}, z_{n+1}))}((0,\dots,0),0) \\ &= \frac{1}{\boldsymbol{\alpha}!}\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial\mathbf{z}^{\alpha}}\log\frac{1}{1-P_{n+1}((\mathbf{z},0))}((0,\dots,0)) \\ &= \frac{1}{\boldsymbol{\alpha}!}\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial\mathbf{z}^{\boldsymbol{\alpha}}}\log\frac{1}{1-P_n(\mathbf{z})}((0,0,\dots 0)) = d^n_{\boldsymbol{\alpha}}, \end{aligned}$$

and the induction hypothesis implies that $d^{n+1}_{\boldsymbol{\alpha}'} = d^n_{\boldsymbol{\alpha}}$ is a polynomial in the variables b_T, $T \neq \emptyset$, included in $[n+1]\backslash\{n+1\} = [n]$, with non-negative coefficients. Notice that the result remains valid if any other α_i is null, since then $d^n_{\boldsymbol{\alpha}}$ is a polynomial in the variables b_T, $T \neq \emptyset$, included in $[n+1]\backslash\{i\}$, with non-negative coefficients.

Step 2. We now suppose that $\alpha_{n+1} > 0$. Differentiating k times with respect to the last variable, we obtain

$$\frac{\partial^k}{\partial z^k_{n+1}}\log\frac{1}{1-P_{n+1}((\mathbf{z}, z_{n+1}))} = (k-1)!\left(\frac{Q_n(\mathbf{z})}{1-P_{n+1}((\mathbf{z}, z_{n+1}))}\right)^k. \tag{3.2}$$

Taking $z_{n+1} = 0$ in (3.2), this result becomes

$$\frac{\partial^k}{\partial z^k_{n+1}}\log\frac{1}{1-P_{n+1}((\mathbf{z},0))} = (k-1)!\left(\frac{Q_n(\mathbf{z})}{1-P_n(\mathbf{z})}\right)^k. \tag{3.3}$$

We also have

$$\frac{\partial^k}{\partial z^k_{n+1}}(\mathbf{z}, z_{n+1})^{\boldsymbol{\alpha}'} = \begin{cases} (\alpha_{n+1})_k(\mathbf{z}, z_{n+1})^{(\boldsymbol{\alpha},\alpha_{n+1}-k)} = (\alpha_{n+1})_k\mathbf{z}^{\boldsymbol{\alpha}} z^{\alpha_{n+1}-k}_{n+1} & \textit{if } \alpha_{n+1} \geqslant k, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Differentiating k times with respect to the last variable, we deduce

$$\begin{aligned} \frac{\partial^k}{\partial z^k_{n+1}}\log\frac{1}{1-P_{n+1}((\mathbf{z}, z_{n+1}))} &= \sum_{\boldsymbol{\alpha}'\in\mathbb{N}^{n+1}_0\backslash\{\mathbf{0}\}} d^{n+1}_{\boldsymbol{\alpha}'}\frac{\partial^k}{\partial z^k_{n+1}}(\mathbf{z}, z_{n+1})^{\boldsymbol{\alpha}'} \\ &= \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n_0;\alpha_{n+1}\geqslant k} d^{n+1}_{(\boldsymbol{\alpha},\alpha_{n+1})}(\alpha_{n+1})_k\mathbf{z}^{\boldsymbol{\alpha}} z^{\alpha_{n+1}-k}_{n+1}. \end{aligned}$$

Taking $z_{n+1} = 0$, we obtain

$$\frac{\partial^k}{\partial z_{n+1}^k}\log\frac{1}{1-P_{n+1}((\mathbf{z},z_{n+1}))}\bigg|_{z_{n+1}=0} = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n:\alpha_{n+1}=k} d^{n+1}_{(\boldsymbol{\alpha},\alpha_{n+1})}(\alpha_{n+1})_{\mathrm{k}}\mathbf{z}^{\boldsymbol{\alpha}} \tag{3.5}$$

$$= \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} k!d^{n+1}_{(\boldsymbol{\alpha},k)}\mathbf{z}^{\boldsymbol{\alpha}} = k!\left(\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} d^{n+1}_{(\boldsymbol{\alpha},k)}\mathbf{z}^{\boldsymbol{\alpha}}\right).$$

Then using (3.3) and (3.5), we have

$$k!\left(\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} d^{n+1}_{(\boldsymbol{\alpha},k)}\mathbf{z}^{\boldsymbol{\alpha}}\right) = (k-1)!\left(\frac{Q_n(\mathbf{z})}{1-P_n(\mathbf{z})}\right)^k. \tag{3.6}$$

Dividing by $k!$ and letting $k=\alpha_{n+1}$, we obtain

$$\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} d^{n+1}_{(\boldsymbol{\alpha},\alpha_{n+1})}\mathbf{z}^{\boldsymbol{\alpha}} = \frac{1}{\alpha_{n+1}}\left(\frac{Q_n(\mathbf{z})}{1-P_n(\mathbf{z})}\right)^{\alpha_{n+1}}. \tag{3.7}$$

Putting $k=1$ in (3.6) gives

$$\frac{Q_n(\mathbf{z})}{1-P_n(\mathbf{z})} = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} d^{n+1}_{(\boldsymbol{\alpha},1)}\mathbf{z}^{\boldsymbol{\alpha}}.$$

We now substitute this into (3.7):

$$\begin{aligned}\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} d^{n+1}_{(\boldsymbol{\alpha},\alpha_{n+1})}\mathbf{z}^{\boldsymbol{\alpha}} &= \frac{1}{\alpha_{n+1}}\left(\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n} d^{n+1}_{(\boldsymbol{\alpha},1)}\mathbf{z}^{\boldsymbol{\alpha}}\right)^{\alpha_{n+1}}\\ &= \frac{1}{\alpha_{n+1}}\prod_{i=1}^{\alpha_{n+1}}\left(\sum_{\boldsymbol{\beta}_i\in\mathbb{N}_0^n} d^{n+1}_{(\boldsymbol{\beta}_i,1)}\mathbf{z}^{\boldsymbol{\beta}_i}\right)\\ &= \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n}\left(\frac{1}{\alpha_{n+1}}\sum_{\boldsymbol{\beta}_1+\ldots+\boldsymbol{\beta}_{\alpha_{n+1}}=\boldsymbol{\alpha}:\boldsymbol{\beta}_i\in\mathbb{N}_0^n}\left(\prod_{i=1}^{\alpha_{n+1}} d^{n+1}_{(\boldsymbol{\beta}_i,1)}\right)\right)\mathbf{z}^{\boldsymbol{\alpha}}.\end{aligned}$$

Therefore we have proved that

$$d^{n+1}_{(\boldsymbol{\alpha},\alpha_{n+1})} = \frac{1}{\alpha_{n+1}}\sum_{\boldsymbol{\beta}_1+\ldots+\boldsymbol{\beta}_{\alpha_{n+1}}=\boldsymbol{\alpha}:\boldsymbol{\beta}_i\in\mathbb{N}_0^n}\left(\prod_{i=1}^{\alpha_{n+1}} d^{n+1}_{(\boldsymbol{\beta}_i,1)}\right). \tag{3.8}$$

For instance, for $n=1,\ \alpha_2=2,\ \alpha=\alpha_1=3,$ we have

$$d^2_{(3,2)} = \frac{1}{2}\sum_{\beta_1+\beta_2=3}\left(\prod_{i=1}^{2} d^2_{(\beta_i,1)}\right) = \frac{1}{2}\sum_{\beta_1=0}^{3} d^2_{(\beta_1,1)}d^2_{(3-\beta_1,1)}).$$

In the same way, a similar result holds for any index j in $[n]$: if for $\boldsymbol{\beta}_i$ in $\mathbb{N}_0^n$ we denote by $(\boldsymbol{\beta}_i)_k$ the kth component of $\boldsymbol{\beta}_i$, we have

$$d^{n+1}_{(\alpha_1,\ldots,\alpha_{j-1},\alpha_j,\alpha_{j+1},\ldots,\alpha_{n+1})} = \frac{1}{\alpha_j}\sum_{\boldsymbol{\beta}_1+\ldots+\boldsymbol{\beta}_{\alpha_j}=(\alpha_1,\ldots,\alpha_{j-1},\alpha_{j+1},\ldots,\alpha_{n+1}):\boldsymbol{\beta}_i\in\mathbb{N}_0^n}\left(\prod_{i=1}^{\alpha_j} d^{n+1}_{((\boldsymbol{\beta}_i)_1,\ldots,(\boldsymbol{\beta}_i)_{j-1},1,(\boldsymbol{\beta}_i)_j,\ldots,(\boldsymbol{\beta}_i)_n)}\right). \quad (3.9)$$

It remains to be observed that if we use (3.8), we obtain a polynomial with non-negative coefficients in the $d^{n+1}_{(\boldsymbol{\beta}_i,1)}$, where $\boldsymbol{\beta}_1+\ldots+\boldsymbol{\beta}_{\alpha_{n+1}}=\boldsymbol{\alpha}$ and $\boldsymbol{\beta}_i\in\mathbb{N}_0^n$. Thus we obtain terms of the type $d^{n+1}_{((\boldsymbol{\beta}_i)_1,\ldots,(\boldsymbol{\beta}_i)_n,1)}$, with $(\boldsymbol{\beta}_l)_j\leqslant\alpha_j$ for l in $[\alpha_{n+1}]$ because $(\boldsymbol{\beta}_1)_j+(\boldsymbol{\beta}_2)_j+\ldots+(\boldsymbol{\beta}_{\alpha_{n+1}})_j=\alpha_j$, the $(\boldsymbol{\beta}_l)_j$ being non-negative integers. Three cases can be distinguished:

1. At least one of $(\boldsymbol{\beta}_l)_j$, $j\in[n]$, is null. In this case step 1 leads to the result.
2. $(\boldsymbol{\beta}_l)_j=1$ for all j in $[n]$. In this case $d^{n+1}_{((\boldsymbol{\beta}_i)_1,\ldots,(\boldsymbol{\beta}_i)_n,1)}=d^{n+1}_{(1,1,\ldots,1,1)}=b_{[n+1]}$ according to (3.1).
3. There is an non-zero integer k such that $k\leqslant n$, $(\boldsymbol{\beta}_l)_k>1$ and $(\boldsymbol{\beta}_l)_j=1$ for $j>k$, that is, $d^{n+1}_{((\boldsymbol{\beta}_i)_1,\ldots,(\boldsymbol{\beta}_i)_n,1)}=d^{n+1}_{((\boldsymbol{\beta}_i)_1,\ldots,(\boldsymbol{\beta}_i)_k,1,\ldots,1)}$.

Using formula (3.9) for the kth variable for these last terms, we obtain

$$d^{n+1}_{((\boldsymbol{\beta}_i)_1,\ldots,(\boldsymbol{\beta}_i)_k,1,\ldots,1)} = \frac{1}{(\boldsymbol{\beta}_i)_k}\sum_{\boldsymbol{\gamma}_1+\ldots+\boldsymbol{\gamma}_{(\boldsymbol{\beta}_i)_k}=((\boldsymbol{\beta}_i)_1,\ldots,(\boldsymbol{\beta}_i)_{k-1},1,\ldots,1):\boldsymbol{\gamma}_l\in\mathbb{N}_0^n}\left(\prod_{l=1}^{(\boldsymbol{\beta}_i)_k} d^{n+1}_{((\boldsymbol{\gamma}_l)_1,\ldots,(\boldsymbol{\gamma}_l)_{k-1},1,(\boldsymbol{\gamma}_l)_k,\ldots,(\boldsymbol{\gamma}_l)_n)}\right).$$

The relations $\boldsymbol{\gamma}_1+\ldots+\boldsymbol{\gamma}_{(\boldsymbol{\beta}_i)_k}=(\alpha_1,\ldots,\alpha_{k-1},1,\ldots,1)$ and $\boldsymbol{\gamma}_l\in\mathbb{N}_0^n$ prove that $\sum_{l=1}^{(\boldsymbol{\beta}_i)_k}(\boldsymbol{\gamma}_l)_m=1$ for any m in $\{k,\ldots,n\}$. This implies that for any m in $\{k,\ldots,n\}$, only one of the $(\boldsymbol{\gamma}_l)_m$, for l in $[(\boldsymbol{\beta}_i)_k]$, is equal to 1, the others being equal to 0. Thus we have a polynomial with non-negative coefficients with terms of the type $d^{n+1}_{((\boldsymbol{\gamma}_l)_1,\ldots,(\boldsymbol{\gamma}_l)_{k-1},1,(\boldsymbol{\gamma}_l)_{k},\ldots,(\boldsymbol{\gamma}_l)_n)}$. If one of $(\boldsymbol{\gamma}_l)_m$, for m in $\{k,\ldots,n\}$, is zero the induction hypothesis applies, otherwise we are in the case $d^{n+1}_{((\boldsymbol{\gamma}_l)_1,\ldots,(\boldsymbol{\gamma}_l)_{k-1},1,1,\ldots,1)}$. Since $k-1<k$, it is thus evident that by repeating a finite number of times this process we shall obtain a polynomial, in $b_{[n+1]}$ and b_T for $T\in\mathfrak{B}_n^*$ and with cardinality at most n, having non-negative coefficients, that is, a polynomial in the variables b_T, $T\in\mathfrak{B}_{n+1}^*$, with non-negative coefficients. This completes the proof of Theorem 1. □

As a consequence we can state our main result.

Theorem 2. *Let* $P(\mathbf{z})=\sum_{T\in\mathfrak{B}_n^*}a_T\mathbf{z}^T$, *and suppose that* $(1-P(\mathbf{z}))^{-\lambda}=\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^n}c_{\boldsymbol{\alpha}}(\lambda)\mathbf{z}^{\boldsymbol{\alpha}}$. *Then* $c_{\boldsymbol{\alpha}}(\lambda)\geqslant 0$ *for all positive* λ *if and only if* b_T, *given by* (3.1), *is non negative for all* $T\in\mathfrak{B}_n^*$.

Proof. We show that, for all $\boldsymbol{\alpha}\in\mathbb{N}_0^n\backslash\{\mathbf{0}\}$, $d_{\boldsymbol{\alpha}}\geqslant 0$ if and only if, for all $T\in\mathfrak{B}_n^*$, $b_T\geqslant 0$. Since $d_{\boldsymbol{\alpha}}$ is a polynomial in the variables b_T, $T\in\mathfrak{B}_n^*$, with non-negative coefficients, $b_T\geqslant 0$ for all T in $\mathfrak{B}_n^*$ implies $d_{\boldsymbol{\alpha}}\geqslant 0$. Conversely, since $d_{\mathbf{1}_T}=b_T$, $d_{\boldsymbol{\alpha}}\geqslant 0$ for all $\boldsymbol{\alpha}$ implies $d_{\mathbf{1}_T}=b_T\geqslant 0$ and the equivalence is proven. □

4. Examples

Let $n = 2$. The conditions are $b_{\{1\}} = a_{\{1\}} \geqslant 0$, $b_{\{2\}} = a_{\{2\}} \geqslant 0$ and

$$b_{\{1,2\}} = \sum_{l=1}^{2} (l-1)! \sum_{\mathcal{T} \in \mathbb{B}^l_{\{1,2\}}} a_{\mathcal{T}} = a_{\{1,2\}} + a_{\{1\}} a_{\{2\}} \geqslant 0,$$

that is, $a_{\{1\}} \geqslant 0$, $a_{\{2\}} \geqslant 0$ and $a_{\{1,2\}} \geqslant -a_{\{1\}} a_{\{2\}}$ (see Griffiths and Milne 1987, Theorem 4; Bar-Lev *et al.* 1994, Proposition 3.1).

Now let $n = 3$. Here

$$1 - P((x, y, z)) = 1 + \frac{1-\rho}{\rho}(x + y + z) + \frac{\rho - 2}{\rho}(xy + yz + zx) + \frac{3-\rho}{\rho} xyz.$$

In this case

$$a_{\{1\}} = a_{\{2\}} = a_{\{3\}} = \frac{\rho - 1}{\rho}, \qquad a_{\{1,2\}} = a_{\{1,3\}} = a_{\{2,3\}} = \frac{2-\rho}{\rho}, \qquad a_{\{1,2,3\}} = \frac{\rho - 3}{\rho};$$

$$b_{\{1\}} = b_{\{2\}} = b_{\{3\}} = \frac{\rho - 1}{\rho}, \qquad b_{\{1,2\}} = a_{\{1,2\}} + a_{\{1\}} a_{\{2\}} = \frac{2-\rho}{\rho} + \frac{\rho-1}{\rho} \times \frac{\rho - 1}{\rho} = \frac{1}{\rho^2}.$$

Hence we have $b_{\{1,2\}} = b_{\{1,3\}} = b_{\{2,3\}} > 0$ and

$$\begin{aligned} b_{\{1,2,3\}} &= \sum_{l=1}^{3} (l-1)! \sum_{\mathcal{T} \in \mathbb{B}^l_{\{1,2,3\}}} a_{\mathcal{T}} \\ &= a_{\{1,2,3\}} + a_{\{1\}} a_{\{2,3\}} + a_{\{2\}} a_{\{1,3\}} + a_{\{3\}} a_{\{1,2\}} + 2 a_{\{1\}} a_{\{2\}} a_{\{3\}} \\ &= \frac{\rho - 3}{\rho} + 3 \frac{\rho - 1}{\rho} \frac{2-\rho}{\rho} + 2 \frac{\rho-1}{\rho} \frac{\rho-1}{\rho} \frac{\rho-1}{\rho} = -\frac{2}{\rho^3} < 0. \end{aligned}$$

Therefore there exist values of λ such that the Taylor expansion of

$$\left(1 + \frac{1-\rho}{\rho}(x+y+z) + \frac{\rho-2}{\rho}(xy + yz + zx) + \frac{3-\rho}{\rho} xyz\right)^{-\lambda}$$

has negative coefficients (see Askey and Gasper 1977).

5. Application to the Griffiths and Milne case

This section compares our necessary and sufficient condition for infinite divisibility to the results obtained by Griffiths and Milne (1987) in the case where $\mathbf{Q}$ is a $n \times n$ real matrix and, writing $\mathbf{Z} = \mathrm{diag}(z_1, \ldots, z_n)$,

$$f(z_1, \ldots, z_n) = \frac{\det(\mathbf{I}_n - \mathbf{Q})}{\det(\mathbf{I}_n - \mathbf{QZ})} \tag{5.1}$$

is a probability generating function for certain choices of $\mathbf{Q}$. Note that $\mathbf{Q}$ is not necessarily symmetric. Griffiths and Milne find a necessary and sufficient condition for infinite divisibility on $\mathbf{Q}$ that they describe as follows. We assume that $q_{ij}q_{ji} \geqslant 0$ for $i \neq j$. With $\{1, \ldots, n\}$ as vertices, put a green edge between vertices $i \neq j$ if $q_{ij} + q_{ji} < 0$ and a red one if $q_{ij} + q_{ji} > 0$. A circuit of length k is a sequence $(v_1, \ldots, v_k)$ of *distinct* vertices such that $\{v_i, v_{i+1}\}$ is an edge (red or green) for all $i = 1, \ldots, k$, with the convention $v_{k+1} = v_1$. It is said to be elementary if it has no chords, namely if $\{v_i, v_j\}$ is not an edge for $|i - j| > 1$. Their Theorem 2 establishes that f is the generating function of an infinitely divisible distribution on $\mathbb{N}_0^n$ if and only if $\mathbf{Q}$ satisfies the following remarkable conditions:

1. The eigenvalues of $\mathbf{Q}$ are in the open unit disc $\{z \in \mathbb{C};\ |z| < 1\}$.
2. For all i in $[n]$, $q_{ii} \geqslant 0$; and for $i \neq j$, $q_{ij}q_{ji} \geqslant 0$.
3. Every elementary circuit has an even number of green edges.

In order to apply our Theorem 2 to the polynomial $P(z_1, \ldots, z_n) = 1 - \det(\mathbf{I}_n - \mathbf{QZ})$, we have to compute the b_T. Actually, we shall prove the following result.

Theorem 3. *Let T be a non-empty subset of $[n]$ and let $\mathcal{C}_T$ be the set of all circular permutations of T. Then*

$$b_T = \sum_{c \in \mathcal{C}_T} \prod_{t \in T} q_{tc(t)} = k^{-1} \sum_{\{i_1, \ldots, i_k\} = T} q_{i_1 i_2} \ldots q_{i_{k-1} i_k} q_{i_k i_1}. \tag{5.2}$$

For a proof of Theorem 3 we need several propositions which are interesting in their own right. Let S be a non-empty set. We denote by $\Pi^S = \bigcup_{T \in \mathfrak{B}^*(S)} \Pi_T$ the union of the sets of the partitions of T for all elements T in $\mathfrak{B}^*(S)$. The set of all partitions of T is ordered as follows: we write $\mathcal{T} \leqslant \mathcal{S}$ if and only if every block of $\mathcal{T}$ is contained in a block of $\mathcal{S}$ (see Stanley 1999, p. 116). We denote by μ_{Π_T} the corresponding Möbius function. Let f and g be two mappings of Π^S into $\mathbb{C}$. We say that f and g are connected by the Möbius inversion formula if and only if, for all T in $\mathfrak{B}^*(S)$, we have

$$g(\mathcal{S}) = \sum_{\mathcal{T} \leqslant \mathcal{S}} f(\mathcal{T}) \Leftrightarrow f(\mathcal{S}) = \sum_{\mathcal{T} \leqslant \mathcal{S}} \mu_{\Pi_T}(\mathcal{T}, \mathcal{S}) g(\mathcal{T}). \tag{5.3}$$

A mapping f from Π^S into $\mathbb{C}$ is called multiplicative if and only if, for all partitions $\mathcal{T} = \{T_1, \ldots, T_k\}$ in Π^S, denoted by $T_1 \bullet \ldots \bullet T_k$, we have $f(\mathcal{T}) = \prod_{\mathrm{i}=1}^{\mathrm{k}} f(T_i)$.

Proposition 4. *Let f and g be two mappings from Π^S into $\mathbb{C}$ that are connected by the Möbius inversion formula. Then f is multiplicative if and only if g is multiplicative.*

Proof. Let us suppose that f is multiplicative. Let $\mathcal{S} = S_1 \bullet \ldots \bullet S_k$ be a partition of T. If $\widehat{0}$ denotes the partition formed by the singletons of T, then $\Pi_{S_1} \times \ldots \times \Pi_{S_k}$ is isomorphic to

$\left[\widehat{0}, \mathcal{S}\right]$ by $\varphi : \Pi_{S_1} \times \ldots \times \Pi_{S_k} \to \left[\widehat{0}, \mathcal{S}\right]$, $\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_k) \mapsto \mathcal{T}_1 \bullet \ldots \bullet \mathcal{T}_\mathrm{k}$. Since f is multiplicative then, for any element $\mathcal{T} = \mathcal{T}_1 \bullet \ldots \bullet \mathcal{T}_\mathrm{k}$ in $\left[\widehat{0}, \mathcal{S}\right]$, we have

$$f(\mathcal{T}) = \prod_{i=1}^{k} f(\mathcal{T}_i).$$

For all T in $\mathfrak{B}^*(S)$ and for all $\mathcal{S} = S_1 \bullet \ldots \bullet S_k$ in Π_T, the Möbius inversion formula implies

$$\begin{aligned} g(\mathcal{S}) &= \sum_{\mathcal{T} \leqslant S_1 \bullet \ldots \bullet S_k} f(\mathcal{T}) = \sum_{\mathcal{T} = \mathcal{T}_1 \bullet \ldots \bullet \mathcal{T}_\mathrm{k} \leqslant \mathrm{S}_1 \bullet \ldots \bullet \mathrm{S}_\mathrm{k}} \mathrm{f}(\mathcal{T}_1 \bullet \ldots \bullet \mathcal{T}_\mathrm{k}) \\ &= \sum_{(\mathcal{T}_1, \ldots, \mathcal{T}_k) \in \Pi_{S_1} \times \ldots \times \Pi_{S_k}} \prod_{i=1}^{k} f(\mathcal{T}_i) = \prod_{i=1}^{k} \left[\sum_{\mathcal{T}_i \leqslant \{S_i\}} f(\mathcal{T}_i) \right] \\ &= \prod_{i=1}^{k} g(\{S_i\}) = \prod_{i=1}^{k} g(S_i). \end{aligned}$$

Conversely, let us suppose that g is multiplicative. For all T in $\mathfrak{B}^*(S)$ and for all $\mathcal{S} = S_1 \bullet \ldots \bullet S_k$ in Π_T, we know (see Stanley 1999, p. 128) that

$$\mu_{\Pi_n | [\widehat{0}, \mathcal{S}]} = \mu_{[\widehat{0}, \mathcal{S}]} = \mu_{\Pi_{S_1}} \times \ldots \times \mu_{\Pi_{S_k}} = \prod_{\mathrm{i}=1}^{\mathrm{k}} \mu_{\Pi_{S_i}}.$$

Hence we have, by the Möbius inversion formula,

$$\begin{aligned} f(\mathcal{S}) &= \sum_{\mathcal{T} \in [\widehat{0}; \mathcal{S}]} g(\mathcal{T}) \mu_{\Pi_T}(\mathcal{T}; \mathcal{S}) = \sum_{(\mathcal{T}_1, \ldots, \mathcal{T}_k) \in \Pi_{S_1} \times \ldots \times \Pi_{sS_k}} \prod_{i=1}^{k} g(\mathcal{T}_i) \mu_{\Pi_{S_i}}(\mathcal{T}_i; \{S_i\}) \\ &= \prod_{i=1}^{k} \left(\sum_{\mathcal{T}_i \leqslant \{S_i\}} g(\mathcal{T}_i) \mu_{\Pi_{S_i}}(\mathcal{T}_i; \{S_i\}) \right) = \prod_{i=1}^{k} f(\{S_i\}) = \prod_{i=1}^{k} f(S_i). \end{aligned}$$

□

Proposition 5. *Let* $\mathbf{Q} = (q_{i,j})_{(i,j) \in [n]^2}$ *be a real* $n \times n$ *matrix and let* $\mathbf{Z} = \mathrm{diag}(z_1, \ldots, z_n)$, *where* $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$. *Let* P *be the polynomial defined by* $1 - P(\mathbf{z}) = 1 - \sum_{T \in \mathfrak{B}_n^*} a_T \mathbf{z}^T = |\mathbf{I}_n - \mathbf{QZ}|$, *where* $\mathbf{I}_n$ *denotes the identity matrix of order n. If* $\mathbf{Q}_T = (q_{ij})_{(i,j) \in T^2}$ *and* $\mathfrak{q}_T = |\mathbf{Q}_T|$, *then*

$$a_T = (-1)^{|T|-1} \mathfrak{q}_T. \tag{5.4}$$

Proof. We have

$$|\mathbf{I}_n - \mathbf{QZ}| = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n} \left(\delta_{\sigma(i),i} - q_{\sigma(i),i} z_i \right),$$

where $\mathfrak{S}_n$ denotes the set of all permutations of $[n]$ and $\varepsilon(\sigma)$ the signature of σ. Thus the coefficient of $\mathbf{z}^T$ in $P(\mathbf{z})$ is

$$\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(j)=j \text{ if } j\notin T}} \varepsilon(\sigma)\prod_{i\in T}(-q_{\sigma(i),i}) = (-1)^{|T|}\sum_{\sigma\in\mathcal{S}_T}\varepsilon(\sigma)\prod_{i\in T} q_{\sigma(i),i} = (-1)^{|T|}|\mathbf{Q}_T|.$$

□

Proposition 6. *Let T be in $\mathfrak{B}_n^*$, $\mathcal{T} = T_1 \bullet T_2 \bullet \ldots \bullet T_l$ be in Π_T and $\mathfrak{q}_{\mathcal{T}} = \prod_{i=1}^l \mathfrak{q}_{T_i}$. Then*

$$b_T = (-1)^{|T|-1}\sum_{\mathcal{T}\in\Pi_T}(-1)^{|\mathcal{T}|-1}(|\mathcal{T}|-1)!\mathfrak{q}_{\mathcal{T}}. \tag{5.5}$$

Proof. Using (3.1) and (5.4), we obtain

$$\begin{aligned} b_T &= \sum_{l=1}^{|T|}(l-1)! \sum_{\mathcal{T}=T_1\bullet T_2\bullet\ldots\bullet T_l\in\Pi_T^l}\prod_{i=1}^{l}(-1)^{|T_i|-1}\mathfrak{q}_{T_i} \\ &= (-1)^{|T|-1}\sum_{l=1}^{|T|}\sum_{\mathcal{T}=T_1\bullet T_2\bullet\ldots\bullet T_l\in\Pi_T^l}(-1)^{l-1}(l-1)!\prod_{i=1}^{l}\mathfrak{q}_{T_i} \\ &= (-1)^{|T|-1}\sum_{\mathcal{T}\in\Pi_T}(-1)^{|\mathcal{T}|-1}(|\mathcal{T}|-1)!\mathfrak{q}_{\mathcal{T}}. \end{aligned}$$

□

Proof of Theorem 3. Let $\mathcal{T} = T_1 \bullet T_2 \bullet \ldots \bullet T_l$ be a partition of T. If $\sigma = c_1 \circ c_2 \circ \ldots \circ c_l$, where $c_i \in \mathcal{C}_{T_i}$, the set of all circular permutations of T_i, then we say that σ is of support $\mathcal{T}$. The set of all permutations of support $\mathcal{T}$ is denoted $\mathfrak{S}_{\mathcal{T}}$ and the set of all permutations of T is denoted $\mathfrak{S}_T$. Let f be the multiplicative mapping on $\Pi^{[n]}$ defined for all T in $\mathfrak{B}_n^*$ by

$$f(T) = (-1)^{|T|-1}\sum_{c\in\mathcal{C}_T}\prod_{i\in T} q_{ic(i)}. \tag{5.6}$$

Let $\mathcal{T} = T_1 \bullet T_2 \bullet \ldots \bullet T_l$ be in $\Pi^{[n]}$. We have

$$\begin{aligned} f(\mathcal{T}) &= \prod_{m=1}^{l}\left((-1)^{|T_m|-1}\sum_{c_m\in\mathcal{C}_{T_m}}\prod_{i\in T_m} q_{ic_m(i)}\right) \\ &= \left(\prod_{m=1}^{l}(-1)^{|T_m|-1}\right)\left(\sum_{(c_1,\ldots,c_m)\in\mathcal{C}_{T_1}\times\ldots\times\mathcal{C}_{T_m}}\prod_{m=1}^{l}\prod_{i\in T_m} q_{i(c_1\circ c_2\circ\ldots\circ c_l)(i)}\right) \\ &= \left(\prod_{m=1}^{l}(-1)^{|T_m|-1}\right)\left(\sum_{\sigma\in\mathfrak{S}_{\mathcal{T}}}\prod_{i\in T} q_{i\sigma(i)}\right) = \sum_{\sigma\in\mathfrak{S}_{\mathcal{T}}}\varepsilon(\sigma)\prod_{i\in T} q_{i\sigma(i)}. \end{aligned} \tag{5.7}$$

Let g be the mapping defined on $\Pi^{[n]}$, connected to f by the Möbius inversion formula. Then we have

$$g(\mathcal{S}) = \sum_{\mathcal{T}\leqslant\mathcal{S}}\sum_{\sigma\in\mathfrak{S}_{\mathcal{T}}}\varepsilon(\sigma)\prod_{i\in T} q_{i\sigma(i)}. \tag{5.8}$$

Taking $\mathcal{S} = \{S\}$ in (5.8), we obtain

$$g(S) = g(\{S\}) = \sum_{\sigma\in\mathfrak{S}_S}\varepsilon(\sigma)\prod_{i\in S} q_{i\sigma(i)} = |\mathbf{Q}_S| = \mathfrak{q}_S. \tag{5.9}$$

Let $\mathcal{S} = S_1 \bullet S_2 \bullet \ldots \bullet S_k$ be a partition of S. By Proposition 4 we have

$$g(\mathcal{S}) = \prod_{i=1}^{k} g(S_i) = \prod_{i=1}^{k}\mathfrak{q}_{S_i} = \mathfrak{q}_{\mathcal{S}}. \tag{5.10}$$

The following result is well known (see Rota 1964):

$$\mu(\mathcal{T},\mathcal{S}) = (-1)^{|\mathcal{T}|-|\mathcal{S}|}\prod_{i=1}^{k}(n_i-1)!, \qquad \text{for } \mathcal{T}\leqslant\mathcal{S},$$

where n_i is the number of blocks of $\mathcal{T}$ contained in S_i, for all i in $[k]$.

Using (5.2) and (5.10), we obtain

$$\begin{aligned} f(T) &= \sum_{\mathcal{T}\leqslant\{T\}}\mu(\mathcal{T},\{T\})g(\mathcal{T}) \\ &= \sum_{\mathcal{T}\in\Pi_T}(-1)^{|\mathcal{T}|-1}(|\mathcal{T}|-1)!\mathfrak{q}_{\mathcal{T}} \\ &= (-1)^{|T|-1}b_T. \end{aligned} \tag{5.11}$$

Therefore, using (5.6) and (5.11), we conclude that

$$b_T = (-1)^{|T|-1}f(T) = \sum_{c\in\mathcal{C}_T}\prod_{i\in T} q_{ic(i)}.$$

Griffiths and Milne (1987) give the following result which leads to a necessary and sufficient condition for $|\mathbf{I}_n - \mathbf{Q}||\mathbf{I}_n - \mathbf{QZ}|^{-1}$ to be the probability generating function of an infinitely divisible distribution. For $\mathbf{j} = (j_1, \ldots, j_n)$ in $\mathbb{N}_0^n \backslash \{\mathbf{0}\}$, the coefficient of $\mathbf{z}^{\mathbf{j}} = z_1^{j_1} \ldots z_n^{j_n}$ in the Taylor expansion of $\log|\mathbf{I}_n - \mathbf{QZ}|^{-1}$ is

$$k^{-1} \sum_{(i_1,\ldots,i_k)\in[n]} q_{i_1 i_2} \ldots q_{i_{k-1} i_k} q_{i_k i_1}, \tag{5.12}$$

where $k = j_1 + \ldots + j_n$ and the number of indices $i_1, \ldots, i_k$ equal to l is j_l, $l = 1, 2, \ldots, n$. Now we take $\mathbf{j} = \mathbf{1}_T$. In this case $k = j_1 + \ldots + j_n = \sum_{i=1}^n \mathbf{1}_T(i) = |T|$, and the number of indices $i_1, \ldots, i_k$ equal to l is $\mathbf{1}_T(l)$, that is, is equal to 1 if l is in T and to 0 otherwise. Thus we obtain once and only once all the elements of T. This means that the mapping $j \mapsto i_j$ is a bijection σ from $[|T|]$ into T. Let us denote by $\mathfrak{S}_{k,T}$ the set of all bijections from $[k] = [|T|]$ into T. We have

$$d_{\mathbf{1}_T} = k^{-1} \sum_{\sigma\in\mathfrak{S}_{k,T}} q_{\sigma(1)\sigma(2)} \cdots {}_{\sigma(k-1)\sigma(k)} q_{\sigma(k)\sigma(1)} = k^{-1} \sum_{c=(\sigma(1),\ldots,\sigma(k)):\sigma\in\mathfrak{S}_{k,T}} \prod_{i\in T} q_{i,c(i)},$$

where c is the cycle $(\sigma(1), \ldots, \sigma(k))$ of T. If $c = (i_1, \ldots, i_k)$ is a cycle of T, there are exactly k bijections σ of $\mathfrak{S}_{k,T}$ such that the two cycles $(\sigma(1), \ldots, \sigma(k))$ and $(i_1, \ldots, i_k)$ are equal. Indeed let s be the cycle $(1, \ldots, k)$ of $[k]$. If σ is such a bijection the others are $\sigma \circ s^m$ for m in $[k-1]$. For all bijections σ satisfying $(\sigma(1), \ldots, \sigma(k)) = c$, we have $q_{\sigma(1)\sigma(2)} \cdots q_{\sigma(k-1)\sigma(k)} q_{\sigma(k)\sigma(1)} = \prod_{i\in T} q_{i,c(i)}$ and

$$d_{\mathbf{1}_T} = k^{-1} k \sum_{c\in\mathcal{C}_T} \prod_{i\in T} q_{i,s(i)} = \sum_{c\in\mathcal{C}_T} \prod_{i\in T} q_{i,c(i)} = b_T$$

according to the first part of (5.2). □

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