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Directional phantom distribution functions for stationary random fields

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We give necessary and sufficient conditions for the existence of a phantom distribution function for a stationary random field on a regular lattice. We also introduce a less demanding notion of a directional phantom distribution, with potentially broader area of applicability. Such approach leads to sectorial limit properties, a phenomenon well-known in limit theorems for random fields. An example of a stationary Gaussian random field is provided showing that the two notions do not coincide. Criteria for the existence of the corresponding notions of the extremal index and the sectorial extremal index are also given.

Keywords: Stationary random fields; extreme value limit theory; phantom distribution function; extremal index; Gaussian random fields

1. Introduction and announcement of results

1.1. Phantom distribution functions for sequences

The notion of a phantom distribution function was introduced by O'Brien [24]. Let $\{X_n : n \in \mathbb{Z}\}$ be a stationary sequence with a marginal distribution function F and partial maxima $M_n := \max\{X_k : 1 \le k \le n\}$, $n \in \mathbb{N}$. We say that a distribution function G is a *phantom distribution function for* $\{X_n\}$, if

$$\sup_{x\in\mathbb{R}} \left| P(M_n \le x) - G(x)^n \right| \xrightarrow[n\to\infty]{} 0.$$

This means that G completely describes asymptotic properties (in law) of partial maxima $\{X_n\}$. G is also involved in description of asymptotics of higher order statistic of $\{X_n\}$ (see [15] and [27]).

If G can be chosen in the form $G(x) = F^\theta(x)$, that is, if for some $\theta \in (0, 1]$

$$\sup_{x\in\mathbb{R}} \left| P(M_n \le x) - P(X_0 \le x)^{\theta n} \right| \xrightarrow[n\to\infty]{} 0,$$

then, following Leadbetter [19], we call θ the extremal index of $\{X_n\}$. The extremal index is a popular tool in the stochastic extreme value limit theory (see, e.g., [20]). There exist, however, important classes of stationary sequences which admit a *continuous* phantom distribution function, while their extremal index is 0 (see [7,19]). The latter means only that their partial maxima increase essentially slower than in the independent case, but does not bring any quantitative information on the growth of M_n. Such phenomenon occurs, for example, when Lindley's process has subexponential innovations [1] or when the continuous target distribution of the random walk Metropolis algorithm has heavy tails [25].

Existence of a phantom distribution function is a quite common property. O'Brien [24] and Rootzén [26] give explicit formulas for phantom distribution functions of some Markov chains exhibiting Harris

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recurrence. Jakubowski [14], Theorem 16, provides a method of construction of a phantom distribution function for instantaneous functions of a Markov chain. Doukhan et al. [7], Theorem 6, show, that any α-mixing sequence with *continuous* marginals admits a continuous phantom distribution function. General Theorem 2, *ibid.*, asserts that a stationary sequence $\{X_n\}$ admits a continuous phantom distribution function if, and only if, there exists a sequence $\{v_n\}$ and $\gamma \in (0,1)$ such that

$$\mathrm{P}(M_n \le v_n) \xrightarrow[n\to\infty]{} \gamma,$$

and for each $T > 0$ the following *Condition* $\mathbf{B}_T(\{v_n\})$ is fulfilled:

$$\sup_{\substack{p,q\in\mathbb{N},\\ p+q\le T\cdot n}} \left|\mathrm{P}(M_{p+q} \le v_n) - \mathrm{P}(M_p \le v_n)\,\mathrm{P}(M_q \le v_n)\right| \xrightarrow[n\to\infty]{} 0.$$

Notice that *Condition* $\mathbf{B}_T(\{v_n\})$ can be satisfied even by non-ergodic sequences (see Theorem 4, *ibid.*). *Condition* $\mathbf{B}_T(\{v_n\})$ was introduced in [13].

Another interesting issue is that there are "user-friendly" criteria of existence of a phantom distribution function for arbitrary (non-stationary) sequences – see [14] and [18], Theorem 3. Such results are particularly useful in investigating Markov chains "starting at the point".

1.2. Phantom distribution functions for random fields

As the previous section shows, the theory of phantom distribution functions for random *sequences* is essentially closed. It is therefore surprising that the corresponding theory of phantom distributions for *random fields over* $\mathbb{Z}^d$ is still far from being complete.

Let $\mathbb{Z}^d$ be the d-dimensional lattice built on integers with the standard (coordinatewise) partial order $\le$. Let $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ be a d-dimensional *stationary* random field with a marginal distribution function F and partial maxima defined for $\mathbf{j}, \mathbf{n} \in \mathbb{Z}^d$ by the formulae

$$M_{\mathbf{j},\mathbf{n}} := \max\{X_{\mathbf{k}} : \mathbf{j} \le \mathbf{k} \le \mathbf{n}\}, \quad \text{if } \mathbf{j} \le \mathbf{n}, \qquad M_{\mathbf{j},\mathbf{n}} := -\infty \quad \text{if } \mathbf{j} \not\le \mathbf{n}.$$

It is also convenient to define

$$M_{\mathbf{n}} := M_{\mathbf{1},\mathbf{n}}, \quad \mathbf{n} \in \mathbb{Z}^d.$$

Of course, $M_{\mathbf{n}}$ is of interest only if $\mathbf{n} \in \mathbb{N}^d$ (here and in the sequel we distinguish between $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$).

It seems that the first paper that mentions the notion of a phantom distribution function in our rectangular setting is [17] and we will follow this paper.

Definition 1.1. We will say that G is a phantom distribution function for $\{X_{\mathbf{n}}\}$, if

$$\sup_{x\in\mathbb{R}} \left| P(M_{\mathbf{n}} \le x) - G(x)^{\mathbf{n}^*} \right| \to 0 \quad \text{as } \mathbf{n} \to \boldsymbol{\infty} \text{ (coordinatewise)}, \tag{1}$$

where $\mathbf{n}^* = n_1 \cdot n_2 \cdot \ldots \cdot n_d$, if $\mathbf{n} = (n_1, n_2, \dots, n_d)$.

Theorem 4.3 *ibid.* states that m-dependent random fields as well as moving maxima, moving averages and Gaussian fields satisfying Berman's condition admit a phantom distribution function in the above, strong sense. Another family of interesting examples, exploring the idea of a *tail field* in the context of the extremal index can be found in [29].

Note that (1) describes the asymptotic behavior of $M_{\mathbf{n}}$ *regardless* of the way in which $\mathbf{n}$ grows to $\boldsymbol{\infty} = (\infty, \infty, \ldots, \infty)$. To make this statement precise, let us define a *monotone curve in* $\mathbb{N}^d$ as a map $\boldsymbol{\psi} : \mathbb{N} \to \mathbb{N}^d$ such that $\boldsymbol{\psi}(n) \to \boldsymbol{\infty}$, for $n = 1, 2, \ldots$ $\boldsymbol{\psi}(n) \leq \boldsymbol{\psi}(n+1)$ and $\boldsymbol{\psi}(n) \neq \boldsymbol{\psi}(n+1)$ (hence $\{\boldsymbol{\psi}(n)^*\}$ is strictly increasing) and, as $n \to \infty$,

$$\frac{\boldsymbol{\psi}(n)^*}{\boldsymbol{\psi}(n+1)^*} \to 1. \tag{2}$$

Definition 1.2. We will say that G is a phantom distribution function for $\{X_{\mathbf{n}}\}$ *along* a monotone curve $\boldsymbol{\psi}$, if

$$\sup_{x \in \mathbb{R}} \left| P(M_{\boldsymbol{\psi}(n)} \leq x) - G(x)^{\boldsymbol{\psi}(n)^*} \right| \xrightarrow[n \to \infty]{} 0. \tag{3}$$

Any function G satisfying (3) will be denoted by $G_{\boldsymbol{\psi}}$. Within such terminology, we have the following proposition.

Proposition 1.3. *A stationary random field* $\{X_{\mathbf{n}}\}$ *admits a continuous phantom distribution function* G *if, and only if, there is a continuous* G *that is a phantom distribution function for* $\{X_{\mathbf{n}}\}$ *along* every *monotone curve.*

Another consequence of (1) is that if x has the property that $G(x)^{\mathbf{n}^*}$ is a "good" approximation of $P(M_{\mathbf{n}} \leq x)$, then it is equally good for all other points $\mathbf{m}$ with $\mathbf{m}^* = \mathbf{n}^*$. In other words, such x is a function of the class $L_k = \{\mathbf{n} \in \mathbb{N}^d; \mathbf{n}^* = k\}$ rather, than of $\mathbf{n}$ alone. We formalize this observation by introducing the notion of a *strongly monotone field of levels*. We will say that $v_{(\cdot)} : \mathbb{N}^d \to \mathbb{R}^1$ is strongly monotone, if $v_{\mathbf{m}} \leq v_{\mathbf{n}}$ whenever $\mathbf{m}^* \leq \mathbf{n}^*$. This implies, in particular, that $v_{\mathbf{m}} = v_{\mathbf{n}}$, if $\mathbf{m}^* = \mathbf{n}^*$. Moreover, there exists a non-decreasing sequence $\{u_n\}$ such that

$$v_{\mathbf{n}} = u_{\mathbf{n}^*}, \quad \mathbf{n} \in \mathbb{N}^d,$$

and conversely, every non-decreasing sequence $\{u_n\}$ defines a strongly monotone field of levels through the above formula.

We are now able to give a multidimensional analog of [7], Theorem 2.

Theorem 1.4. *Let* $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ *be a stationary random field. Then* $\{X_{\mathbf{n}}\}$ *admits a continuous phantom distribution function if, and only if, the following two conditions are satisfied.*

(i) *There exist* $\gamma \in (0, 1)$ *and a* strongly monotone *field of levels* $\{v_{\mathbf{n}}; \mathbf{n} \in \mathbb{N}^d\}$ *such that*

$$P(M_{\mathbf{n}} \leq v_{\mathbf{n}}) \to \gamma \quad \text{as } \mathbf{n} \to \boldsymbol{\infty}. \tag{4}$$

(ii) *For every monotone curve* $\boldsymbol{\psi}$ *and every* $T > 0$ *the following* Condition $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ *holds.*

$$\beta_T^{\boldsymbol{\psi}}(n) := \max_{\mathbf{p}(1)+\mathbf{p}(2) \leq T\boldsymbol{\psi}(n)} \Bigl| P(M_{\mathbf{p}(1)+\mathbf{p}(2)} \leq v_{\boldsymbol{\psi}(n)}) - \prod_{\mathbf{i} \in \{1,2\}^d} P(M_{(p_1(i_1), p_2(i_2), \ldots, p_d(i_d))} \leq v_{\boldsymbol{\psi}(n)}) \Bigr| \xrightarrow[n \to \infty]{} 0.$$

(*The quantities* $\mathbf{p}(1)$ *and* $\mathbf{p}(2)$ *under maximum take values in* $\mathbb{N}_0^d$).

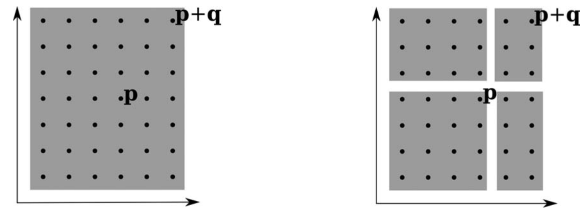

Figure 1. Breaking probabilities into blocks as a consequence of Condition $\mathbf{B}^{\boldsymbol{\psi} T}(\{v_{\boldsymbol{\psi}(n)}\})$, for $d = 2$.

Condition $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ looks complicated but it is based on a simple idea. We shall illustrate it in the two-dimensional case. Notice that for $d = 2$, we have

$$\beta_T^{\boldsymbol{\psi}}(n) = \max_{\mathbf{p}+\mathbf{q} \le T\boldsymbol{\psi}(n)} \Big| P(M_{\mathbf{p}+\mathbf{q}} \le v_{\boldsymbol{\psi}(n)}) - P(M_{\mathbf{p}} \le v_{\boldsymbol{\psi}(n)}) P(M_{(p_1,q_2)} \le v_{\boldsymbol{\psi}(n)}) P(M_{(q_1,p_2)} \le v_{\boldsymbol{\psi}(n)}) P(M_{\mathbf{q}} \le v_{\boldsymbol{\psi}(n)}) \Big|$$

and, moreover, by the stationarity,

$$\begin{aligned} P(M_{(p_1,q_2)} \le v_{\boldsymbol{\psi}(n)}) &= P(M_{(1,p_2+1),(p_1,p_2+q_2)} \le v_{\boldsymbol{\psi}(n)}), \\ P(M_{(q_1,p_2)} \le v_{\boldsymbol{\psi}(n)}) &= P(M_{(p_1+1,1),(p_1+q_1,p_2)} \le v_{\boldsymbol{\psi}(n)}), \\ P(M_{\mathbf{q}} \le v_{\boldsymbol{\psi}(n)}) &= P(M_{\mathbf{p+1},\mathbf{p+q}} \le v_{\boldsymbol{\psi}(n)}). \end{aligned}$$

It follows that if $\beta_T^{\boldsymbol{\psi}}(n) \to 0$, as $n \to \infty$, then $P(M_{\mathbf{p}+\mathbf{q}} \le v_{\boldsymbol{\psi}(n)})$ can be approximated by the product of the four probabilities for maxima over disjoint blocks, as in Figure 1.

By convention, if some coordinate of $\mathbf{p}$ or $\mathbf{q}$ is 0, then $P(M_{\mathbf{p}+\mathbf{q}} \le v_{\boldsymbol{\psi}(n)})$ breaks into smaller number of blocks (for $d = 2$ into 2 or 1 block).

Remark 1.5. Models exhibiting local dependence (like m-dependent or max-m-approximable random fields) admit a continuous phantom distribution function by [17], Theorem 4.3, and so, by our Theorem 1.4, satisfy Condition $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$.

Remark 1.6. Readers familiar with mixing conditions may dislike the shape of Condition $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ for there is no "separation of blocks" like in [21], Coordinatewise mixing, or Ling [22], Condition A1. Apart from the more complicated form of these conditions (that would be overwhelming in d-dimensional considerations), they are essentially not easier in verification. We find the form of Condition $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ very useful in theoretical consideration. As a good example of how to check Condition $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ (in one dimension) may serve Theorems 6–9 in [7].

Remark 1.7. The framework of maxima over rectangles may seem to be too restrictive for applications. It has, however, substantial advantages. First, it preserves the one-dimensional idea of complete description of asymptotics of maxima in terms of a single field of levels (asymptotic quantiles). Second, in view of its simplicity, it is a natural candidate to be used in description of local properties of maxima of *locally stationary* random fields. Finally, it brings a non-trivial illustration of possible difficulties in transmission of one-dimensional results to spatial considerations.

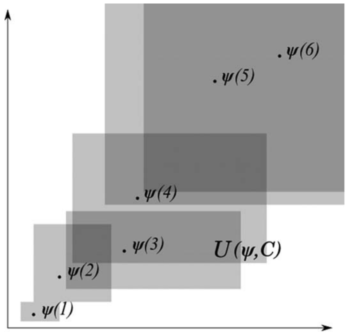

Figure 2. The shaded area is the set $U(\boldsymbol{\psi}, C) \subset \mathbb{R}^2$ for $C = 2$.

Remark 1.8. Given a random field on $\mathbb{Z}^d$ it is natural to consider quantities $M(A) = \max_{\mathbf{n}\in A} X_{\mathbf{n}}$, where $A \subset \mathbb{Z}^d$ belongs to a suitable class of subsets $\mathcal{A}$. Operating with quantities $M(A)$ is quite easy when $\{X_{\mathbf{n}}\}$ are independent, but passing to weakly dependent fields requires much care. We refer to [6] and [3] for discussion of possible complications and to [16] for instruction how to deal with the boundary problems in case of arbitrary sets A.

1.3. Directional and sectorial phantom distribution functions

Suppose that F is continuous. Choose $\gamma \in (0, 1)$ and define the following field of levels (in fact, quantiles):

$$v_{\mathbf{n}} = \inf\{x : P(M_{\mathbf{n}} \le x) = \gamma\}.$$

Then $\{v_{\mathbf{n}}\}$ is non-decreasing, we have $P(M_{\mathbf{n}} \le v_{\mathbf{n}}) \to \gamma$, but there is no reason to expect that it is strongly monotone (see Section 1.4).

It follows that the basic observation developed in [7] for sequences fails for phantom distribution functions for random fields (by our Theorem 1.4). This signalizes a serious difficulty and suggests that the theory of phantom distribution functions (and of the extremal index) in the sense of strong definition (1) is restricted to random fields with really short-range dependencies (numerous examples of which are mentioned in the previous section).

The theory of *directional* phantom distribution functions, that is developed below, is free of such drawbacks.

Let $\{\boldsymbol{\psi}(n)\}$ be a monotone curve. We define the class $\mathcal{U}_{\boldsymbol{\psi}}$ of monotone curves, being a kind of a "neighbourhood" of $\boldsymbol{\psi}$, as follows. A monotone curve $\boldsymbol{\varphi}$ belongs to $\mathcal{U}_{\boldsymbol{\psi}}$ if and only if for some constant $C \ge 1$ and for almost all $n \in \mathbb{N}$

$$\boldsymbol{\varphi}(n) \in U(\boldsymbol{\psi}, C) := \bigcup_{j\in\mathbb{N}} \prod_{i=1}^{d} \left[C^{-1}\psi_i(j), C\psi_i(j)\right].$$

An example of $U(\boldsymbol{\psi}, C)$ is shown in Figure 2.

Definition 1.9. Let $\{\boldsymbol{\psi}(n)\}$ be a monotone curve. We will say that a distribution function G is the $\boldsymbol{\psi}$-directional phantom distribution function for $\{X_{\mathbf{n}}\}$, if G is a phantom distribution function for $\{X_n\}$

along every monotone curve belonging to the set $\mathcal{U}_{\boldsymbol{\psi}}$. We shall denote the $\boldsymbol{\psi}$-directional phantom distribution function by $G_{\boldsymbol{\psi}}$.

Note that we already used the notation $G_{\boldsymbol{\psi}}$ to denote the phantom distribution function *along* $\boldsymbol{\psi}$. But there is no ambiguity. As we shall see in Theorem 1.12 below any phantom distribution function *along* $\boldsymbol{\psi}$ is automatically the $\boldsymbol{\psi}$-directional phantom distribution function for $\{X_{\mathbf{n}}\}$ and conversely.

Remark 1.10. Let $\boldsymbol{\Delta}(n) = (n, n, \dots, n)$, $n \in \mathbb{N}$, denote the diagonal map. Observe that $\boldsymbol{\varphi}$ belongs to $\mathcal{U}_{\boldsymbol{\Delta}}$ if, and only if, $\varphi_1(n), \varphi_2(n), \dots, \varphi_d(n)$ are of the same order, that is, $1/C \leq \varphi_i(n)/\varphi_j(n) < C$ for some $C \geq 1$, all $i, j \in \{1, 2, \dots, d\}$ and almost all $n \in \mathbb{N}$. Therefore, the following definition is natural.

Definition 1.11. If G is a $\boldsymbol{\Delta}$-directional phantom distribution function, we call it a *sectorial* phantom distribution function.

Theorem 1.12. *Let* $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ *be a stationary random field and let* $\boldsymbol{\psi}$ *be a monotone curve.*
The following statements (i)–(iii) *are equivalent.*

(i) $\{X_{\mathbf{n}}\}$ *admits a continuous phantom distribution function along* $\boldsymbol{\psi}$.
(ii) $\{X_{\mathbf{n}}\}$ *admits a continuous* $\boldsymbol{\psi}$*-directional phantom distribution function.*
(iii) *There exist* $\gamma \in (0, 1)$ *and a* non-decreasing *sequence of levels* $\{v_{\boldsymbol{\psi}(n)}\}$, $n \in \mathbb{N}$, *such that*

$$P(M_{\boldsymbol{\psi}(n)} \leq v_{\boldsymbol{\psi}(n)}) \xrightarrow[n\to\infty]{} \gamma, \tag{5}$$

and for every $T > 0$ *Condition* $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ *holds.*

Some comments are relevant here.

First, the notion of $\boldsymbol{\psi}$-directional phantom distribution function is essentially weaker than that of global phantom distribution function. In Section 1.4, we construct a stationary Gaussian random field $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^2\}$ that admits a sectorial phantom distribution function while the global phantom distribution function does not exist for $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^2\}$.

Second, we do not know whether similar examples are common in practice. We would like, however, to stress the fact that knowing sectorial (or other $\boldsymbol{\psi}$-directional) phantom distribution function can be sufficient in many problems, because the family $\mathcal{U}_{\boldsymbol{\psi}}$ is quite large. In fact, "sectorial" results arise in several areas of the theory of random fields. For instance, Gut [12] gives strong laws for i.i.d. random fields indexed by a *sector* and Gadidov [11] deals with such framework for U-statistics.

Third, finding a sectorial (or $\boldsymbol{\psi}$-directional) phantom distribution function is essentially easier than finding a global one, as Theorem 1.13, stated below for general non-stationary random fields, shows. Notice that we can directly adopt all the definitions to the non-stationary setting.

Theorem 1.13. *Let* $\{Z_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ *be an* arbitrary *random field with partial maxima* $\{M_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}^d}$ *and let* $\boldsymbol{\psi}$ *be a monotone curve.*

Then $\{Z_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ *admits a* $\boldsymbol{\psi}$*-directional continuous phantom distribution function if, and only if, there exist* $\gamma \in (0, 1)$ *and a non-decreasing sequence of levels* $\{v_{\boldsymbol{\psi}(n)}\}$ *such that*

$$P(M_{(\lfloor q_1\psi_1(n)\rfloor, \lfloor q_2\psi_2(n)\rfloor, \dots, \lfloor q_d\psi_d(n)\rfloor)} \leq v_{\boldsymbol{\psi}(n)}) \xrightarrow[n\to\infty]{} \gamma^{q_1 q_2 \cdots q_d}. \tag{6}$$

for every d-tuple $\mathbf{q} = (q_1, q_2, \dots, q_d) \in \mathbb{Q}^d$, *where* $\mathbb{Q}$ *is a dense subset of* $\mathbb{R}^+$. (*Here and in what follows* $\lfloor\cdot\rfloor$ *stands for the floor function.*)

Remark 1.14. In the case of sectorial phantom distribution function (i.e., if $\boldsymbol{\psi} = \boldsymbol{\Delta}$) relation (6) takes especially simple form

$$P(M_{(\lfloor q_1 n\rfloor, \lfloor q_2 n\rfloor, \ldots, \lfloor q_d n\rfloor)} \le v_{\boldsymbol{\Delta}(n)}) \xrightarrow[n\to\infty]{} \gamma^{q_1 q_2 \cdots q_d}, \quad (q_1, q_2, \ldots, q_d) \in \mathbb{Q}^d.$$

Remark 1.15. Theorem 1.13 is a multidimensional counterpart of [18], Theorem 3, (see also [14], Theorem 2, Corollary 5).

1.4. Example

1.4.1. *Tools for analysis of extremes of Gaussian families*

The analysis of the example we are going to construct requires the whole power of classic methods developed in sixties and seventies of the twentieth century and summarized in an exhaustive manner in [20], Part II. In this theory, the central position is occupied by the so-called normal comparison lemma ([20], Theorem 4.2.1) which we will use in the form of the Berman inequality restated below as (7).

Let $(W_1, W_2, \ldots, W_n)$ and $(Z_1, Z_2, \ldots, Z_n)$ be two standardized normal random vectors, i.e. $EW_j = EZ_j = 0$ and $EW_j^2 = EZ_j^2 = 1$, $j = 1, 2, \ldots, n$. Let $\Lambda^W_{i,j}$ (resp. $\Lambda^Z_{i,j}$) be the covariance of W_i and W_j (resp. Z_i and Z_j), $1 \le i, j \le n$. Finally, let $\omega_{i,j} = \max\{|\Lambda^W_{i,j}|, |\Lambda^Z_{i,j}|\}$, $1 \le i, j \le n$ and assume that

$$\max_{i\ne j} \omega_{i,j} = \delta < 1.$$

Then we have

$$\begin{aligned} &\Big| P\Big(\max_{1\le j\le n} W_j \le u\Big) - P\Big(\max_{1\le j\le n} Z_j \le u\Big)\Big| \\ &\quad \le L(\delta) \sum_{1\le i<j\le n} \big|\Lambda^W_{i,j} - \Lambda^Z_{i,j}\big| \exp\Big(-\frac{u^2}{1+\omega_{i,j}}\Big), \end{aligned} \tag{7}$$

where $L(\delta) = 1/(2\pi\sqrt{1-\delta^2})$. In particular, if $Z_1, Z_2, \ldots, Z_n$ are independent then we obtain

$$\Big| P\Big(\max_{1\le j\le n} W_j \le u\Big) - \Phi(u)^n\Big| \le L(\delta) \sum_{1\le i<j\le n} |r_{i,j}| \exp\Big(-\frac{u^2}{1+|r_{i,j}|}\Big), \tag{8}$$

where $r_{i,j} = \Lambda^W_{i,j}$ and $\Phi(x)$ is the distribution function of a standard normal random variable.

Given these tools it was possible to demonstrate possible distributional limits of partial maxima for stationary standardized Gaussian sequences $\{X_j\}$ with the covariance sequence $r_n = EX_nX_0$, $n = 0, 1, 2, \ldots$. In particular,

if $r_n \ln n \to 0$, then under some centering and normalization M_n converge to a Gumbel distribution (in fact: Φ is a phantom distribution function for $\{X_j\}$) – see [20], Theorem 4.3.3;

if $r_n \ln n \to c \in (0, \infty)$ then under the same centering and normalization as above M_n converge in law to the convolution of Gumbel and normal distributions (but no phantom distribution function exists) – see [20], Corollary 6.5.2;

if both $r_n \to 0$ *and* $r_n \ln n \to \infty$ *monotonically* then under *some* centering and normalization M_n converge to the normal law (and no phantom distribution function exists) – see [20], Theorem 6.6.4.

1.4.2. *The idea*

It was proved in [17], Section 3.2, that a stationary Gaussian random field $\{X_{(i,j)}\}_{(i,j)\in\mathbb{Z}^2}$ with the covariance field

$$r_{i,j} = E(X_{(i,j)}X_{(0,0)}), \quad (i,j) \in \mathbb{N}_0^2,$$

satisfying the two-dimensional Berman condition

$$r_{i,j}\ln(i\cdot j) \to 0 \quad \text{as } (i,j) \to (\infty,\infty),$$

admits a (global) phantom distribution function Φ.

We are going to provide an example of a stationary Gaussian random field with the covariance field satisfying

$$r_{i,j} \sim C\frac{\ln\ln|i|}{\ln|i|}\frac{1}{\ln|j|} \quad \text{as } (i,j) \to (\infty,\infty), \tag{9}$$

for some $C > 0$. Such a field satisfies the Berman condition $r_{n,n}\ln n^2 \to 0$ along the diagonal. On the other hand, we have

$$r_{\lfloor n/\ln n\rfloor,\lfloor \ln n\rfloor}\ln n \to C, \quad \text{as } n \to \infty,$$

as well as

$$r_{\lfloor n/\ln\ln n\rfloor,\lfloor \ln\ln n\rfloor}\ln n \to +\infty \quad \text{as } n \to \infty,$$

that is, the Berman condition is violated along some monotone curves.

By the analogy with the three cases discussed in the previous paragraph one may guess that we have a phantom distribution function (namely Φ) along the diagonal (hence $\{X_{(i,j)}\}$ admits a sectorial phantom distribution function) and that no phantom distribution function is suitable along some monotone curves (hence, there is no global phantom distribution function). This guess is right, but the covariance structure of random variables contributing to the maxima over rectangles is more complicated than in the one-dimensional case. Therefore the transmission to random fields is not automatic and we have to perform carefully all computations.

1.4.3. *The construction of a random field*

We shall construct a stationary Gaussian random field $\mathbf{X} = \{X_{(i,j)}, (i,j) \in \mathbb{Z}^2\}$ with mean zero, unit variance and covariance function $EX_{(i,j)}X_{(0,0)} = r_{i,j}$ of the form

$$r_{i,j} = \eta_1(i)\eta_2(j),$$

where $\eta_1(\theta)$ and $\eta_2(\theta)$ are characteristic functions of symmetric distributions on $\mathbb{R}^1$ (hence, real functions).

Both η_1 and η_2 will be defined according to Polya's recipe (see, e.g., [9], p. 509).

Take $\gamma_1 > 0$ and consider the polygon connecting points

$$(0,1), \left(1, \gamma_1\left(27\frac{\ln(\ln 27)}{\ln 27} - 26\frac{\ln(\ln 28)}{\ln 28}\right)\right), \left(28, \gamma_1\frac{\ln(\ln 28)}{\ln 28}\right), \left(29, \gamma_1\frac{\ln(\ln 29)}{\ln 29}\right), \ldots$$

It is a graph of a function $\eta_1^{\gamma_1}$ defined on $\mathbb{R}^+$. Let Γ_1 consists of γ_1 with the property that $\eta_1^{\gamma_1}$ is a *convex* function on $\mathbb{R}^+$. This set is non-empty. Indeed, $f(x) = \ln(\ln(x))/\ln(x)$ is a convex function on $(27-\kappa, +\infty)$, for some $\kappa > 0$. Therefore, $\eta_1^{\gamma_1}$ is convex on $[27, +\infty)$ for every $\gamma_1 > 0$ and the problem is reduced to simple geometrical considerations involving the first three points in the above series. A direct calculation shows that $\gamma_1 = 0.3 \in \Gamma_1$.

In a similar way, we construct functions $\eta_2^{\gamma_2}$ on the base of points

$$(0,1), \left(1, \gamma_2\left(\frac{2}{\ln 2} - \frac{1}{\ln 3}\right)\right), \left(3, \gamma_2 \frac{1}{\ln 3}\right), \left(4, \gamma_2 \frac{1}{\ln 4}\right), \ldots .$$

Functions $\eta_2^{\gamma_2}$ are convex for $\gamma_2 \in \Gamma_2 \subset (0, +\infty)$, with $\gamma_2 = 0.1 \in \Gamma_2$.

Let $\Gamma \subset \Gamma_1 \times \Gamma_2$ consists of points (γ_1, γ_2) satisfying additionally

$$\gamma_1 > 1/4, \quad \gamma_1\left(27 \frac{\ln(\ln 27)}{\ln 27} - 26 \frac{\ln(\ln 28)}{\ln 28}\right) < \gamma_2\left(\frac{2}{\ln 2} - \frac{1}{\ln 3}\right) < \frac{1 - 2\gamma_1}{1 + 2\gamma_1}. \tag{10}$$

Again direct calculation shows that $(0.3, 0.1) \in \Gamma$ and so Γ is non-empty.

Let $(\gamma_1, \gamma_2) \in \Gamma$. Set $\eta_1(\theta) = \eta_1^{\gamma_1}(\theta)$, $\eta_2(\theta) = \eta_2^{\gamma_2}(\theta)$, $\theta \geq 0$ and extend η_1 and η_2 to even functions by reflection. By Polya's recipe both η_1 and η_2 are characteristic functions. In particular, both $\{\eta_1(i)\}_{i \in \mathbb{Z}}$ and $\{\eta_2(j)\}_{j \in \mathbb{Z}}$ are positive definite sequences.

It follows that $r_{i,j} = \eta_1(i)\eta_2(j)$ is a covariance field on $\mathbb{Z}^2$. Moreover, by (10) it satisfies

$$\delta := \sup_{(i,j) \in \mathbb{Z}^2 \setminus \{(0,0)\}} r_{i,j} < \frac{1 - 2\gamma_1}{1 + 2\gamma_1} < \frac{1}{3}, \tag{11}$$

and for i and j with sufficiently large absolute values we have even more than (9):

$$r_{i,j} = \gamma_1 \gamma_2 \frac{\ln \ln |i|}{\ln |i|} \frac{1}{\ln |j|}.$$

1.4.4. Φ *is a sectorial phantom distribution function*

We shall prove that

$$\sup_{x \in \mathbb{R}} \left| P(M_{\mathbf{n}} \leq x) - \Phi(x)^{n^2} \right| \xrightarrow[n \to \infty]{} 0, \tag{12}$$

where $\mathbf{n} = (n, n) = \boldsymbol{\Delta}(n)$, $M_{\mathbf{n}} = \max_{(i,j) \in [1,n] \times [1,n]} X_{(i,j)}$. Applying Theorem 1.12, we will conclude that Φ is a $\boldsymbol{\Delta}$-directional (or sectorial) phantom distribution function for $\mathbf{X}$.

As in [20], Section 4.3, in order to prove (12) it is sufficient to show that for every $c > 0$

$$P\big(M_{\mathbf{n}} \leq u_{\mathbf{n}}(c)\big) = \Phi\big(u_{\mathbf{n}}(c)\big)^{n^2} + o(1),$$

where levels $\{u_{\mathbf{n}}(c)\}$ are such that $n^2(1 - \Phi(u_{\mathbf{n}}(c))) \to c$. Note that for n large enough

$$\exp\left(-\frac{(u_{\mathbf{n}}(c))^2}{2}\right) = \frac{\sqrt{2\pi} c u_{\mathbf{n}}(c)}{n^2}\big(1 + o(1)\big) \leq \frac{2\sqrt{\pi} c u_{\mathbf{n}}(c)}{n^2} = K(c) \frac{u_{\mathbf{n}}(c)}{n^2} \tag{13}$$

and that

$$u_{\mathbf{n}}(c) \sim \sqrt{4 \ln n} \quad \text{as } n \to \infty. \tag{14}$$

We have by (8)

$$\left| P\big(M_{\mathbf{n}} \leq u_{\mathbf{n}}(c)\big) - \Phi\big(u_{\mathbf{n}}(c)\big)^{n^2} \right|$$

$$\le L(\delta) \sum_{\substack{(i,j),(k,l)\in\{1,2,\dots,n\}^2\\(i,j)\neq(k,l)}} \operatorname{Cov}(X_{(i,j)}, X_{(k,l)}) \exp\left(-\frac{(u_{\mathbf{n}}(c))^2}{1+r_{i-k,j-l}}\right)$$

$$\le 4L(\delta)n^2 \sum_{\substack{0\le i,j\le n\\(i,j)\neq(0,0)}} r_{i,j} \exp\left(-\frac{(u_{\mathbf{n}}(c))^2}{1+r_{i,j}}\right), \tag{15}$$

where we have used the stationarity and the fact that $r_{i,j} > 0$, $i, j \in \mathbb{Z}$. Repeating the steps of the proof of [20], Lemma 4.3.2, choose α, $0 < \alpha < \frac{1-3\delta}{1+\delta}$, (it is possible by (11)) and split the sum in the last line of (15) in two parts $\Sigma_1(n) = \sum_{(i,j)\in A_n}$ and $\Sigma_2(n) = \sum_{(i,j)\in B_n}$, where

$$A_n = \{\lceil n^\alpha \rceil, \dots, n\} \times \{\lceil n^\alpha \rceil, \dots, n\} \quad \text{and} \quad B_n = \{0, 1, \dots, n\}^2 \setminus (A_n \cup \{\mathbf{0}\}).$$

(Here and in the sequel $\lceil\cdot\rceil$ denotes the ceiling function). First, let us find the asymptotics of the part involving $\Sigma_2(n)$. We have for large n

$$\begin{aligned}
4L(\delta)n^2\Sigma_2(n) &\le 4L(\delta)n^2\big(2n^{1+\alpha} - (\lceil n^\alpha\rceil - 1)^2\big) \exp\left(-\frac{(u_{\mathbf{n}}(c))^2}{1+\delta}\right)\\
&\le 8L(\delta)K(c)^{\frac{2}{1+\delta}} n^{3+\alpha} \left(\frac{u_{\mathbf{n}}(c)}{n^2}\right)^{\frac{2}{1+\delta}} \quad \text{by (13)}\\
&\sim 8L(\delta)K(c)^{\frac{2}{1+\delta}} (4\ln n)^{\frac{1}{1+\delta}} n^{\alpha+3-\frac{4}{1+\delta}} \to 0 \quad \text{by (14) and the choice of } \alpha.
\end{aligned}$$

Next, let us notice that for $i, j \ge \lceil n^\alpha\rceil$ and n large enough

$$r_{i,j} \le \frac{\ln(\ln n^\alpha)}{(\ln n^\alpha)^2} \le \alpha^{-2} \frac{\ln\ln n}{(\ln n)^2}.$$

Therefore, setting $\delta_n' = \sup_{i,j\in A_n} r_{i,j}$ and using (14) we obtain that $\delta_n'(u_{\mathbf{n}}(c))^2 \to 0$, as $n \to \infty$. Keeping this relation in mind, we can proceed as follows.

$$\begin{aligned}
4L(\delta)n^2\Sigma_1(n) &= 4L(\delta)n^2 \sum_{(i,j)\in A_n} r_{i,j} \exp\left(-\frac{(u_{\mathbf{n}}(c))^2}{1+r_{i,j}}\right)\\
&= 4L(\delta)n^2 \exp\big(-(u_{\mathbf{n}}(c))^2\big) \sum_{(i,j)\in A_n} r_{i,j} \exp\left(\frac{(u_{\mathbf{n}}(c))^2 r_{i,j}}{1+r_{i,j}}\right)\\
&\le 4L(\delta)\big(K(c)\big)^2 n^2 \left(\frac{u_{\mathbf{n}}(c)}{n^2}\right)^2 n^2\delta_n' \exp\big(\delta_n'(u_{\mathbf{n}}(c))^2\big) \quad \text{by (13)}\\
&= 4L(\delta)\big(K(c)\big)^2 \delta_n' (u_{\mathbf{n}}(c))^2 \exp\big(\delta_n'(u_{\mathbf{n}}(c))^2\big) \xrightarrow[n\to\infty]{} 0.
\end{aligned}$$

1.4.5. *There is no global phantom distribution function*

Let us consider the monotone curve

$$\boldsymbol{\psi}(n) = \big(\lfloor n/\ln n\rfloor, \lfloor \ln n\rfloor\big), \quad n \in \mathbb{N}.$$

By Proposition 1.3, it is enough to show that Φ *is not* a phantom distribution function for $\{X_{(i,j)}, (i,j) \in \mathbb{Z}^2\}$ along $\boldsymbol{\psi}$.

We will show first that

$$\sup_{x\in\mathbb{R}} \big| P(M_{\boldsymbol{\psi}(n)} \le x) - P(\widetilde{M}_n \le x) \big| \xrightarrow[n\to\infty]{} 0, \tag{16}$$

where for each $n \in \mathbb{N}$, $\widetilde{M}_n$ is the maximum of $\boldsymbol{\psi}(n)^*$ standard normal random variables $\xi_1, \xi_2, \dots, \xi_{\boldsymbol{\psi}(n)^*}$ with $\rho_n = \operatorname{cov}(\xi_i, \xi_j) = \frac{\gamma_1\gamma_2}{\ln n}$, $i \neq j$. As in the case of (12), we have to prove that

$$P\big(M_{\psi(n)} \le w_n(c)\big) = P\big(\widetilde{M}_n \le w_n(c)\big) + o(1),$$

for sequences of levels $\{w_n(c)\}$ such that $P(\widetilde{M}_n \le w_n(c)) \to c \in (0,1)$. Later we shall show that $\{w_n(c)\}$ satisfies

$$\exp\left(-\frac{w_n(c)^2}{2}\right) \le K'(c)\frac{w_n(c)}{n} \quad \text{and} \quad w_n(c) \sim \sqrt{2\ln n}. \tag{17}$$

By virtue of (7), and similarly as in the case of (15),

$$\begin{aligned} &\big|P\big(M_{\psi(n)} \le w_n(c)\big) - P\big(\widetilde{M}_n \le w_n(c)\big)\big| \\ &\quad \le 4L(\delta)n \sum_{(i,j)\in D_n} |r_{i,j} - \rho_n| \exp\left(-\frac{(w_n(c))^2}{1+\omega_{i,j}}\right), \end{aligned}$$

where $D_n = \{(i,j) : 0 \le i \le \frac{n}{\ln n}, 0 \le j \le \ln n\} \setminus \{(0,0)\}$ and $\omega_{i,j} = \max\{r_{i,j}, \rho_n\} = r_{i,j}$ on D_n. Let us split the set of indices D_n in three smaller parts, $D_n = D_n^{(1)} \sqcup D_n^{(2)} \sqcup D_n^{(3)}$, where $D_n^{(1)} = \{(i,j) : 0 \le i \le n^\alpha, 0 \le j \le \ln n\} \setminus \{(0,0)\}$, $D_n^{(2)} = \{(i,j) : n^\alpha < i \le \frac{n}{\ln n}, 0 \le j \le (\ln n)^\beta\}$ and $D_n^{(3)} = \{(i,j) : n^\alpha < i \le \frac{n}{\ln n}, (\ln n)^\beta < j \le \ln n\}$, where the parameters α and β will be chosen later.

By (11), we have $\delta < (1-2\gamma_1)/(1+2\gamma_1)$, or, equivalently, $2\gamma_1 < (1-\delta)/(1+\delta)$. So we can find α satisfying

$$2\gamma_1 < \alpha < \frac{1-\delta}{1+\delta}.$$

From (17), we have as $n \to \infty$,

$$\begin{aligned} n \sum_{(i,j)\in D_n^{(1)}} |r_{i,j} - \rho_n| \exp\left(-\frac{(w_n(c))^2}{1+r_{i,j}}\right) &\le n n^\alpha \ln n \exp\left(-\frac{(w_n(c))^2}{1+\delta}\right) \\ &\le \big(K'(c)\big)^{\frac{2}{1+\delta}} n^{\alpha+1} \ln n \left(\frac{w_n(c)}{n}\right)^{\frac{2}{1+\delta}} \\ &\sim \big(\sqrt{2}K'(c)\big)^{\frac{2}{1+\delta}} n^{\alpha - \frac{1-\delta}{1+\delta}} (\ln n)^{\frac{2+\delta}{1+\delta}} \to 0. \end{aligned}$$

The estimate for the term related to the sum over $(i,j) \in D_n^{(2)}$ is a bit more challenging. For indices $(i,j) \in D_n^{(2)}$ we have $|r_{i,j} - \rho_n| \le r_{i,j} \le \frac{\gamma_1}{\alpha}\frac{\ln\ln n}{\ln n} =: \delta_n$. Therefore, we obtain

$$n \sum_{(i,j)\in D_n^{(2)}} |r_{i,j} - \rho_n| \exp\left(-\frac{(w_n(c))^2}{1+r_{i,j}}\right)$$

$$
\begin{aligned}
&\le \frac{\gamma_1}{\alpha} n \frac{n}{\ln n} (\ln n)^{\beta} \frac{\ln\ln n}{\ln n} \exp\left(-\frac{(w_n(c))^2}{1+\delta_n}\right) \\
&\le \frac{\gamma_1}{\alpha} \big(K'(c)\big)^2 n^2 (\ln n)^{\beta-2} \ln\ln n \left(\frac{\sqrt{2\ln n}}{n}\right)^2 n^{2\delta_n} \\
&= \frac{\gamma_1}{\alpha} \big(K'(c)\big)^2 (\ln n)^{\beta-1} \ln\ln n \exp\left(2\frac{\gamma_1}{\alpha}\frac{\ln\ln n}{\ln n}\ln n\right) \\
&= \frac{\gamma_1}{\alpha} \big(K'(c)\big)^2 (\ln n)^{\beta+2\gamma_1/\alpha-1} \ln\ln n. \qquad (18)
\end{aligned}
$$

Because $\gamma_1 < \alpha/2$, we can find a positive β satisfying the inequality $\beta + 2\gamma_1/\alpha - 1 < 0$. For such β the expression in (18) tends to 0.

It remains to show that the term related to the sum over $(i,j) \in D_n^{(3)}$ vanishes as $n \to \infty$. Denote $\delta_n' = \max_{(i,j)\in D_n^{(3)}} r_{i,j}$ and notice that $\delta_n' \le \frac{\gamma_1\gamma_2}{\alpha\beta}/\ln n$. We need a special decomposition.

$$
\begin{aligned}
&n \sum_{(i,j)\in D_n^{(3)}} |r_{i,j} - \rho_n| \exp\left(-\frac{(w_n(c))^2}{1+r_{i,j}}\right) \le n \exp\left(-\frac{(w_n(c))^2}{1+\delta_n'}\right) \sum_{(i,j)\in D_n^{(3)}} (r_{i,j} - \rho_n) \\
&= \left\{\frac{n^2}{\ln n} \exp\left(-\frac{(w_n(c))^2}{1+\delta_n'}\right)\right\} \cdot \left\{\frac{\ln n}{n} \sum_{(i,j)\in D_n^{(3)}} (r_{i,j} - \rho_n)\right\} = I_1(n) \cdot I_2(n).
\end{aligned}
$$

Using (17) we obtain the boundedness of $\{I_1(n)\}$.

$$
\begin{aligned}
I_1(n) &= \frac{n^2}{\ln n} \exp\left(-\frac{(w_n(c))^2}{1+\delta_n'}\right) \le \big(K'(c)\big)^2 \frac{n^2}{\ln n} \left(\frac{w_n(c)}{n}\right)^2 \left(\frac{n}{w_n(c)}\right)^{2\frac{\gamma_1\gamma_2}{\alpha\beta}/\ln n} \\
&\sim \big(K'(c)\big)^2 \frac{n^2}{\ln n} \frac{2\ln n}{n^2} e^{2\frac{\gamma_1\gamma_2}{\alpha\beta}} \big(1 + o(1)\big) = O(1).
\end{aligned}
$$

We will conclude the proof of (16) by showing that $I_2(n) \to 0$ as $n \to \infty$. We have

$$
\begin{aligned}
\frac{\ln n}{n} \sum_{(i,j)\in D_n^{(3)}} (r_{i,j} - \rho_n) &= \frac{\ln n}{n} \sum_{(i,j)\in D_n^{(3)}} r_{i,j} - \frac{\ln n}{n}\left(\frac{n}{\ln n} - n^{\alpha}\right)\big(\ln n - (\ln n)^{\beta}\big)\rho_n \\
&= \gamma_1\gamma_2 \frac{\ln n}{n} \left(\sum_{i=n^{\alpha}}^{n/\ln n} \frac{\ln\ln i}{\ln i}\right)\left(\sum_{j=(\ln n)^{\beta}}^{\ln n} \frac{1}{\ln j}\right) - \gamma_1\gamma_2\Big(1 + O\big((\ln n)^{\beta-1}\big)\Big).
\end{aligned}
$$

We shall estimate the two sums appearing above. By integration by parts, we have for $1 < a < b$

$$
\int_a^b \ln t \frac{e^t}{t}\,dt \le \frac{a}{a-1} \ln b \frac{e^b}{b} \quad \text{and} \quad \int_a^b \frac{e^t}{t}\,dt \le \frac{a}{a-1}\frac{e^b}{b}.
$$

Therefore

$$
\sum_{i=n^{\alpha}}^{n/\ln n} \frac{\ln\ln i}{\ln i} \le \int_{n^{\alpha/2}}^{n/\ln n} \frac{\ln\ln y}{\ln y}\,dy = \int_{\alpha\ln n/2}^{\ln n - \ln\ln n} \ln t \frac{e^t}{t}\,dt
$$

$$\leq \frac{\alpha \ln n/2}{\alpha \ln n/2 - 1} \ln(\ln n - \ln\ln n) \frac{e^{\ln n - \ln\ln n}}{\ln n - \ln\ln n} = \frac{n \ln\ln n}{(\ln n)^2}\left(1 + O\left(\frac{\ln\ln n}{\ln n}\right)\right).$$

Similarly

$$\sum_{j=(\ln n)^\beta}^{\ln n} \frac{1}{\ln j} \leq \int_{(\ln n)^{\beta/2}}^{\ln n} \frac{1}{\ln y}\, dy = \int_{\beta \ln\ln n/2}^{\ln\ln n} \frac{e^t}{t}\, dt$$

$$\leq \frac{\beta \ln\ln n/2}{\beta \ln\ln n/2 - 1} \frac{e^{\ln\ln n}}{\ln\ln n} = \frac{\ln n}{\ln\ln n}\left(1 + O\left(\frac{1}{\ln\ln n}\right)\right).$$

Finally, we get

$$I_2(n) \leq \gamma_1\gamma_2 \left\{\left(1 + O\left(\frac{1}{\ln\ln n}\right)\right)\left(1 + O\left(\frac{\ln\ln n}{\ln n}\right)\right) - \left(1 + O\left((\ln n)^{\beta-1}\right)\right)\right\} \xrightarrow[n\to\infty]{} 0.$$

To complete the proof of (16), we have to verify (17).

Proposition 1.16. *There exists a continuous strictly increasing distribution function H such that for every $x \in \mathbb{R}$*

$$P\left(a_n(\widetilde{M}_n - b_n) \leq x\right) \xrightarrow[n\to\infty]{} H(x),$$

where

$$a_n = \sqrt{2\ln n}, \qquad b_n = \sqrt{2\ln n} - \frac{\ln\ln n + \ln(4\pi)}{2\sqrt{2\ln n}}, \quad n \in \mathbb{N}.$$

For each $c \in (0,1)$, let $x = x(c)$ be such that $H(x) = c$ and let $y_n(c) = x(c)/a_n + b_n$.
If $P(\widetilde{M}_n \leq w_n(c)) \to c \in (0,1)$, then $|w_n(c) - y_n(c)| = o(1/\sqrt{\ln n})$ and $\{w_n(c)\}$ satisfies (17).

Proof. The proof of the first part of the proposition coincides, in fact, with a part of the proof of [20], Theorem 6.5.1, (see also [23]). But these results deal basically with *partial maxima of stationary sequences*, while here we have a complicated covariance structure of a random field. Therefore, we provide a complete argument.

We may and do assume that $\boldsymbol{\psi}(n)^* = n$. By the definition, $\widetilde{M}_n$ is equal in law to $\sqrt{1-\rho_n}\widehat{M}_n + \sqrt{\rho_n}\zeta$, where $\widehat{M}_n$ is the maximum of a sequence of n independent standard normal random variables and ζ is standard normal independent of $\widehat{M}_n$. We thus obtain

$$\begin{aligned} P\left(a_n(\widetilde{M}_n - b_n) \leq x\right) &= P(\sqrt{1-\rho_n}\widehat{M}_n + \sqrt{\rho_n}\zeta \leq x/a_n + b_n) \\ &= \int_{-\infty}^{\infty} P\left(\widehat{M}_n \leq (1-\rho_n)^{-1/2}(x/a_n + b_n - \sqrt{\rho_n}z)\right)\varphi(z)\, dz \\ &= \int_{-\infty}^{\infty} \left(\Phi\left((1-\rho_n)^{-1/2}(x/a_n + b_n - \sqrt{\rho_n}z)\right)\right)^n \varphi(z)\, dz \\ &\longrightarrow \int_{-\infty}^{+\infty} \exp\left(-\exp(-x - \gamma_1\gamma_2 + \sqrt{2\gamma_1\gamma_2}z)\right)\varphi(z)\, dz =: H(x), \end{aligned}$$

because (see the proof of [20], Theorem 6.5.1)

$$\left(1 - \frac{\gamma_1\gamma_2}{\ln n}\right)^{-1/2}\left(x/a_n + b_n - \sqrt{\frac{\gamma_1\gamma_2}{\ln n}}z\right) = \frac{x + \gamma_1\gamma_2 - \sqrt{2\gamma_1\gamma_2}z}{a_n} + b_n + o\left((a_n)^{-1}\right).$$

Assume that $P(\widetilde{M}_n \le w_n(c)) \to c \in (0,1)$. Consider levels $y_n(c) = x(c)/a_n + b_n$. Let $x' < x(c) < x''$. We have eventually

$$\frac{x' - x(c)}{a_n} = x'/a_n + b_n - y_n(c) \le w_n(c) - y_n(c) \le x''/a_n + b_n - y_n(c) = \frac{x'' - x(c)}{a_n}.$$

Because x' and x'' can be chosen arbitrarily close,

$$\left|w_n(c) - y_n(c)\right| = o\big((a_n)^{-1}\big).$$

This clearly implies (17). □

Given (16), it is not difficult to prove that $\Phi(x)$ *is not* a phantom distribution function for $\{X_{(i,j)}\}$ along $\boldsymbol{\psi}$. Because $H(x)$ does not coincide with the Gumbel standardized distribution H_0, we have $H_0(x_0) \neq H(x_0)$ for some x_0. And we have proved that $P(\widetilde{M}_n \le x_0/a_n + b_n) \to H(x_0)$, while we know that $\Phi(x_0/a_n + b_n)^n \to H_0(x_0)$.

1.5. Extremal indices

We will use the results of the previous sections to provide a complete theory of the extremal index for maxima of random fields in the rectangular setting. Recall that F stands for the marginal distribution function of $\{X_{\mathbf{n}}\}$.

Definition 1.17. We say that $\theta \in (0,1]$ is *the extremal index* for $\{X_{\mathbf{n}}\}$, if the function G given by $G(x) := P(X_{\mathbf{0}} \le x)^\theta$, $x \in \mathbb{R}$, is a phantom distribution function for $\{X_{\mathbf{n}}\}$.

If $G(x) := P(X_{\mathbf{0}} \le x)^\theta$, for some $\theta \in (0,1]$, is a $\boldsymbol{\psi}$-directional (resp. sectorial) phantom distribution function for $\{X_{\mathbf{n}}\}$, then we say that θ is the $\boldsymbol{\psi}$-directional (resp. sectorial) extremal index for $\{X_{\mathbf{n}}\}$.

Remark 1.18. This definition of the (global) extremal index is taken from [17]. We note that a "more classical" definition of the (global) extremal index for random fields was proposed in [4], see also [28] and [10]. These papers, however, did not bring conclusive results. For instance, the formula for calculating the extremal index proposed in [10] does not work for a simple 1-dependent random field given in [17], Example 5.5.

Examples of calculation of the global extremal index for a variety of random fields on the lattice $\mathbb{Z}^d$ can be found in [2] (moving averages and moving maxima), [17] (models with local dependence) and [29] (regularly varying random fields). Some related work for Gaussian random fields is given in [22].

If we know a lot about the structure of a model it is possible to define other extremal-index-like notions. As examples can serve papers [5] and [8]. We do not know, however, whether the techniques developed here can be adopted to the setting of these papers.

Remark 1.19. As the example provided in Section 1.4 shows, the notion of the sectorial extremal index is essentially weaker than the notion of the (global) extremal index. Indeed, the random field considered in this example has the *sectorial extremal index* $\theta = 1$, while the (global) extremal index does not exist.

Within the theory of phantom distribution functions, we have nice criteria for the existence of the extremal index and the sectorial extremal index.

Theorem 1.20. *Let $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ be a stationary random field. Then $\{X_{\mathbf{n}}\}$ has the extremal index $\theta \in (0, 1]$ if, and only if, there exist $\gamma_{or}, \gamma_{in} \in (0, 1)$ and a* strongly monotone *field of levels $\{v_{\mathbf{n}}; \mathbf{n} \in \mathbb{N}^d\}$ such that*

$$P(M_{\mathbf{n}} \le v_{\mathbf{n}}) \to \gamma_{or}, \qquad F(v_{\mathbf{n}})^{\mathbf{n}^*} \to \gamma_{in} \quad \text{as } \mathbf{n} \to \boldsymbol{\infty}, \qquad \theta = \frac{\ln \gamma_{or}}{\ln \gamma_{in}}, \tag{19}$$

and for every monotone curve $\boldsymbol{\psi}$ and every $T > 0$ Condition $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ *holds.*

Theorem 1.21. *Let $\{X_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ be a stationary random field. Then $\{X_{\mathbf{n}}\}$ has the $\boldsymbol{\psi}$-directional extremal index $\theta \in (0, 1]$ if, and only if, there exist $\gamma_{or}, \gamma_{in} \in (0, 1)$ and a* non-decreasing *sequence of levels $\{v_{\boldsymbol{\psi}(n)}\}$, $n \in \mathbb{N}$, such that*

$$P(M_{\boldsymbol{\psi}(n)} \le v_{\boldsymbol{\psi}(n)}) \xrightarrow[n\to\infty]{} \gamma_{or}, \qquad F(v_{\boldsymbol{\psi}(n)})^{n^d} \xrightarrow[n\to\infty]{} \gamma_{in}, \qquad \theta = \frac{\ln \gamma_{or}}{\ln \gamma_{in}}, \tag{20}$$

and for every $T > 0$ Condition $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ holds.

Remark 1.22. Upon substitution $\boldsymbol{\psi} = \boldsymbol{\Delta}$, Theorem 1.21 gives criteria for the existence of the sectorial extremal index $\theta \in (0, 1]$.

By analogy to the case of sequences we can introduce also the value of the extremal index 0 to describe the situation when the maxima of the random field increase essentially slower than in the independent case. It is reasonable here to comply with formulas (19) and (20).

Definition 1.23. Suppose that $\{X_{\mathbf{n}}\}$ admits a continuous phantom distribution function (resp. $\boldsymbol{\psi}$-directional continuous phantom distribution function). If there exists a strongly monotone field $\{v_{\mathbf{n}}\}$ (resp. non-decreasing sequence $\{v_{\boldsymbol{\psi}(n)}\}$) of levels such that $\gamma_{or} \in (0, 1)$ *and* $\gamma_{in} = 0$ or $\gamma_{in} \in (0, 1)$ *and* $\gamma_{or} = 1$, then we say that the extremal index (resp. the $\boldsymbol{\psi}$-directional extremal index) of $\{X_{\mathbf{n}}\}$ is 0.

Remark 1.24. Similarly as in [7], Theorem 3, one can show that the extremal index 0 reflects tail properties of the phantom distribution function and the marginal distribution and therefore does not depend on the particular choice of the sequence of levels used in calculation of γ_{or} and γ_{in}.

The following example modifies the construction given in [7], Theorem 4, and shows that the above definition is not empty. It also demonstrates the efficiency of Theorem 1.13.

Example 1.25. We shall construct $\{X_{\mathbf{n}}\}$ as a mixture of i.i.d. random fields on $\mathbb{Z}^2$. Let $\Omega = \mathbb{N} \times \mathbb{R}^{\mathbb{N}^2}$ and let $\Pi((k, (x_{\mathbf{j}})_{\mathbf{j}\in\mathbb{N}^2}) = k$, $X_{\mathbf{n}}((k, (x_{\mathbf{j}})_{\mathbf{j}\in\mathbb{N}^2})) = x_{\mathbf{n}}$, for $\mathbf{n} \in \mathbb{N}^2$. Choose a strictly increasing sequence $\{u_n\} \in \mathbb{R}$ and for $k \in \mathbb{N}$ define a purely jump distribution function F_k by

$$F_k(x) = \begin{cases} 0 & \text{if } x < u_{k^2}, \\ 1 - \dfrac{1}{n} & \text{if } u_n \le x < u_{n+1}, n \ge k^2. \end{cases}$$

Now set

$$\mathrm{P}(\Pi = k) = \frac{1}{k(k+1)}, \quad k = 1, 2, \ldots,$$

and define the conditional distribution of $\{X_{\mathbf{n}}\}$ given $\Pi = k$ as the product $\mu_k^{\otimes \mathbb{N}^2}$, where the probability measure μ_k corresponds to the distribution function F_k.

Given $\{u_n\}$ one can define a strongly monotone field of levels by the formula

$$v_{(m,n)} = u_{m \cdot n}, \quad m, n \geq 1.$$

For $q_1, q_2 > 0$ and a monotone curve $\boldsymbol{\psi}$ we have

$$P(M_{(\lfloor q_1 \psi_1(n) \rfloor, \lfloor q_2 \psi_2(n) \rfloor)} \leq v_{\boldsymbol{\psi}(n)}) = \sum_{k=1}^{\infty} P(\Pi = k) F_k^{\lfloor q_1 \psi_1(n) \rfloor \cdot \lfloor q_2 \psi_2(n) \rfloor}(u_{\boldsymbol{\psi}(n)^*}),$$

and if $\boldsymbol{\psi}(n)^* \geq k^2$

$$F_k^{\lfloor q_1 \psi_1(n) \rfloor \cdot \lfloor q_2 \psi_2(n) \rfloor}(u_{\boldsymbol{\psi}(n)^*}) = \left(1 - \frac{1}{\boldsymbol{\psi}(n)^*}\right)^{\lfloor q_1 \psi_1(n) \rfloor \cdot \lfloor q_2 \psi_2(n) \rfloor} \xrightarrow[n \to \infty]{} e^{-q_1 q_2}.$$

Therefore

$$P(M_{(\lfloor q_1 \psi_1(n) \rfloor, \lfloor q_2 \psi_2(n) \rfloor)} \leq v_{\boldsymbol{\psi}(n)}) \xrightarrow[n \to \infty]{} e^{-q_1 q_2}, \quad q_1, q_2 > 0. \tag{21}$$

By Theorem 1.13 $\{X_{\mathbf{n}}\}$ admits a continuous phantom distribution function *along every monotone curve*, hence by Theorem 1.12 Condition $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ holds for every monotone curve $\boldsymbol{\psi}$ and every $T > 0$. And relation (4) in Theorem 1.4 is given by (21) with $q_1 = q_2 = 1$. It follows that $\{X_{\mathbf{n}}\}$ admits a *global* continuous phantom distribution function.

We shall show that the global extremal index of $\{X_{\mathbf{n}}\}$ is 0. We have $\gamma_{or} = e^{-1}$, so it is enough to show that $\gamma_{in} = 0$, or that

$$\boldsymbol{\psi}(n)^* P(X_{(1,1)} > v_{\boldsymbol{\psi}(n)}) \xrightarrow[n \to \infty]{} \infty,$$

along any monotone curve $\boldsymbol{\psi}$. We have, indeed,

$$\begin{aligned}
\boldsymbol{\psi}(n)^* P(X_{(1,1)} > v_{\boldsymbol{\psi}(n)}) &= \sum_{k=1}^{\infty} P(\Pi = k) \boldsymbol{\psi}(n)^* \big(1 - F_k(v_{\boldsymbol{\psi}(n)})\big) \\
&= \sum_{k=1}^{\infty} P(\Pi = k) \boldsymbol{\psi}(n)^* \Big(I\big(k > \sqrt{\boldsymbol{\psi}(n)^*}\big) + \big(1/\boldsymbol{\psi}(n)^*\big) I\big(k \leq \sqrt{\boldsymbol{\psi}(n)^*}\big)\Big) \\
&= \boldsymbol{\psi}(n)^* \left(\sum_{k=\lfloor \sqrt{\boldsymbol{\psi}(n)^*} \rfloor + 1}^{\infty} \frac{1}{k(k+1)} \right) + \sum_{k=1}^{\lfloor \sqrt{\boldsymbol{\psi}(n)^*} \rfloor} \frac{1}{k(k+1)} \\
&\geq \frac{\boldsymbol{\psi}(n)^*}{\lfloor \sqrt{\boldsymbol{\psi}(n)^*} \rfloor + 1} \xrightarrow[n \to \infty]{} \infty.
\end{aligned}$$

2. Auxiliary results and proofs of theorems and propositions

2.1. Proof of Proposition 1.3

Clearly, if G is a phantom distribution function for $\{X_{\mathbf{n}}\}$, then it is a phantom distribution function for $\{X_{\mathbf{n}}\}$ along every monotone curve. So assume the latter property and suppose that G *does not* satisfy

(1). It follows that there exists a number $\varepsilon > 0$, a monotone sequence $\mathbf{m}(n) \to \infty$ and a sequence $\{x_n\}$ such that

$$\left|P(M_{\mathbf{m}(n)} \le x_n) - G(x_n)^{\mathbf{m}(n)^*}\right| > \varepsilon, \quad n \in \mathbb{N}.$$

The point is that $\mathbf{m}(n)$ need not satisfy (2) and so it is not a monotone curve according to our definition. But we can always find a monotone curve $\boldsymbol{\psi}(n)$ such that $\mathbf{m}(n) = \boldsymbol{\psi}(m_n)$ for some increasing sequence $\{m_n\}$. Indeed, let us begin with $\mathbf{m}(1)$ and connect it with $\mathbf{m}(2)$ by a sequence of points that in each step increases only by one in one coordinate. Then proceed the same way with points $\mathbf{m}(2)$ and $\mathbf{m}(3)$, etc. The obtained map $\boldsymbol{\psi}(\cdot) : \mathbb{N} \to \mathbb{N}^d$ satisfies (2). And G cannot be a phantom distribution function for $\boldsymbol{\psi}$.

2.2. The mixing-like condition

Let $\beta_T^{\boldsymbol{\psi}}(n,k)$ for $n, k \in \mathbb{N}$, $k \ge 2$, be defined as

$$\beta_T^{\boldsymbol{\psi}}(n,k) := \sup_{\mathbf{p}(1)+\cdots+\mathbf{p}(k) \le T\boldsymbol{\psi}(n)} \Bigg| P(M_{\mathbf{p}(1)+\cdots+\mathbf{p}(k)} \le v_{\boldsymbol{\psi}(n)}) - \prod_{\mathbf{i} \in \{1,\ldots,k\}^d} P(M_{(p_1(i_1),\ldots,p_d(i_d))} \le v_{\boldsymbol{\psi}(n)}) \Bigg|,$$

where $\mathbf{p}(1), \mathbf{p}(2), \ldots, \mathbf{p}(k)$ take values in $\mathbb{N}_0^d$. Then $\beta_T^{\boldsymbol{\psi}}(n,2) = \beta_T^{\boldsymbol{\psi}}(n)$ is the term appearing in the definition of Condition $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$. We are able to control the growth of $\beta_T^{\boldsymbol{\psi}}(n,k)$.

Lemma 2.1. *The following inequality holds.*

$$\beta_T^{\boldsymbol{\psi}}(n,k) \le k^d \beta_T^{\boldsymbol{\psi}}(n), \quad k \ge 2. \tag{22}$$

Proof. Let us take $k \ge 3$ and $\mathbf{p}(1), \mathbf{p}(2), \ldots, \mathbf{p}(k) \in \mathbb{N}_0^d$ satisfying the assumption $\mathbf{p}(1) + \mathbf{p}(2) + \cdots + \mathbf{p}(k) \le T\boldsymbol{\psi}(n)$. Define $\mathbf{q}(1) := \mathbf{p}(1)$, $\mathbf{q}(2) := \mathbf{p}(2)$, $\ldots$, $\mathbf{q}(k-2) := \mathbf{p}(k-2)$, $\mathbf{q}(k-1) := \mathbf{p}(k-1) + \mathbf{p}(k)$, so that

$$\mathbf{q}(1) + \mathbf{2}(2) + \cdots + \mathbf{q}(k-1) = \mathbf{p}(1) + \mathbf{p}(2) + \cdots + \mathbf{p}(k) \le T\boldsymbol{\psi}(n).$$

Then we obtain the following estimate.

$$\begin{aligned}
&\Bigg| P(M_{\mathbf{p}(1)+\cdots+\mathbf{p}(k)} \le v_{\boldsymbol{\psi}(n)}) - \prod_{\mathbf{j} \in \{1,2,\ldots,k\}^d} P(M_{(p_1(j_1),\ldots,p_d(j_d))} \le v_{\boldsymbol{\psi}(n)}) \Bigg| \\
&\quad \le \Bigg| P(M_{\mathbf{q}(1)+\mathbf{q}(2)+\cdots+\mathbf{q}(k-1)} \le v_{\boldsymbol{\psi}(n)}) - \prod_{\mathbf{i} \in \{1,2,\ldots,k-1\}^d} P(M_{(q_1(i_1),\ldots,q_d(i_d))} \le v_{\boldsymbol{\psi}(n)}) \Bigg| \\
&\qquad + \Bigg| \prod_{\mathbf{i} \in \{1,2,\ldots,k-1\}^d} P(M_{(q_1(i_1),\ldots,q_d(i_d))} \le v_{\boldsymbol{\psi}(n)}) - \prod_{\mathbf{j} \in \{1,2,\ldots,k\}^d} P(M_{(p_1(j_1),\ldots,p_d(j_d))} \le v_{\boldsymbol{\psi}(n)}) \Bigg| \\
&\quad \le \beta_T^{\boldsymbol{\psi}}(n,k-1) + |\Pi_1 - \Pi_2|.
\end{aligned}$$

Let $\mathcal{D}_k(r)$ consists of all $\mathbf{i} = (i_1, i_2, \ldots, i_d) \in \{1, 2, \ldots, k-1\}^d$ such that the number of s with the property that $i_s = k-1$ equals r. Next, for $\mathbf{i} \in \mathcal{D}_k(r)$ define $\mathcal{E}_k(r, \mathbf{i})$ as the set of $\mathbf{j} = (j_1, j_2, \ldots, j_d) \in \{1, 2, \ldots, k\}^d$ such that $j_s = i_s$, if $i_s \neq k-1$ and $j_s \in \{k-1, k\}$, if $i_s = k-1$. Let us observe that for $\mathbf{i} \in \mathcal{D}_k(0)$ we have $\mathcal{E}_k(0, \mathbf{i}) = \{\mathbf{i}\}$ and that for each $\mathbf{i} = (i_1, i_2, \ldots, i_d) \in \mathcal{D}_k(r)$

$$\left| P(M_{(q_1(i_1),\ldots,q_d(i_d))} \leq v_{\boldsymbol{\psi}(n)}) - \prod_{\mathbf{j} \in \mathcal{E}_k(r,\mathbf{i})} P(M_{(p_1(j_1),\ldots,p_d(j_d))} \leq v_{\boldsymbol{\psi}(n)}) \right| \leq \beta_T^{\boldsymbol{\psi}}(n).$$

Taking into account these relations and using the obvious expansions:

$$\Pi_1 = \prod_{r=0}^{d} \prod_{\mathbf{i} \in \mathcal{D}_k(r)} P(M_{(q_1(i_1),\ldots,q_d(i_d))} \leq v_{\boldsymbol{\psi}(n)}),$$

$$\Pi_2 = \prod_{r=0}^{d} \prod_{\mathbf{i} \in \mathcal{D}_k(r)} \prod_{\mathbf{j} \in \mathcal{E}_k(r,\mathbf{i})} P(M_{(p_1(j_1),\ldots,p_d(j_d))} \leq v_{\boldsymbol{\psi}(n)}),$$

we obtain that

$$\begin{aligned} |\Pi_1 - \Pi_2| &\leq \beta_T^{\boldsymbol{\psi}}(n) \sum_{r=1}^{d} \#\mathcal{D}_k(r) = \big((k-1)^d - (k-2)^d\big) \beta_T^{\boldsymbol{\psi}}(n) \\ &= \left(\sum_{r=0}^{d-1} (k-1)^{d-1-r} (k-2)^r \right) \beta_T^{\boldsymbol{\psi}}(n) \leq d(k-1)^{d-1} \beta_T^{\boldsymbol{\psi}}(n). \end{aligned}$$

It follows that for $k \geq 3$

$$\beta_T^{\boldsymbol{\psi}}(n, k) \leq \beta_T^{\boldsymbol{\psi}}(n, k-1) + d(k-1)^{d-1} \beta_T^{\boldsymbol{\psi}}(n).$$

Iterating the above relation, we get (22). □

Lemma 2.2. *Let* $\mathbf{N}(n) = (N_1(n), N_2(n), \ldots, N_d(n)) \in \mathbb{N}^d$, $\mathbf{N}(n) \to \boldsymbol{\infty}$. *Suppose that* $q_1, q_2, \ldots, q_d \in \mathbb{N}$ *are such that for some* $T_0 > 0$, $(q_1 N_1(n), q_2 N_2(n), \ldots, q_d N_d(n)) \leq T_0 \boldsymbol{\psi}(n)$, $n \in \mathbb{N}$. *If Condition* $\mathbf{B}_{T_0}^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ *holds, then we have, as* $n \to \infty$,

$$\begin{aligned} &P(M_{(q_1 N_1(n), q_2 N_2(n), \ldots, q_d N_d(n))} \leq v_{\boldsymbol{\psi}(n)}) \\ &\quad = P(M_{(N_1(n), N_2(n), \ldots, N_d(n))} \leq v_{\boldsymbol{\psi}(n)})^{q_1 q_2 \cdots q_d} + o(1). \end{aligned} \tag{23}$$

Proof. Fix $n \in \mathbb{N}$. We can represent $(q_1 N_1(n), q_2 N_2(n), \ldots, q_d N_d(n))$ as the sum of $s = q_1 + q_2 + \cdots + q_d$ specific components, namely q_1 components $(N_1(n), 0, \ldots, 0)$, q_2 components $(0, N_2(n), 0, \ldots, 0)$, etc. Keeping the order, let us denote these components by $\mathbf{p}(1), \mathbf{p}(2), \ldots, \mathbf{p}(s)$. By Lemma 2.1, as $n \to \infty$,

$$P(M_{(q_1 N_1(n), q_2 N_2(n), \ldots, q_d N_d(n))} \leq v_{\boldsymbol{\psi}(n)}) - \prod_{\mathbf{i} \in \{1,\ldots,s\}^d} P(M_{(p_1(i_1),\ldots,p_d(i_d))} \leq v_{\boldsymbol{\psi}(n)}) \to 0.$$

It remains to identify

$$\prod_{\mathbf{i} \in \{1,\ldots,s\}^d} P(M_{(p_1(i_1),\ldots,p_d(i_d))} \leq v_{\boldsymbol{\psi}(n)})$$

with

$$P(M_{(N_1(n),N_2(n),\ldots,N_d(n))} \leq v_{\boldsymbol{\psi}(n)})^{q_1 q_2 \ldots q_d}.$$

Consider a typical term $\mathbf{P_i} = P(M_{(p_1(i_1),\ldots,p_d(i_d))} \leq v_{\boldsymbol{\psi}(n)})$, $\mathbf{i} \in \{1, 2, \ldots, s\}^d$. If some coordinate $p_j(i_k)$ is 0, then $\mathbf{P_i} = 1$, for we have $\max \varnothing = -\infty$ by the well-known convention. If all coordinates are non-zero, then $p_1(i_1) = N_1(n)$, $p_2(i_2) = N_2(n), \ldots, p_d(i_d) = N_d(n)$, $\mathbf{P_i} = P(M_{(N_1(n),\ldots,N_d(n))} \leq v_{\boldsymbol{\psi}(n)})$ and this can be achieved in $q_1 q_2 \ldots q_d$ ways. □

Corollary 2.3. *In assumptions of Lemma 2.2, if Condition $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ is satisfied for every $T > 0$, then (23) holds for any $q_1, q_2, \ldots, q_d \in \mathbb{N}$.*

Corollary 2.4. *Suppose that $\{\mathbf{N}(n)\} \subset \mathbb{N}^d$, $\mathbf{N}(n) \to \boldsymbol{\infty}$, $\{\mathbf{k}(n)\} \subset \mathbb{N}^d$ and for some $T_0 > 0$*

$$\big(k_1(n)\mathbf{N}_1(n), k_2(n)\mathbf{N}_2(n), \ldots, k_d(n)\mathbf{N}_d(n)\big) \leq T_0 \boldsymbol{\psi}(n), \quad n \in \mathbb{N}.$$

If Condition $\mathbf{B}_{T_0}^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ holds and $(k_1(n) + \cdots + k_d(n))^d \beta_{T_0}^{\boldsymbol{\psi}}(n) \to 0$, as $n \to \infty$, then

$$P(M_{(k_1(n)\mathbf{N}_1(n),k_2(n)\mathbf{N}_2(n),\ldots,k_d(n)\mathbf{N}_d(n))} \leq v_{\boldsymbol{\psi}(n)}) = P(M_{\mathbf{N}(n)} \leq v_{\boldsymbol{\psi}(n)})^{\mathbf{k}(n)^*} + o(1).$$

Proof. Proof follows by a careful inspection of the proof of Lemma 2.2. □

Recall that $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}^1$. We extend this functions to vectors $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ in a natural way:

$$\lfloor \mathbf{x} \rfloor := \big(\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \ldots, \lfloor x_d \rfloor\big).$$

The next fact is of independent interest and therefore for the future purposes we state it as a theorem.

Theorem 2.5. *Let $\{\mathbf{N}(n)\} \subset \mathbb{N}^d$, $\mathbf{N}(n) \to \boldsymbol{\infty}$ and satisfies $\mathbf{N}(n) \leq T_0 \boldsymbol{\psi}(n)$, $n \in \mathbb{N}$, for some $T_0 > 0$. Let Condition $\mathbf{B}_{T_0(1+\varepsilon)}^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ holds, for some $\varepsilon > 0$.*

Suppose that $k_n \to \infty$ in such a way that as $n \to \infty$ both $k_n^d \beta_{T_0}^{\boldsymbol{\psi}}(n) \to 0$ and $k_n = o(N_i(n))$, $i = 1, 2, \ldots, d$.

Then, as $n \to \infty$,

$$P(M_{\mathbf{N}(n)} \leq v_{\boldsymbol{\psi}(n)}) = P(M_{(\lfloor N_1(n)/k_n \rfloor, \lfloor N_2(n)/k_n \rfloor, \ldots, \lfloor N_d(n)/k_n \rfloor)} \leq v_{\boldsymbol{\psi}(n)})^{k_n^d} + o(1) \tag{24}$$

$$= \exp\big(-k_n^d P(M_{(\lfloor N_1(n)/k_n \rfloor, \lfloor N_2(n)/k_n \rfloor, \ldots, \lfloor N_d(n)/k_n \rfloor)} > v_{\boldsymbol{\psi}(n)})\big) + o(1). \tag{25}$$

Proof. From Corollary 2.4, we obtain that

$$P(M_{\mathbf{N}(n)} \leq v_{\boldsymbol{\psi}(n)}) \leq P(M_{(k_n \lfloor N_1(n)/k_n \rfloor, k_n \lfloor N_2(n)/k_n \rfloor, \ldots, k_n \lfloor N_d(n)/k_n \rfloor)} \leq v_{\boldsymbol{\psi}(n)})$$

$$= P(M_{(\lfloor N_1(n)/k_n \rfloor, \lfloor N_2(n)/k_n \rfloor, \ldots, \lfloor N_d(n)/k_n \rfloor)} \leq v_{\boldsymbol{\psi}(n)})^{k_n^d} + o(1) =: V_n.$$

To get the other bound, for each $n \in \mathbb{N}$ find numbers $l_{n,1}, l_{n,2}, \ldots, l_{n,d}$ in $\mathbb{N}$ such that

$$(k_n + l_{n,i} - 1)\lfloor N_i(n)/k_n \rfloor \leq N_i(n) < (k_n + l_{n,i})\lfloor N_i(n)/k_n \rfloor, \quad i = 1, 2, \ldots, d.$$

In other words,

$$l_{n,i} = \left\lfloor \frac{N_i(n) - k_n \lfloor N_i(n)/k_n \rfloor}{\lfloor N_i(n)/k_n \rfloor} \right\rfloor + 1,$$

what implies

$$l_{n,i} = o(k_n), \quad i = 1, 2, \ldots, d. \tag{26}$$

This in turn implies that for large n

$$(k_n + l_{n,i}) \lfloor N_i(n)/k_n \rfloor \le T_0(1 + \varepsilon_0)\boldsymbol{\psi}(n), \quad i = 1, 2, \ldots, d.$$

Therefore, we can again apply Corollary 2.4.

$$\begin{aligned}
&P(M_{\mathbf{N}(n)} \le v_{\boldsymbol{\psi}(n)}) \\
&\quad \ge P(M_{((k_n+l_{n,1})\lfloor N_1(n)/k_n\rfloor,(k_n+l_{n,2})\lfloor N_2(n)/k_n\rfloor,\ldots,(k_n+l_{n,d})\lfloor N_d(n)/k_n\rfloor)} \le v_{\boldsymbol{\psi}(n)}) \\
&\quad = P(M_{(\lfloor N_1(n)/k_n\rfloor,\lfloor N_2(n)/k_n\rfloor,\ldots,\lfloor N_d(n)/k_n\rfloor)} \le v_{\boldsymbol{\psi}(n)})^{\prod_{i=1}^d (k_n+l_{n,i})} + o(1) =: U_n.
\end{aligned}$$

From (26), we get $U_n - V_n = o(1)$ and so (24) holds.

Relation (25) is equivalent to (24), since $(a_m)^m - \exp(-m(1-a_m)) \to 0$, as $m \to \infty$, for arbitrary $\{a_m\} \subset [0, 1]$. □

Proposition 2.6. *Let $\{\mathbf{R}(n)\} \subset \mathbb{R}_+^d$, $\mathbf{R}(n) \to \infty$ and $q_1, q_2, \ldots, q_d \in \mathbb{N}$. Suppose that for some $T_0 > 0$*

$$\big(q_1 R_1(n), q_1 R_2(n), \ldots, q_d R_d(n)\big) \le T_0 \boldsymbol{\psi}(n), \quad n \in \mathbb{N}.$$

If for some $\varepsilon > 0$ Condition $\mathbf{B}_{T_0(1+\varepsilon)}^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ holds, then, as $n \to \infty$,

$$\begin{aligned}
&P(M_{(\lfloor q_1 R_1(n)\rfloor,\lfloor q_2 R_2(n)\rfloor,\ldots,\lfloor q_d R_d(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)}) \\
&\quad = P(M_{(\lfloor R_1(n)\rfloor,\lfloor R_2(n)\rfloor,\ldots,\lfloor R_d(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)})^{q_1 q_2 \cdots q_d} + o(1).
\end{aligned} \tag{27}$$

Proof. Let us notice first that

$$\begin{aligned}
&P(M_{(\lfloor q_1 R_1(n)\rfloor,\lfloor q_2 R_2(n)\rfloor,\ldots,\lfloor q_d R_d(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)}) \\
&\quad \le P(M_{(q_1\lfloor R_1(n)\rfloor,q_2\lfloor R_2(n)\rfloor,\ldots,q_d\lfloor R_d(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)}) \\
&\quad = P(M_{(\lfloor R_1(n)\rfloor,\lfloor R_2(n)\rfloor,\ldots,\lfloor R_d(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)})^{q_1 q_2 \cdots q_d} + o(1),
\end{aligned}$$

where the last equality holds by Lemma 2.2. Therefore, it is enough to find expressions U_n and V_n such that $V_n - U_n = o(1)$ and $U_n \le P(M_{(\lfloor q_1 R_1(n)\rfloor,\lfloor q_2 R_2(n)\rfloor,\ldots,\lfloor q_d R_d(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)})$, while $V_n \ge P(M_{(q_1\lfloor R_1(n)\rfloor,q_2\lfloor R_2(n)\rfloor,\ldots,q_d\lfloor R_d(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)})$.

Let $r_n \to \infty$ in such a way that $r_n^d \beta_{T_0(1+\varepsilon)}^{\boldsymbol{\psi}}(n) \to 0$ and $r_n = o(R_i(n))$, $i = 1, 2, \ldots, d$. Then for n large enough we have $q_i \lfloor R_i(n)\rfloor \ge (r_n - 1)\lfloor q_i R_i(n)/r_n\rfloor$, $i = 1, 2, \ldots, d$, and therefore by Corollary 2.4

$$\begin{aligned}
&P(M_{(q_1\lfloor R_1(n)\rfloor,q_2\lfloor R_2(n)\rfloor,\ldots,q_d\lfloor R_d(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)}) \\
&\quad \le P(M_{((r_n-1)\lfloor q_1 R_1(n)/r_n\rfloor,(r_n-1)\lfloor q_2 R_2(n)/r_n\rfloor,\ldots,(r_n-1)\lfloor q_d R_d(n)/r_n\rfloor)} \le v_{\boldsymbol{\psi}(n)}) \\
&\quad = P(M_{((\lfloor q_1 R_1(n)/r_n\rfloor,\lfloor q_2 R_2(n)/r_n\rfloor,\ldots,\lfloor q_d R_d(n)/r_n\rfloor)} \le v_{\boldsymbol{\psi}(n)})^{(r_n-1)^d} + o(1) := V_n.
\end{aligned}$$

In order to find U_n we shall proceed like in the proof of Theorem 2.5. Let $s_{n,i} \in \mathbb{N}$, $i = 1, 2, \ldots, d$, be such that

$$(r_n + s_{n,i} - 1)\lfloor q_i R_i(n)/r_n \rfloor \le \lfloor q_i R_i(n) \rfloor < (r_n + s_{n,i})\lfloor q_i R_i(n)/r_n \rfloor, \quad i = 1, 2, \ldots, d,$$

or, equivalently,

$$s_{n,i} = \left\lfloor \frac{\lfloor q_i R_i(n) \rfloor - r_n \lfloor q_i R_i(n)/r_n \rfloor}{\lfloor q_i R_i(n)/r_n \rfloor} \right\rfloor + 1.$$

Applying Corollary 2.4, we get

$$\begin{aligned} &P(M_{(\lfloor q_1 R_1(n) \rfloor, \lfloor q_2 R_2(n) \rfloor, \ldots, \lfloor q_d R_d(n) \rfloor)} \le v_{\boldsymbol{\psi}(n)}) \\ &\quad \ge P(M_{((r_n+s_{n,1})\lfloor q_1 R_1(n)/r_n \rfloor, (r_n+s_{n,2})\lfloor q_2 R_2(n)/r_n \rfloor, \ldots, (r_n+s_{n,d})\lfloor q_d R_d(n)/r_n \rfloor)} \le v_{\boldsymbol{\psi}(n)}) \\ &\quad = P(M_{(\lfloor q_1 R_1(n)/r_n \rfloor, \lfloor q_2 R_2(n)/r_n \rfloor, \ldots, \lfloor q_d R_d(n)/r_n \rfloor)} \le v_{\boldsymbol{\psi}(n)})^{\prod_{i=1}^d (r_n+s_{n,i})} + o(1) =: U_n. \end{aligned}$$

Since $s_{n,i} = o(r_n)$, $i = 1, 2, \ldots, d$, we get $U_n - V_n = o(1)$ and so (27) holds. □

2.3. Fields of monotone levels

In this section, we shall examine previous results in conjunction with properties of the sequence of levels $\{v_{\boldsymbol{\psi}(n)}\}$.

Proposition 2.7. *If Condition $\mathbf{B}_1^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ holds for a monotone sequence of levels $\{v_{\boldsymbol{\psi}(n)}\}$ that satisfies (5), then*

$$v_{\boldsymbol{\psi}(n)} \nearrow F_*,$$

where $F_* = \sup\{x : F(x) < 1\}$.

Proof. If $v_{\boldsymbol{\psi}(n_0)} \ge F_*$ for some n_0, then $P(M_{\boldsymbol{\psi}(n)} \le v_{\boldsymbol{\psi}(n)}) = 1$ for all $n \ge n_0$ and (5) cannot hold. So assume that for some $\eta > 0$ $v_{\boldsymbol{\psi}(n)} \le (1-\eta)F_*$, $n \in \mathbb{N}$. Then for some $a > 0$ we have $P(X_{\mathbf{1}} \le v_{\boldsymbol{\psi}(n)}) \le 1 - a$, $n \in \mathbb{N}$.

Let $k_n \to \infty$ in such a way that $k_n^d \beta_1^{\boldsymbol{\psi}}(n) \to 0$. Then by (24)

$$\begin{aligned} P(M_{\boldsymbol{\psi}(n)} \le v_{\boldsymbol{\psi}(n)}) &\le P(M_{k_n \lfloor \boldsymbol{\psi}(n)/k_n \rfloor} \le v_{\boldsymbol{\psi}(n)}) = P(M_{\lfloor \boldsymbol{\psi}(n)/k_n \rfloor} \le v_{\boldsymbol{\psi}(n)})^{k_n^d} + o(1) \\ &\le P(X_{\mathbf{1}} \le v_{\boldsymbol{\psi}(n)})^{k_n^d} + o(1) \le (1-a)^{k_n^d} + o(1) \to 0. \end{aligned}$$

This again contradicts (5) and so $v_{\boldsymbol{\psi}(n)} \nearrow F_*$. □

Proposition 2.8. *Suppose (5) holds for some monotone sequence of levels $\{v_{\boldsymbol{\psi}(n)}\}$ and some $\gamma \in (0, 1)$ and Condition $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ holds for every $T > 0$.*

(i) *For every d-tuple $\mathbf{t} = (t_1, t_2, \ldots, t_d) \in (0, \infty)^d$,*

$$P(M_{(\lfloor t_1 \psi_1(n) \rfloor, \lfloor t_2 \psi_2(n) \rfloor, \ldots, \lfloor t_d \psi_d(n) \rfloor)} \le v_{\boldsymbol{\psi}(n)}) \xrightarrow[n\to\infty]{} \gamma^{t_1 t_2 \cdots t_d}. \tag{28}$$

(ii) *If a set* $A \subset [0,\infty)^d$ *does not contain any sequence* $\{\mathbf{t}(n)\}$ *with the property that* $t_{i_1}(n) \to \infty$ *and* $t_{i_2}(n) \to 0$ *for some* $i_1 \neq i_2 \in \{1,2,\dots,d\}$, *then*

$$\sup_{\mathbf{t}\in A}\left| P(M_{(\lfloor t_1\psi_1(n)\rfloor, \lfloor t_2\psi_2(n)\rfloor,\dots,\lfloor t_d\psi_d(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)}) - \gamma^{t_1t_2\cdots t_d}\right| \xrightarrow[n\to\infty]{} 0.$$

Proof. First consider $t_1 = 1/q_1, t_2 = 1/q_2, \dots, t_d = 1/q_d$, where $q_1, q_2, \dots, q_d \in \mathbb{N}$. Set $R_i(n) = \boldsymbol{\psi}_i(n)/q_i$. By Proposition 2.6,

$$\gamma \longleftarrow P(M_{\boldsymbol{\psi}(n)} \le v_{\boldsymbol{\psi}(n)}) = P(M_{(\lfloor \psi_1(n)/q_1\rfloor, \lfloor \psi_2(n)/q_2\rfloor,\dots,\lfloor \psi_d(n)/q_d\rfloor)} \le v_{\boldsymbol{\psi}(n)})^{q_1q_2\cdots q_d} + o(1),$$

hence

$$P(M_{(\lfloor \psi_1(n)/q_1\rfloor, \lfloor \psi_2(n)/q_2\rfloor,\dots,\lfloor \psi_d(n)/q_d\rfloor} \le v_{\boldsymbol{\psi}(n)}) \longrightarrow \gamma^{1/(q_1q_2\cdots q_d)} = \gamma^{t_1t_2\cdots t_d}.$$

By another application of Proposition 2.6, we have for $p_1, p_2, \dots, p_d \in \mathbb{N}$,

$$\begin{aligned}
&P(M_{(\lfloor p_1\psi_1(n)/q_1\rfloor, \lfloor p_2\psi_2(n)/q_2\rfloor,\dots,\lfloor p_d\psi_d(n)/q_d\rfloor)} \le v_{\boldsymbol{\psi}(n)}) \\
&\quad = P(M_{(\lfloor \psi_1(n)/q_1\rfloor, \lfloor \psi_2(n)/q_2\rfloor,\dots,\lfloor \psi_d(n)/q_d\rfloor)} \le v_{\boldsymbol{\psi}(n)})^{p_1p_2\cdots p_d} + o(1) \\
&\quad = \gamma^{\frac{p_1p_2\cdots p_d}{q_1q_2\cdots q_d}} + o(1) = \gamma^{t_1t_2\cdots t_d} + o(1),
\end{aligned}$$

if $t_1 = p_1/q_1, t_2 = p_2/q_2, \dots, t_d = p_d/q_d$.

We have proved (28) over the countable dense set $\mathbb{Q}_+^d$. The pointwise convergence over $\mathbb{R}_+^d$ follows then by the monotonicity of maps

$$\mathbf{s} \mapsto P(M_{(\lfloor s_1\psi_1(n)\rfloor, \lfloor s_2\psi_2(n)\rfloor,\dots,\lfloor s_d\psi_d(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)})$$

and the continuity of the limiting map $\mathbf{s} \mapsto \gamma^{s_1s_2\cdots s_d}$.

Part (ii) of Proposition 2.8 is, in fact, a general statement on convergence of monotone functions to a continuous function on $[0,\infty)^d$. For the sake of notational simplicity, we shall restrict our attention to the case $d = 2$. The general case can be proved analogously.

Let $A \subset [0,\infty)^2$ be a set fulfilling the assumptions of part (ii) of Proposition 2.8. Let $\{\mathbf{t}(n)\} \subset A$ be a sequence converging to some $\mathbf{t} \in [0,\infty]^2$. We have to prove that

$$P(M_{(\lfloor t_1(n)\psi_1(n)\rfloor, \lfloor t_2(n)\psi_2(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)}) - \gamma^{t_1(n)t_2(n)} \xrightarrow[n\to\infty]{} 0. \tag{29}$$

We shall consider the following three situations: (a) $\mathbf{t} \in (0,\infty)^2$; (b) $\max\{t_1,t_2\} < \infty$ and $\min\{t_1,t_2\} = 0$; (c) $\max\{t_1,t_2\} = \infty$ and $\min\{t_1,t_2\} > 0$. The case (d) $\max\{t_1,t_2\} = \infty$ and $\min\{t_1,t_2\} = 0$ is excluded by the assumptions on the set A.

Suppose that $\mathbf{t} \in (0,\infty)^2$. Then $(t_1 - \varepsilon, t_2 - \varepsilon) \le (t_1(n), t_2(n)) \le (t_1 + \varepsilon, t_2 + \varepsilon)$ for sufficiently large $n \in \mathbb{N}$ and every $\varepsilon > 0$. By the monotonicity and part (i), we get for small ε

$$\begin{aligned}
\gamma^{(t_1+\varepsilon)(t_2+\varepsilon)} &\longleftarrow P(M_{(\lfloor (t_1+\varepsilon)\psi_1(n)\rfloor, \lfloor (t_2+\varepsilon)\psi_2(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)}) \\
&\le P(M_{(\lfloor t_1(n)\psi_1(n)\rfloor, \lfloor t_2(n)\psi_2(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)}) \\
&\le P(M_{(\lfloor (t_1-\varepsilon)\psi_1(n)\rfloor, \lfloor (t_2-\varepsilon)\psi_2(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)}) \longrightarrow \gamma^{(t_1-\varepsilon)(t_2-\varepsilon)}.
\end{aligned}$$

Hence,

$$\lim_{n\to\infty} P(M_{(\lfloor t_1(n)\psi_1(n)\rfloor, \lfloor t_2(n)\psi_2(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)}) = \gamma^{t_1t_2} = \lim_{n\to\infty} \gamma^{t_1(n)t_2(n)},$$

and condition (29) is satisfied in case (a).

Now consider $\mathbf{t} = (t_1, 0)$ with $t_1 \in [0, \infty)$. Then $\gamma^{t_1(n)t_2(n)} \to 1$. Similarly, for every $\varepsilon > 0$ we have by part (i)

$$\begin{aligned}1 &\geq P(M_{(\lfloor t_1(n)\psi_1(n)\rfloor, \lfloor t_2(n)\psi_2(n)\rfloor)} \leq v_{\boldsymbol{\psi}(n)}) \\ &\geq P(M_{(\lfloor (t_1+\varepsilon)\psi_1(n)\rfloor, \lfloor \varepsilon\psi_2(n)\rfloor)} \leq v_{\boldsymbol{\psi}(n)}) \to \gamma^{(t_1+\varepsilon)\varepsilon}.\end{aligned}$$

Passing with $\varepsilon \to 0$ gives us (29) in case (b).

Next, assume that $\mathbf{t} = (\infty, t_2)$ for some $t_2 \in (0, \infty]$. Then $\gamma^{t_1(n)t_2(n)} \to 0$. Moreover, for all $R > 0$, $\varepsilon > 0$ and sufficiently large $n \in \mathbb{N}$ we have

$$\begin{aligned}0 &\leq P(M_{(\lfloor t_1(n)\psi_1(n)\rfloor, \lfloor t_2(n)\psi_2(n)\rfloor)} \leq v_{\boldsymbol{\psi}(n)}) \\ &\leq P(M_{(\lfloor R\psi_1(n)\rfloor, \lfloor (t_2-\varepsilon)\psi_2(n)\rfloor)} \leq v_{\boldsymbol{\psi}(n)}) \longrightarrow \gamma^{(t_2-\varepsilon)R}.\end{aligned}$$

Passing with $R \to \infty$ gives (29) in case (c) and completes the proof of part (ii) of the proposition. □

2.4. Proof of Theorem 1.4

2.4.1. *Necessity*

Suppose that G is a continuous distribution function. Take $\gamma \in (0, 1)$ and for $\mathbf{n} \in \mathbb{N}^d$ define

$$v_{\mathbf{n}} = \inf\{x : G(x)^{\mathbf{n}^*} = \gamma\}.$$

Then the field of levels $\{v_{\mathbf{n}}\}$ is strongly monotone.

If G is a phantom distribution function for $\{X_{\mathbf{n}}\}$, then

$$P(M_{\mathbf{n}} \leq v_{\mathbf{n}}) = G(v_{\mathbf{n}})^{\mathbf{n}^*} + o(1) = \gamma + o(1),$$

hence condition (i) of the theorem is satisfied.

Next, let $\boldsymbol{\psi}$ be a monotone curve and let $T > 0$. We want to verify Condition $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$. Assume that $\mathbf{p}(n) \to \infty$ and $\mathbf{q}(n) \to \infty$ satisfy additionally

$$\mathbf{p}(n) + \mathbf{q}(n) \leq T\boldsymbol{\psi}(n), \quad n \in \mathbb{N}.$$

Passing to a subsequence, if necessary, we can assume that

$$\frac{p_i(n)}{\psi_i(n)} \to s_i \in [0, T], \qquad \frac{q_i(n)}{\psi_i(n)} \to t_i \in [0, T], \quad i = 1, 2, \dots, d.$$

We have

$$P(M_{\mathbf{p}(n)+\mathbf{q}(n)} \leq v_{\boldsymbol{\psi}(n)}) = G(v_{\boldsymbol{\psi}(n)})^{(\mathbf{p}(n)+\mathbf{q}(n))^*} = G(v_{\boldsymbol{\psi}(n)})^{\boldsymbol{\psi}(n)^* \frac{(\mathbf{p}(n)+\mathbf{q}(n))^*}{\boldsymbol{\psi}(n)^*}} \longrightarrow \gamma^{\prod_{i=1}^d (s_i+t_i)}.$$

Consider the following expansion.

$$\prod_{i=1}^d (s_i + t_i) = \sum_{I_0 \subset \{1,2,\dots,d\}} \prod_{i \in I_0} s_i \times \prod_{j \notin I_0} t_j = \sum_{I_0 \subset \{1,2,\dots,d\}} \Pi_{I_0}.$$

It is clear that each term $\gamma^{\Pi_{I_0}}$ is a common limit for both $G(v_{\boldsymbol{\psi}(n)})^{\mathbf{r}(n)^*}$ and $P(M_{\mathbf{r}(n)} \leq v_{\boldsymbol{\psi}(n)})$, where

$$r_i(n) = \begin{cases} p_i(n) & \text{if } i \in I_0; \\ q_i(n) & \text{if } i \notin I_0. \end{cases}$$

We have proved that the difference between the two expressions appearing in Condition $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ tends to zero.

The same is also true if some coordinate of $\mathbf{p}(n)$ or $\mathbf{q}(n)$ remains bounded along a subsequence, since then the corresponding terms in the expansion converge to 1. Indeed, suppose that for example, $p_1(n) \leq K$, $n \in \mathbb{N}$. Then for large n

$$\begin{aligned} \lim_{n\to\infty} P(M_{(p_1(n),\ldots,p_d(n))} \leq v_{\boldsymbol{\psi}(n)}) &\geq \lim_{n\to\infty} P(M_{(\lfloor\varepsilon\psi_1(n)\rfloor,\lfloor T\psi_2(n)\rfloor,\ldots,\lfloor T\psi_d(n)\rfloor)} \leq v_{\boldsymbol{\psi}(n)}) \\ &= \lim_{n\to\infty} G(v_{\boldsymbol{\psi}(n)})^{\varepsilon T^{d-1}\boldsymbol{\psi}(n)^*} = \gamma^{\varepsilon T^{d-1}} \nearrow 1 \quad \text{as } \varepsilon \to 0. \end{aligned}$$

2.4.2. *Sufficiency*

Let $\{v_{\mathbf{n}}\}$ be a strongly monotone field of levels such that $P(M_{\mathbf{n}} \leq v_{\mathbf{n}}) \longrightarrow \gamma$, for some $\gamma \in (0, 1)$.

We shall show that along every monotone curve $\boldsymbol{\psi}(n)$ there exists a continuous phantom distribution function $G_{\boldsymbol{\psi}}$ and that all these functions are strictly tail-equivalent in the sense of [7], that is, if $G_{\boldsymbol{\psi}'}$ and $G_{\boldsymbol{\psi}''}$ are phantom distribution functions along monotone curves $\boldsymbol{\psi}'$ and $\boldsymbol{\psi}''$, respectively, then

$$(G_{\boldsymbol{\psi}'})_* = (G_{\boldsymbol{\psi}''})_* \quad \text{and} \quad \frac{1 - G_{\boldsymbol{\psi}'}(x)}{1 - G_{\boldsymbol{\psi}''}(x)} \to 1 \quad \text{as } x \to (G_{\boldsymbol{\psi}'})_*-.$$

Applying [7], Proposition 1, p. 700, one gets that

$$\sup_{x\in\mathbb{R}} \left| G_{\boldsymbol{\psi}'}(x)^n - G_{\boldsymbol{\psi}''}(x)^n \right| \longrightarrow 0. \tag{30}$$

If (30) holds for all pairs $\boldsymbol{\psi}'$ and $\boldsymbol{\psi}''$, then it is enough to set $G = G_{\boldsymbol{\Delta}}$, where $\boldsymbol{\Delta}(n) = (n, n, \ldots, n)$.

So let us take any monotone curve $\boldsymbol{\psi}(n)$ and assume that Condition $\mathbf{B}_T^{\boldsymbol{\psi}}(\{v_{\boldsymbol{\psi}(n)}\})$ holds for every $T > 0$.

We define $G_{\boldsymbol{\psi}}$ by the following formula.

$$G_{\boldsymbol{\psi}}(x) := \begin{cases} 0 & \text{if } x < v_{\boldsymbol{\psi}(1)}; \\ \gamma^{1/\boldsymbol{\psi}(n)^*} & \text{if } x \in [v_{\boldsymbol{\psi}(n)}, v_{\boldsymbol{\psi}(n+1)}); \\ 1 & \text{if } x \geq v_\infty := \sup\{v_{\boldsymbol{\psi}(n)} : n \in \mathbb{N}\}. \end{cases} \tag{31}$$

Notice that by Lemma 2.7 $v_{\boldsymbol{\psi}(n)} \nearrow F_* = (G_{\boldsymbol{\psi}})_*$.

We want to prove that for every sequence $\{x_n\} \subset \mathbb{R}$

$$P(M_{\boldsymbol{\psi}(n)} \leq x_n) - G_{\boldsymbol{\psi}}(x_n)^{\boldsymbol{\psi}(n)^*} \longrightarrow 0.$$

It is easy to see that the only nontrivial case is when $x_n \nearrow (G_{\boldsymbol{\psi}})_*$. For each $n \in \mathbb{N}$, let m_n be such that $v_{\boldsymbol{\psi}(m_n)} \leq x_n < v_{\boldsymbol{\psi}(m_n+1)}$ and let

$$t_1(n) = \frac{\psi_1(n)}{\psi_1(m_n)}, \qquad t_2(n) = \frac{\psi_2(n)}{\psi_2(m_n)}, \qquad \ldots, \qquad t_d(n) = \frac{\psi_d(n)}{\psi_d(m_n)}.$$

By the monotonicity of $\boldsymbol{\psi}(n)$, for given n either $t_1(n), t_2(n), \dots, t_d(n) \le 1$, or $t_i(n) \ge 1, i = 1, 2, \dots, d$, so that the set $A = \{\mathbf{t}(n) = (t_1(n), t_2(n), \dots, t_d(n)); n \in \mathbb{N}\}$ satisfies the assumption of part (ii) in Proposition 2.8. Consequently

$$
\begin{aligned}
P(M_{\boldsymbol{\psi}(n)} \le x_n) &\ge P(M_{\boldsymbol{\psi}(n)} \le v_{\boldsymbol{\psi}(m_n)}) \\
&= P(M_{(t_1(n)\psi_1(m_n), t_2(n)\psi_2(m_n), \dots, t_d(n)\psi_d(m_n))} \le v_{\boldsymbol{\psi}(m_n)}) \\
&= \gamma^{t_1(n)\cdot t_2(n)\cdots t_d(n)} + o(1).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
P(M_{\boldsymbol{\psi}(n)} \le x_n) &\le P(M_{\boldsymbol{\psi}(n)} \le v_{\boldsymbol{\psi}(m_n+1)}) \\
&= \gamma^{t_1(n)\cdot t_2(n)\cdots t_d(n) \frac{\boldsymbol{\psi}(m_n)^*}{\boldsymbol{\psi}(m_n+1)^*}} + o(1) = \gamma^{t_1(n)\cdot t_2(n)\cdots t_d(n)} + o(1).
\end{aligned}
$$

Therefore

$$
P(M_{\boldsymbol{\psi}(n)} \le x_n) = \gamma^{\frac{\boldsymbol{\psi}(n)^*}{\boldsymbol{\psi}(m_n)^*}} + o(1) = G_{\boldsymbol{\psi}}(x_n)^{\boldsymbol{\psi}(n)^*} + o(1),
$$

and our claim follows. It remains to replace the purely discontinuous distribution function $G_{\boldsymbol{\psi}}$ with another that is continuous and strictly tail-equivalent to $G_{\boldsymbol{\psi}}$. This can be done following for example, [7], pp. 703–704.

Remark 2.9. Note that so far we have used only the monotonicity of levels $\{v_{\boldsymbol{\psi}}\}$!

In order to prove the strict tail-equivalence of all $G_{\boldsymbol{\psi}}$, we need a slight improvement of [7], Proposition 1.

Lemma 2.10. *Let $\{\phi(n)\} \subset \mathbb{N}$ be increasing and such that $\phi(n)/\phi(n+1) \to 1$. If two distribution functions G and H satisfy*

$$
\lim_{n\to\infty} G(v_n)^{\phi(n)} = \lim_{n\to\infty} H(v_n)^{\phi(n)} = \gamma \in (0, 1),
$$

for some non-decreasing sequence of levels $\{v_n\}$, then G and H are strictly tail-equivalent.

Proof. We mimic [7], p. 701. Let $x_n \nearrow G_* = H_*$ and let m_n be such that $v_{m_n} \le x_n < v_{m_n+1}$, $n \in \mathbb{N}$. Then

$$
\phi(m_n)\big(1 - G(v_{m_n+1})\big) \le \phi(m_n)\big(1 - G(x_n)\big) \le \phi(m_n)\big(1 - G(v_{m_n})\big).
$$

Then both $\phi(m_n)(1 - G(v_{m_n})) \longrightarrow -\log\gamma$ and

$$
\phi(m_n)\big(1 - G(v_{m_n+1})\big) = \frac{\phi(m_n)}{\phi(m_n+1)}\phi(m_n+1)\big(1 - G(v_{m_n+1})\big) \longrightarrow -\log\gamma,
$$

and so $\phi(m_n)(1 - G(x_n)) \longrightarrow -\log\gamma$. But we can repeat this procedure for H equally well. Therefore

$$
\lim_{n\to\infty} \frac{1 - G(x_n)}{1 - H(x_n)} = \lim_{n\to\infty} \frac{\phi(m_n)(1 - G(x_n))}{\phi(m_n)(1 - H(x_n))} = 1. \qquad \square
$$

Let $G_{\boldsymbol{\psi}'}$ and $G_{\boldsymbol{\psi}''}$ be phantom distribution functions defined by (31) for monotone curves $\boldsymbol{\psi}'$ and $\boldsymbol{\psi}''$.

By the very definition $G_{\boldsymbol{\psi}'}(v_{\boldsymbol{\psi}'(n)})^{\boldsymbol{\psi}'(n)^*} \longrightarrow \gamma$. So it is enough to show that also

$$G_{\boldsymbol{\psi}''}(v_{\boldsymbol{\psi}'(n)})^{\boldsymbol{\psi}'(n)^*} \longrightarrow \gamma.$$

Let m_n be such that $\boldsymbol{\psi}''(m_n)^* \leq \boldsymbol{\psi}'(n)^* < \boldsymbol{\psi}''(m_n+1)^*$. Clearly, we have

$$\lim_{n\to\infty} \frac{\boldsymbol{\psi}''(m_n)^*}{\boldsymbol{\psi}'(n)^*} = \lim_{n\to\infty} \frac{\boldsymbol{\psi}''(m_n+1)^*}{\boldsymbol{\psi}'(n)^*} = 1. \tag{32}$$

Since $v_{\mathbf{n}}$ is *strongly* monotone, we have also $v_{\boldsymbol{\psi}''(m_n)} \leq v_{\boldsymbol{\psi}'(n)} \leq v_{\boldsymbol{\psi}''(m_n+1)}$, hence

$$G_{\boldsymbol{\psi}''}(v_{\boldsymbol{\psi}''(m_n)})^{\boldsymbol{\psi}'(n)^*} \leq G_{\boldsymbol{\psi}''}(v_{\boldsymbol{\psi}'(n)})^{\boldsymbol{\psi}'(n)^*} \leq G_{\boldsymbol{\psi}''}(v_{\boldsymbol{\psi}''(m_n+1)})^{\boldsymbol{\psi}'(n)^*}.$$

By (32) the first and the third terms converge to γ, and so $G_{\boldsymbol{\psi}'}$ and $G_{\boldsymbol{\psi}''}$ are strictly tail-equivalent. This completes the proof of Theorem 1.4.

2.5. Proof of Theorem 1.12

Implication (ii) $\Rightarrow$ (i) is a matter of definitions. Implication (i) $\Rightarrow$ (iii) can be proved the same way as the necessity in Section 2.4.1 (with obvious modifications).

We may also profit from the proof of Theorem 1.4 in the proof of implication (iii) $\Rightarrow$ (ii). Let $\boldsymbol{\psi}$ be a monotone curve satisfying assumption (iii) of Theorem 1.12. By Remark 2.9 function $G_{\boldsymbol{\psi}}$ defined by (31) is a phantom distribution function for $\{X_{\mathbf{n}}\}$ *along* $\boldsymbol{\psi}$. We want to show that it is also a phantom distribution function for $\{X_{\mathbf{n}}\}$ along *any other* $\boldsymbol{\varphi} \in \mathcal{U}_{\boldsymbol{\psi}}$, i.e. that for any $x_n \nearrow (G_{\boldsymbol{\psi}})_* = F_*$ we have

$$P(M_{\boldsymbol{\varphi}(n)} \leq x_n) - G_{\boldsymbol{\psi}}(x_n)^{\boldsymbol{\varphi}(n)^*} \longrightarrow 0.$$

For each $n \in \mathbb{N}$, let m_n be such that $v_{\boldsymbol{\psi}(m_n)} \leq x_n < v_{\boldsymbol{\psi}(m_n+1)}$ and let

$$t_1(n) = \frac{\varphi_1(n)}{\psi_1(m_n)}, \qquad t_2(n) = \frac{\varphi_2(n)}{\psi_2(m_n)}, \qquad \ldots, \qquad t_d(n) = \frac{\varphi_d(n)}{\psi_d(m_n)}.$$

We are going to show that the set $A = \{\mathbf{t}(n) = (t_1(n), t_2(n), \ldots, t_d(n));\, n \in \mathbb{N}\}$ satisfies the assumption of part (ii) in Proposition 2.8. By the definition of the class $\mathcal{U}_{\boldsymbol{\psi}}$, let $C \geq 1$ be such that for almost all $n \in \mathbb{N}$

$$\boldsymbol{\varphi}(n) \in \bigcup_{j\in\mathbb{N}} \prod_{i=1}^{d} \big[C^{-1}\psi_i(j), C\psi_i(j)\big].$$

This means that for $n \geq n_0$ there is $j_n \to \infty$ such that

$$C^{-1}\psi_i(j_n) \leq \varphi_i(n) \leq C\psi_i(j_n), \quad i = 1, 2, \ldots, d.$$

Depending on whether $j_n \leq m_n$ or $j_n \geq m_n$ we get that either $t_1(n), t_2(n), \ldots, t_d(n) \leq C$ or $t_1(n), t_2(n), \ldots, t_d(n) \geq C^{-1}$. Hence, we may apply Proposition 2.8 (ii) and we can estimate

$$\begin{aligned}
P(M_{\boldsymbol{\varphi}(n)} \leq x_n) &\geq P(M_{\boldsymbol{\varphi}(n)} \leq v_{\boldsymbol{\psi}(m_n)}) \\
&= P(M_{(t_1(n)\psi_1(m_n), t_2(n)\psi_2(m_n), \ldots, t_d(n)\psi_d(m_n))} \leq v_{\boldsymbol{\psi}(m_n)}) \\
&= \gamma^{t_1(n)\cdot t_2(n)\cdots t_d(n)} + o(1),
\end{aligned}$$

and

$$P(M_{\boldsymbol{\varphi}(n)} \le x_n) \le P(M_{\boldsymbol{\varphi}(n)} \le v_{\boldsymbol{\psi}(m_n+1)})$$
$$= \gamma^{t_1(n)\cdot t_2(n)\cdots t_d(n)\frac{\boldsymbol{\psi}(m_n)^*}{\boldsymbol{\psi}(m_n+1)^*}} + o(1) = \gamma^{t_1(n)\cdot t_2(n)\cdots t_d(n)} + o(1).$$

Therefore,

$$P(M_{\boldsymbol{\varphi}(n)} \le x_n) = \gamma^{\frac{\boldsymbol{\varphi}(n)^*}{\boldsymbol{\psi}(m_n)^*}} + o(1) = G_{\boldsymbol{\psi}}(x_n)^{\boldsymbol{\varphi}(n)^*} + o(1),$$

and Theorem 1.12 follows.

2.6. Proof of Theorem 1.13

Functions

$$f_n(\mathbf{s}) = P(M_{(\lfloor s_1\psi_1(n)\rfloor, \lfloor s_2\psi_2(n)\rfloor, \dots, \lfloor s_d\psi_d(n)\rfloor)} \le v_{\boldsymbol{\psi}(n)}), \quad n \in \mathbb{N}, \qquad f_\infty(\mathbf{s}) = \gamma^{s_1 s_2 \cdots s_d},$$

are non-increasing on $[0, +\infty)^d$ and f_∞ is continuous. Therefore, we can apply part (ii) of Proposition 2.8 and deduce that $\{f_n\}$ converges to f_∞ uniformly on any set A possessing the property described therein.

Suppose that $\boldsymbol{\varphi} \in \mathcal{U}_{\boldsymbol{\psi}}$, that is, there exist $C \ge 1$, n_0 and $j_n \to \infty$, defined for $n \ge n_0$, such that

$$C^{-1}\psi_i(j_n) \le \varphi_i(n) \le C\psi_i(j_n), \quad i = 1, 2, \dots, d, n \ge n_0.$$

Then we can proceed as in the proof of Theorem 1.12 and verify that G defined by (31) is a phantom distribution function for $\{Z_{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ along $\boldsymbol{\varphi}$. This proves the sufficiency part. The necessity follows as in the general case.

2.7. Proof of Theorems 1.20 and 1.21

In view of Theorems 1.4 and 1.12 and Definition 1.17, it is enough to show that F^θ is tail equivalent to a phantom distribution function G (resp. a $\boldsymbol{\psi}$-directional phantom distribution function $G_{\boldsymbol{\psi}}$) for $\{X_{\mathbf{n}}\}$.

But by (20), we have

$$G(v_{\mathbf{n}})^{\mathbf{n}^*} \to \gamma_{or}, \qquad \big(F^\theta(v_{\mathbf{n}})\big)^{\mathbf{n}^*} = \big(F(v_{\mathbf{n}})^{\mathbf{n}^*}\big)^\theta \to \gamma_{in}^\theta = \gamma_{or},$$

and we can apply Lemma 2.10. The reasoning leading to Theorem 1.21 differs only by notation.

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