Minimal $\mathcal{L}^p$-densities with prescribed marginals

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We derive sharp lower bounds for $\mathcal{L}^p$-functions on the n-dimensional unit hypercube in terms of their p-ths marginal moments. Such bounds are the unique solutions of a system of constrained nonlinear integral equations depending on the marginals. For square-integrable functions, the bounds have an explicit expression in terms of the second marginals moments.

Keywords: Banach spaces; integral equations; multivariate distributions; sharp estimates

1. Introduction

This paper obtains lower bounds of $\mathcal{L}^p$-functions on the n-dimensional unit hypercube in terms of moments of their marginals (i.e., one-dimensional projections). The lower bound is identified as the solution of a system of constrained nonlinear integral equations, where the marginals appear as inhomogeneous data. In the Hilbertian case $p = 2$ (and only in such a case), the integral equations become linear and admit an explicit solution (Section 2.3), implying that any square-integrable function $g : \mathbb{R}^n \ni \xi \mapsto g(\xi) \in \mathbb{R}$ such that $\int g(\xi)\, d\xi = 1$, satisfies the bound

$$\int g^2(\xi)\, d\xi \geq \left(\sum_{i=1}^{n} \int g_i(\xi_i)^2\, d\xi_i \right) - (n-1), \qquad (1.1)$$

where $\xi = (\xi_i)_{1 \leq i \leq n}$ and the marginal g_i is the integral of g with respect to all but the i-ths argument ξ_i, $1 \leq i \leq n$. Henceforth, all integrals are understood on the unit (hyper)cube of the respective integration variable, unless explicitly stated otherwise.

This bound is reminiscent of the lower Fréchet–Hoeffding bound [7], Theorem 2.2.3, for a copula $C = C(\xi)$ (i.e., the joint cumulative distribution function of n-dimensional random variable with uniform marginals),

$$C(\xi) \geq \max \left\{ \sum_{i=1}^{n} \xi_i - (n-1), 0 \right\}. \qquad (1.2)$$

However, a closer inspection shows that such a bound is rather different from the ones considered in this paper. First, (1.2) is a point-wise bound (which reduces to state mere positivity on the set $\{ \sum_{i=1}^{n} \xi_i < n-1 \}$), while (1.1) is a norm bound. Second, while copula bounds are – by construction – independent of marginal distributions, the norm bound established in this paper depends critically on the marginals densities considered.

The results of this paper are originally motivated by questions arising in the optimisation of options portfolios [4], which are equivalent, by convex duality, to the minimisation of the second moment of

the stochastic discount factor, subject to constraints on its marginals. This paper investigates the more challenging setting of an arbitrary moment (as opposed to the second moment), which leads to two related issues. First, the problem cannot be tackled with Hilbert space techniques, but requires arguments from Banach space theory. Second, while the second moment leads to linear first-order conditions and a minimal density that is additive across different variables, the general case considered here entails as first-order conditions nonlinear integral equations and a resulting more complex nonlinear decomposition of the minimal density.

The rest of the paper is organised as follows. Section 2 sets up the general $\mathcal{L}^p$-problem and offers a heuristic derivation of the equations governing the lower estimate. The rigorous proof of the sharp lower bound follows in Theorem 2.5. This general statement is then applied to the case $p = 2$ in Section 2.3, and results in Theorem 2.7. Section 2.4 reinterprets Theorem 2.5 as a regularity result for nonlinear integral equations. Section 3 provides a numerical study of the minimal solutions to the integral equations in dimension two, discussing how the optimisers depart from the linear case $p = 2$. Finally, Section 4 explains the probabilistic implications for portfolio selection.

2. $\mathcal{L}^p$-bounds

Let $1 < p < \infty$ and $n \geq 2$. Denote by $\mathcal{L}^p([0, 1]^n)$ the space of equivalence classes of Lebesgue-measurable functions f on the unit hypercube $[0, 1]^n$, for which $\|f\|_p := (\int |f(\xi)|^p \, d\xi)^{\frac{1}{p}} < \infty$. $\mathcal{L}^p$-spaces are strictly convex[1] in that for any $0 < t < 1$ and $f, g \in \mathcal{L}^p([0, 1]^n)$ such that $\|f\|_p = \|g\|_p = 1$ and $f \neq g$, it holds that $\|tf + (1 - t)g\|_p < 1$. As $\mathcal{L}^p$-spaces are also reflexive, the following Banach-space analog [5], Corollary 5.1.19, of a familiar Hilbert space result [8], Theorem 4.10, holds:

Theorem 2.1. *If a normed space is rotund and reflexive, then each of its nonempty, convex, closed subsets has a unique minimal element of smallest norm.*

Henceforth, for $1 \leq i \leq n$ denote by ξ_i the ith coordinate of $\xi \in \mathbb{R}^n$ and by ξ_i^c the $(n - 1)$-vector, in which the latter is omitted, that is, $\xi_i^c = (\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n)$. We shall integrate functions $f = f(\xi)$ either with respect to ξ_i^c, in which case we write $f_i(\xi_i) := \int f(\xi) \, d\xi_i^c$, or with respect to ξ_i, and then we write $\int f(\xi) \, d\xi_i$.

2.1. Heuristic derivation

To minimise the $\mathcal{L}^p$-norm $\|h\|_p$ subject to the marginal constraints

$$\int h(\xi) \, d\xi_i^c = g_i(\xi_i), \quad 1 \leq i \leq n,$$

consider the Lagrangian

$$L = \frac{1}{p} \int |h(\xi)|^p \, d\xi - \frac{1}{n} \sum_{i=1}^{n} \int \Phi(\xi_i) \left(\int h(\xi) \, d\xi_i^c - g_i(\xi_i) \right) d\xi_i.$$

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[1] This property is also called rotund, cf. [5], Example 1.10.2 and Theorem 1.11.10,

Setting the directional derivatives equal to zero yields the first order conditions

$$\text{sign}\big(h(\xi)\big)\big|h(\xi)\big|^{p-1} = \frac{1}{n}\sum_{i=1}^{n}\Phi_i(\xi_i),$$

whence

$$h(\xi) = \text{sign}\left(\sum_{i=1}^{n}\Phi_i(\xi_i)\right)\left|\frac{1}{n}\sum_{i=1}^{n}\Phi_i(\xi_i)\right|^{\frac{1}{p-1}}.$$

The marginal constraints imply

$$\int \text{sign}\left(\frac{1}{n}\sum_{j=1}^{n}\Phi_j(\xi_j)\right)\left|\frac{1}{n}\sum_{j=1}^{n}\Phi_j(\xi_j)\right|^{\frac{1}{p-1}} d\xi_i^c = g_i(\xi_i), \quad 1 \le i \le n. \tag{2.1}$$

To uniquely identify[2] the Lagrange multipliers Φ_i – which are otherwise determined up to an additive constant – it suffices to impose the conditions

$$\int \Phi_i(\xi_i)\, d\xi_i = 0, \quad 2 \le i \le n. \tag{2.2}$$

Note that these conditions are required only for $i \ge 2$.

2.2. Main result

The discussion begins with a characterisation of minimality in Banach spaces [9], Theorem 4.21:

Lemma 2.2. *Let* $1 < p < \infty$, $f \in \mathcal{L}^p$, *and* Y *be a closed subspace of* $\mathcal{L}^p$. *The following are equivalent*:

1. $\|f\|_p \le \|f + k\|_p$ *for all* $k \in Y$.
2. $\int \text{sign}(f(\xi))|f(\xi)|^{p-1}k(\xi)\, d\xi = 0$ *for all* $k \in Y$.

A function f is *orthogonal* to a subspace Y if it satisfies any of the equivalent statements of Lemma 2.2.

Lemma 2.3. *Let* $1 < p < \infty$, $f \in \mathcal{L}^q([0,1]^n)$, *where* $q = p/(p-1)$, *and denote by*

$$\mathcal{N} := \left\{\phi \in \mathcal{L}^p([0,1]^n)\,\middle|\, \int \phi(\xi)\, d\xi_i^c \equiv 0, \text{for } 1 \le i \le n\right\}. \tag{2.3}$$

The following are equivalent:

1. $\int f(x)\phi(x)\, dx = 0$ *for all* $\phi \in \mathcal{N}$.
2. $f(x) = \frac{1}{n}\sum_{i=1}^{n}\Psi_i(x_i)$, *where* Ψ_i *lie in* $\mathcal{L}^q([0,1])$, $1 \le i \le n$.

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[2]For the uniqueness proof, see the proof of Theorem 2.5.

Proof. The implication (2) $\Rightarrow$ (1) is straightforward. To show that (1) $\Rightarrow$ (2), note that, by Jensen's inequality, for any $\phi \in \mathcal{L}^p([0, 1]^n)$ and any $1 \le i \le n$, $\int \phi(\xi) \, d\xi_i^c \in \mathcal{L}^p([0, 1])$. Hence, for any $\phi \in \mathcal{L}^p([0, 1]^n)$,

$$\widetilde{\phi}(\xi) := \phi(\xi) - \sum_{i=1}^{n} \int \phi(\xi) \, d\xi_i^c + (n-1) \int \phi(\eta) \, d\eta \in \mathcal{N},$$

and Fubini's theorem yields

$$\int \left(f(x) - \sum_{i=1}^{n} \int f(\xi) \, d\xi_i^c + (n-1) \int f(\xi) \, d\xi \right) \phi(x) \, dx = 0.$$

Thus by duality,

$$f(\xi) = \sum_{i=1}^{n} \int f(\xi) \, d\xi_i^c - (n-1) \int f(\xi) \, d\xi \quad \xi\text{-a.e..}$$

The functions $\Phi_i(\xi_i) := n \int f(\xi) \, d\xi_i^c - (n-1) \int f(\xi) \, d\xi$, $1 \le i \le n$, are in $\mathcal{L}^q([0, 1]^n)$, and they sum to f, as claimed. $\qquad \square$

The previous two lemmas combine to the following.

Corollary 2.4. *Let $f \in \mathcal{L}^q([0, 1]^n)$, and $\mathcal{N} \subset \mathcal{L}^p([0, 1]^n)$ as defined in (2.3). The following are equivalent*:

1. $\mathrm{sign}(f)|f|^{1/(p-1)}$ *is orthogonal to $\mathcal{N}$.*
2. $\int f(x)\phi(x) \, dx = 0$ *for all $\phi \in \mathcal{N}$.*
3. $f(x) = \frac{1}{n} \sum_{i=1}^{n} \Psi_i(x_i)$*, where Ψ_i lie in $\mathcal{L}^q([0, 1]^n)$, $1 \le i \le n$.*

Theorem 2.5. *Let $p > 1$. Any $g \in \mathcal{L}^p$ satisfies*

$$\int |g(\xi)|^p \, d\xi \ge \int \left| \overline{\Phi}(\xi) \right|^{\frac{p}{p-1}} d\xi, \tag{2.4}$$

where

$$\overline{\Phi}(\xi) := \frac{1}{n} \sum_{i=1}^{n} \Phi_i(\xi_i)$$

and Φ_i are the unique solutions of the system of integral equations (2.1)–(2.2).
The estimate (2.4) is sharp, in that equality in (2.4) holds, if and only if

$$g(\xi) = \mathrm{sign}(\overline{\Phi}(\xi)) |\overline{\Phi}(\xi)|^{\frac{1}{p-1}}. \tag{2.5}$$

Proof. By Jensen's inequality, $g_i := \int g(\xi) \, d\xi_i^c \in \mathcal{L}^p([0, 1])$ for $1 \le i \le n$, hence the set

$$\mathcal{M} := \left\{ h \in \mathcal{L}^p([0, 1]^n) \middle| \int h(\xi) \, d\xi_i^c = g_i(\xi_i), 1 \le i \le n \right\}$$

is well-defined, and it is non-empty because $g \in \mathcal{M}$. The set is convex, by construction, and it is closed: Let $h_n \in \mathcal{M}$ and $\lim_{n \to \infty} h_n = h$ in $\mathcal{L}^p([0, 1]^n)$. Then the sequence $(h_n)_{n \ge 1}$ is uniformly integrable,

hence by Vitali's convergence theorem, ξ_i-almost everywhere,

$$\int h(\xi)\,d\xi_i^c = \int \lim_{n\to\infty} h_n(\xi)\,d\xi_i^c = \lim_{n\to\infty} \int h_n(\xi)\,d\xi^c = g_i(\xi_i).$$

Denote by h_* the unique element in $\mathcal{M}$ of smallest norm.[3] We claim that $h_* = g$, where g is defined in (2.5). To this end, recall the function space defined in (2.3). By the minimality of h_*, it follows that for any $\varepsilon > 0$ and any $\phi \in \mathcal{N}$

$$\|h_* \pm \varepsilon\phi\|_p^p - \|h_*\|_p^p \geq 0, \tag{2.6}$$

and therefore, by Lemma 2.2[4]

$$\int \operatorname{sign}(h_*(\xi))|h_*(\xi)|^{p-1}\phi(\xi)\,d\xi = 0, \quad \phi \in \mathcal{N}.$$

The implication (1) $\Rightarrow$ (3) in Corollary 2.4 yields

$$\operatorname{sign}(h_*(\xi))|h_*(\xi)|^{p-1} = \overline{\Phi}(\xi), \quad \text{where } \overline{\Phi}(\xi) := \frac{1}{n}\sum_{i=1}^{n}\Phi_i(\xi_i),$$

with measurable functions $\Phi_i(\xi_i)$, $1 \leq i \leq n$, depending on one variable ξ_i only. Because $\operatorname{sign}(h_*) = \operatorname{sign}(\overline{\Phi}(\xi))$, it follows that

$$h_*(\xi_1,\ldots,\xi_n) = \operatorname{sign}(\overline{\Phi}(\xi))|\overline{\Phi}(\xi)|^{\frac{1}{p-1}}$$

and $\overline{\Phi}$ solves the nonlinear integral equations (2.1) for $1 \leq i \leq n$. As these equations involve the sum $\overline{\Phi}$ only, we can satisfy the extra constraints (2.2), by replacing Φ_i by $\Phi_i - \int \Phi_i(\xi_i)\,d\xi_i$ ($2 \leq i \leq n$), if necessary.

It remains to show the uniqueness. Assume that, besides $\overline{\Phi}$, also $\overline{\Psi}(\xi) := \frac{1}{n}\sum_{i=1}^{n}\Psi_i(\xi_i)$ solves (2.1)–(2.2). By Corollary 2.4 (3) $\Rightarrow$ (1), the function $h := \operatorname{sign}(\overline{\Psi})|\overline{\Psi}|^{\frac{1}{p-1}}$ is orthogonal to $\mathcal{N}$ defined in (2.3). Furthermore, by (2.1), $h - h_* \in \mathcal{N}$, hence by definition of orthogonality, $\|h\|_p \leq \|h_*\|_p$. In view of (2.6), $h = h_*$, whence also $\overline{\Psi} = \overline{\Phi}$. As $\overline{\Phi}(\xi) = \frac{1}{n}\sum_{i=1}^{n}\Phi_i(\xi_i) = \frac{1}{n}\sum_{i=1}^{n}\Psi_i(\xi_i) =: \overline{\Psi}(\xi)$ almost everywhere, the extra constraints (2.2) yield, upon integration of $n\overline{\Phi} = n\overline{\Psi}$ with respect to $d\xi_1^c$, that $\Phi_1(\xi_1) = \Psi_1(\xi_1)$ almost everywhere (the rest of the integrals vanish). Applying the constraint for $i = 2$, it follows that

$$\int \Phi_1(\xi_1)\,d\xi_1 + \Phi_2(\xi_2) + 0 = \int \Psi_1(\xi_1)\,d\xi_1 + \Psi_2(\xi_2) + 0 = \int \Phi_1(\xi_1)\,d\xi_1 + \Psi_2(\xi_2),$$

whence $\Phi_2(\xi_2) = \Psi_2(\xi_2)$ ξ_2-almost everywhere. Continuing similarly for $3 \leq i \leq n$, it follows that $\Phi_i = \Psi_i$ ξ_i-almost everywhere for $3 \leq i \leq n$. □

Remark 2.6. Suppose all marginals are densities. If $n - 1$ marginal data are uniform densities, say $g_i(\xi_i) = 1$ for $i \in \mathcal{T} \subset \{1, 2, \ldots, n\}$ with $|T| \geq n - 1$, then the corresponding solutions Φ_i ($i \in T$) are constants. This is easiest to see for $T = \{2, \ldots, n\}$. Then, it can be verified that the solution of (2.1)–(2.2) is of the form $\Phi_i = 0$ for $2 \leq i \leq n$, and $\Phi_1 := \operatorname{sign}(g_1)|g_1|^{p-1}$. If the index set T contains 1, then Φ_1 will be a constant, but not necessarily equals to 0 (see last row of Figure 1).

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[3] See Theorem 2.1, and the paragraph preceding it concerning its applicability for $\mathcal{L}^p$ spaces.
[4] Note that $|h_*|^{p-1} \in L^q$, where $q = \frac{p}{p-1}$, hence the below pairing is finite, by Hölder's inequality.
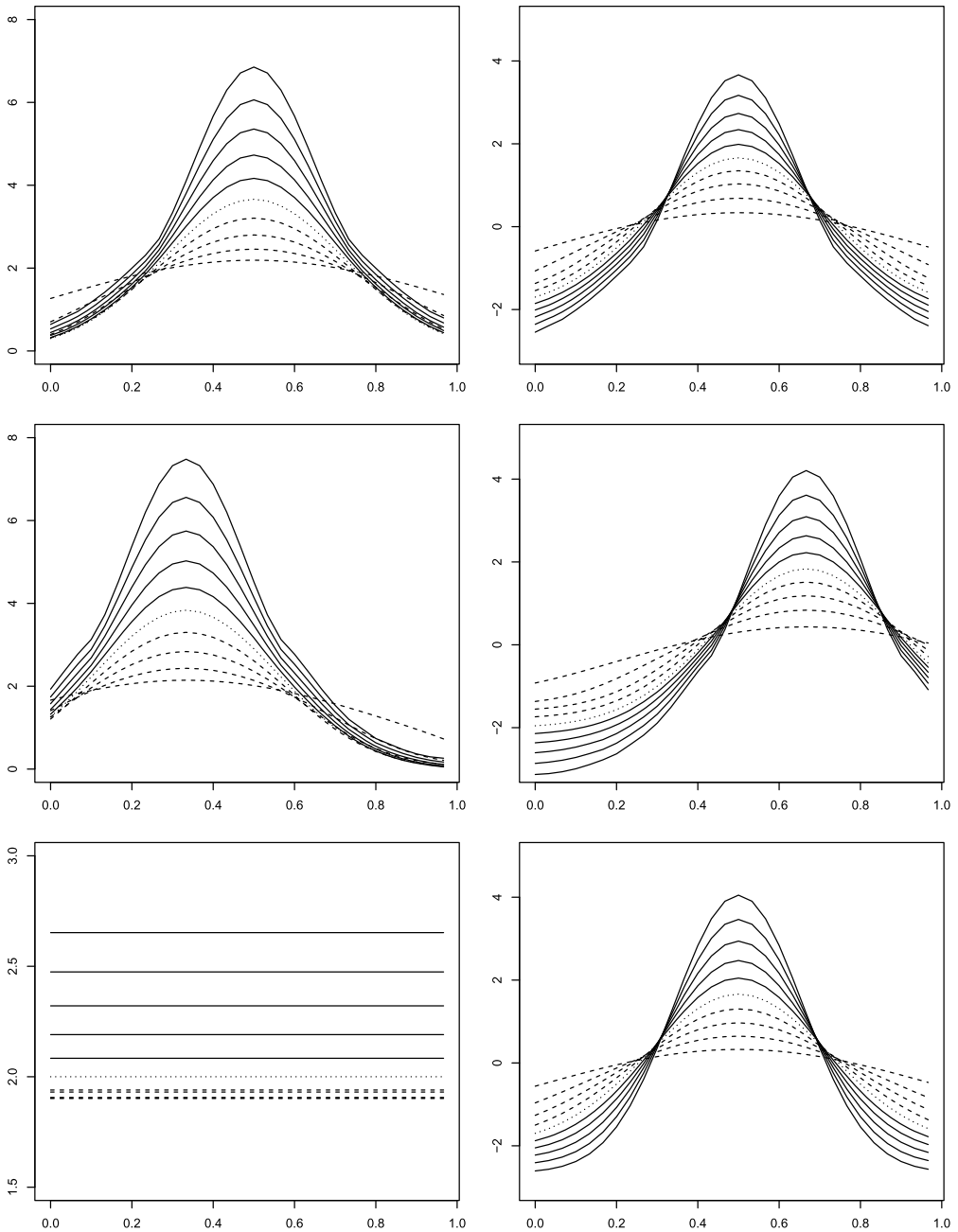
Figure 1. In each row, we depict the solutions of the nonlinear integral equations (2.1)–(2.2); on the left panels Φ_1 and on the right one Φ_2, respectively. The dotted lines depict the explicit solution in the $p = 2$ case; the other lines depart from the Hilbertian case by $\Delta p = 20\%$ steps; the dashed lines depict the solutions for $p = 1.8, 1.6, 1.4$ and 1.2. Similarly, the solid lines depict the numerical solutions for $p = 2.2, 2.4, \ldots, 3$. In the first row, both marginals are Gaussian densities with equal parameter $\mu = 1/2$ and $\sigma^2 = 10\%$, truncated and re-normalised so to have unit mass on $[0, 1]$. In the second row, the means are varied to $\mu = 1/3$ on the left, and $\mu = 2/3$ on the right. In the third row, $g_1 = 1$ (uniform density).

2.3. The $\mathcal{L}^2$-case

For $p = 2$, the system of integral equations (2.1)–(2.2) becomes linear:

$$\left(\int \sum_{j=1}^{n} \Phi_j(\xi_j) \right) d\xi_i^c = n g_i(\xi_i), \quad 1 \leq i \leq n, \tag{2.7}$$

$$\int \Phi_i(\xi_i) \, d\xi_i = 0, \quad 2 \leq i \leq n. \tag{2.8}$$

For $i = 1$, Equation (2.7) yields $\Phi_1(\xi_1) = n g_1(\xi_1)$, and in conjunction with Equation (2.8), it follows that

$$\Phi_i(\xi_i) = n \left(g_i(\xi_i) - \int g_1(\xi_1) \, d\xi_1 \right) = n \left(g_i(\xi_i) - \int g(\xi) \, d\xi \right), \quad 2 \leq i \leq n.$$

In view of Theorem 2.5, it follows the following theorem.

Theorem 2.7. *Any* $g \in \mathcal{L}^2([0, 1]^n)$ *satisfies*

$$\int g^2(\xi) \, d\xi \geq \sum_{i=1}^{n} \int g_i(\xi_i)^2 \, d\xi_i - (n-1) \left(\int g(x) \, dx \right)^2. \tag{2.9}$$

The bound is sharp: *equality holds if and only if g equals*

$$\overline{\Phi}(\xi) := \left(\sum_{i=1}^{n} g_i(\xi_i) \right) - (n-1) \int g(x) \, dx.$$

Remark 2.8. Note that the above includes the case where g has vanishing integral. In this case, the estimate (2.9) lacks the last term $(n-1) \int g(x) \, dx$.

2.4. Regularity of integral equations

One can view Theorem 2.5 as a regularity result for nonlinear integral equations with constraints – the dual problem.

Corollary 2.9. *Let* $n \geq 2$ *and* $p > 1$. *For any* $g \in \mathcal{L}^p([0, 1]^n)$, *there exists a unique solution*

$$\Phi(\xi) = \left(\Phi_1(\xi_1), \Phi_2(\xi_2), \dots, \Phi_n(\xi_n) \right)$$

of the integral equations (2.1)–(2.2) *with data g, satisfying the bound*

$$\|\overline{\Phi}\|_q \leq \|g\|_p,$$

where q denotes conjugate exponent (satisfying $1/p + 1/q = 1$).

3. Numerical examples

For illustration, we study dimension $n = 2$ and consider data $g_1(\xi_1)$ and $g_2(\xi_2)$ of unit integral.

To solve the nonlinear integral equations (2.1)–(2.2), we discrete the involved integrals using a mesh-size of $1/30$, and solve the resulting nonlinear equations with the optim solver on R (minimising residuals), using the exact solution in the $p = 2$ case as starting value; and then increasing (resp. decreasing) from the Hilbertian case by $\Delta p = 10\%$, using repeatedly the previous numerical solution as seed for the solver. Each example involves normal marginal densities, re-normalised so to have unit mass on $[0, 1]$.[5]

Figure 1 below depicts solutions for three different sets of data g_1, g_2, and for a range of $p > 1$. The explicit solution for $p = 2$ (Theorem 2.7)

$$\Phi_1(\xi_1) = 2g_1(\xi_1),$$
$$\Phi_2(\xi_2) = 2\big(g_2(\xi_2) - 1\big).$$

is plotted by dotted lines. The figure generally shows that as $p \downarrow 1$, the solutions Φ_1, Φ_2 get flatter (dashed lines), while as p increases from 2, the solutions have more and more pronounced peaks (solid lines). In the first row of Figure 1, the data is identical, however since $\int \Phi_2(\xi_2) \, d\xi_2 = 0$, Φ_1 and Φ_2 differ by a real constant. Moreover, the second row of the Figure uses inhomogeneous data g_1, g_2, centering around different means. Finally, the last row of Figure 1 confirms the theoretical finding of Remark 2.6: the solution Φ_1 on the left, that corresponds to uniform marginal data g_1, is constant also for $p \neq 2$.

4. Probabilistic implications

The main result in this paper is originally motivated by the problem of selecting the portfolio of options with maximal Sharpe ratio, which is equivalent, by convex duality, to the minimisation of the second moment of the stochastic discount factor, subject to constraints on its marginals (cf. [4], Section 2.2). In this financial application, the baseline measure is described by a probability density $p(\xi)$, which identifies the views of an investor on the joint distribution of the prices of n risky assets $\mathcal{S}^1, \ldots, \mathcal{S}^n$ at some future date T (modelled by random variables $S^1, \ldots, S^n$). Each of these is an underlying asset of European Call and Put options with a continuum of strikes, maturing at T. The density p thus replaces the Lebesgue measure in the present paper. The corresponding $\mathcal{L}^p$-problem ($p > 1$) generalises equations (2.1)–(2.2) to

$$\int \operatorname{sign}\left(\frac{1}{n}\sum_{j=1}^{n}\Phi_j(\xi_j)\right)\left|\frac{1}{n}\sum_{j=1}^{n}\Phi_j(\xi_j)\right|^{\frac{1}{p-1}} p(\xi) \, d\xi_i^c = g_i(\xi_i), \quad 1 \leq i \leq n,$$

$$\int \Phi_i(\xi_i)p_i(\xi_i) \, d\xi_i = 0, \quad 2 \leq i \leq n.$$

The new weight p in these equations, and its n marginal densities p_i constitute only a small modification that can be treated similarly as in Section 2.2. The "inhomogeneous data" g_i ($1 \leq i \leq n$) represents

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[5]It may appear more natural to pick densities that integrate to one by default, e.g. instances of the beta distribution with parameters α, β. However, this seemingly more natural choice leads to similar effects as explained below. For intuitive and for numerical reasons, we found normal data more convenient.

the risk-neutral marginal distributions of asset S^i ($1 \leq i \leq n$) which is, due to [1,3,6] proportional to the second derivative of European Call options prices, with respect to their strike price.

An important implication of the result is that the solution is necessarily of the functional form $g(\xi) = \text{sign}(\overline{\Phi}(\xi)|\overline{\Phi}(\xi)|^{1/(p-1)}$, where $\overline{\Phi}(\xi) = \frac{1}{n} \sum_{j=1}^{n} \Phi_j(\xi_j)$. First, this result casts in a cautionary light the practice of pricing complex derivatives by fitting parametric copulas to risk-neutral marginals. In general, the price obtained from some copula family is neither replicable nor linked to an optimisation objective. Second, the result implies that the maximal random payoff $\overline{\Phi}$ (the primal object) is not an arbitrary function of the the asset prices $S^1, \ldots, S^n$, but it additively separates in each asset, that is, it is of the form

$$\overline{\Phi}(S_1, \ldots, S_n) = \frac{1}{n} \sum_{j=1}^{n} \Phi_j(S_j),$$

and thus can be interpreted as an option portfolio, featuring European options on each of the n individual underlyings. Indeed, according to the Carr-Madan formula [2], each summand Φ_j, if regular enough, can be written as

$$\Phi_j(K) = \Phi_j(K_0) + \Phi'_j(K_0)(K - K_0) + \int_0^{K_0} \Phi''_j(\kappa)(\kappa - K)^+ \, d\kappa$$

$$+ \int_{K_0}^{\infty} \Phi'_j(\kappa)(K - \kappa)^+ \, d\kappa, \quad K \geq 0$$

and thus decomposes into the payoff of $\Phi_j(K_0)$ units of a zero-coupon bond maturing at T (first summand), $\Phi'_j(K_0)$ forward contracts with delivery price K_0 (second summand) and a continuum of European puts (with strike $\kappa \leq K_0$), and European calls (with strikes $\kappa \geq K_0$).[6]

Acknowledgements

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